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THE DISTRIBUTION OF FATOU POINTS OF BOUNDED AND NORMAL ANALYTIC FUNCTIONS

BY

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§ 1. Introduction

A point $\zeta = e^{i\theta}$ on the unit circle $C_z : |z| = 1$ is said to be a Fatou point of a function $f(z)$ defined in the unit disk $D : |z| < 1$ provided there is a constant c such that $f(z)$ has angular limit c in each Stolz angle at ζ; c is called the Fatou value of $f(z)$ at ζ. In case $f(z)$ is a bounded analytic function, then $f(z)$ has c as Fatou value at ζ if and only if $f(z)$ has radial limit $f(\zeta) = c$ at ζ:

$$f(\zeta) = \lim_{r \to 1} f(r\zeta) ;$$

and by Fatou's theorem, $f(\zeta)$ exists for almost every point ζ (i.e. for almost every value of θ, $0 \leq \theta < 2\pi$).

In [6, Theorem 4] Seidel proved the following improvement of a theorem of R. Nevanlinna:

Theorem 0. *Let* $w = f(z)$ *be a bounded analytic function in* D:

$$|f(z)| < 1 .$$

Suppose $f(z)$ *has radial limits of modulus one at almost every point of an open arc* A *on* C_z. *Denote by* E_w *the set of radial limits* $f(\zeta)$, $\zeta \in A$, *that lie on the circle* $C_w : |w| = 1$. *If* $f(z)$ *has a singularity on* A, *then* E_w *is the whole circle* C_w.

In Theorem 0, it follows that E_w has the same property on each subarc of A containing a singularity of $f(z)$. We consider the following question. If E_w is any set on C_w and E_ζ is the set of *all* points ζ on C_z such that $f(\zeta) \in E_w$, what metric properties does E_ζ have in each neighborhood of a singularity ζ of $f(z)$ on A? Theorem 0 only asserts that there is a non-empty subset of E_ζ in each open arc containing ζ.

A bounded analytic function $f(z)$ constructed in [2, Theorem 12], originally due to J. A. Jenkins, serves to illustrate the problem. The function $f(z)$ has radial limits of modulus one at almost every point ζ on C_z (thus $|f(z)| < 1$) and each point ζ is a singularity of $f(z)$. Also, there exists a set K_ζ of measure zero on C_z that satisfies the following: Each point of K_ζ is a Fatou point of $f(z)$, and on each subarc of C_z there is a subset of K_ζ for which the corresponding set of Fatou values covers C_w. Thus, given any set E_w on C_w, if

$$E_{\zeta} = \{\zeta : f(\zeta) \in E_w\},$$

then conceivably E_{ζ} could be of measure zero on some subarc of C_z. However, as to be shown in Theorem 1, this is not the case provided E_w has positive inner Lebesgue measure.

In Section 3 functions of class (U) are considered. By a simple application of a well-known extension of Löwner's lemma, a general inequality on the inner and outer measures of E_{ζ} and E_w can be obtained for functions $f(z)$ of class (U) satisfying $f(0) = 0$. In fact, if E_{ζ} and E_w are measurable, their measures are equal. In Section 4 some limitations on the distribution of the set E_{ζ} are pointed out.

Finally, in Section 5 our results are used to answer negatively a question concerning the distribution of Fatou points of normal analytic functions posed by Bagemihl and Seidel [1, page 10].

§ 2. The distribution of Fatou points

A measurable set E on C_z is said to be *metrically dense at a point* $\zeta \in C_z$ provided

$$m(E \cap A) > 0$$

for every open arc A on C_z containing ζ, where m denotes Lebesgue measure.

Theorem 1. *Let $f(z)$ be a bounded analytic function in D:*

$$|f(z)| < 1 .$$

Suppose $f(z)$ has radial limits of modulus one at almost every point of an open arc A on C_z. Let E_w be any subset of C_w, and define

$$E_{\zeta} = \{\zeta : f(\zeta) \in E_w\}.$$

If E_w is a Borel set, $m(E_w) > 0$, then E_{ζ} is metrically dense at each singularity of $f(z)$ on A.

In general, for each singularity ζ of $f(z)$ on A and each open arc $A_0 \subset A$, $\zeta \in A_0$,

$$m_i(E_{\zeta} \cap A_0) > 0 \quad \textit{if} \quad m_i(E_w) > 0$$

and

$$m_e(E_{\zeta} \cap A_0) < m(A_0) \quad \textit{if} \quad m_e(E_w) < 2\pi ,$$

where m_i and m_e denote inner and outer Lebesgue measure respectively.

Proof. Suppose E_w is a Borel set, $m(E_w) > 0$. Then E_{ζ} is measurable (see [4, Corollary]). Take any singularity ζ of $f(z)$ on A and any open arc $A_0 \subset A$, $\zeta \in A_0$. We can assume $\zeta = 1$, or otherwise consider the function $f(\zeta z)$. Then, by [6, Theorem 8], there exists a point w_0, $|w_0| < 1$, and a sequence $\{z_n\}$, $|z_n| < 1$, $z_n \to 1$, such that

$$f(z_n) = w_0, \quad n = 1, 2, \ldots. \tag{1}$$

The linear transformation

$$\tau = \psi(w) = \frac{w - w_0}{1 - \bar{w}_0 w}$$

carries E_w onto a set E_τ on C_τ: $|\tau| = 1$ satisfying

$$m(E_\tau) > 0 .$$

As $z_n \to 1$, the linear transformations of $|z| \leq 1$ onto itself given by

$$\varphi_n(z) = \frac{z + z_n}{1 + \bar{z}_n z} \tag{2}$$

converge uniformly to 1 on each compact subset of $|z| \leq 1$ that does not contain $z = -1$. For, if $|z + 1| \geq \delta$, $|z| \leq 1$, and z_n is sufficiently close to 1, then

$$|\varphi_n(z) - 1| = \frac{|z(1 - \bar{z}_n) - (1 - z_n)|}{|1 + \bar{z}_n z|} \leq \frac{4}{\delta} |1 - z_n| .$$

Thus each function $\varphi_n(z)$ maps a subarc A_n of C_z onto A_0, where

$$\lim m(A_n) = 2\pi . \tag{3}$$

Consider the sequence

$$\tau = g_n(z) = \psi(f(\varphi_n(z))) .$$

Let

$$F_n = \{\zeta : g_n(\zeta) \in C_\tau - E_\tau\} .$$

Each F_n is measurable since F_n corresponds under $\varphi_n(z)$ to the measurable set F_ζ given by

$$F_\zeta = \{\zeta : f(\zeta) \in C_w - E_w\} ;$$

F_ζ being measurable because E_w is a Borel set [4, Corollary].

By (1) and (2), $g_n(0) = 0$ for each n, and, by Theorem 0, the set of radial limits $g_n(\zeta)$, $\zeta \in F_n \cap A_n$, covers $C_\tau - E_\tau$. An extension of Löwner's lemma [5, page 34] implies

$$\begin{aligned} m(F_n \cap A_n) &\leq m(C_\tau - E_\tau) \\ &= 2\pi - m(E_\tau) , \end{aligned} \tag{4}$$

where $m(E_\tau) > 0$.

It follows that

(5) $$m(F_\zeta \cap A_0) < m(A_0)\,,$$

for otherwise

$$m(F_n \cap A_n) = m(A_n)\,, \quad n = 1, 2, \ldots,$$

and, by (3),

$$\lim m(F_n \cap A_n) = 2\pi\,,$$

which contradicts (4). By Theorem 0, every point of C_w is a radial limit of $f(z)$ on A_0, and so

(6) $$m(E_\zeta \cap A_0) + m(F_\zeta \cap A_0) = m(A_0)\,.$$

By (5) and (6),

$$m(E_\zeta \cap A_0) > 0\,.$$

Thus E_ζ is metrically dense at $\zeta = 1$.

Now choose any singularity ζ of $f(z)$ on A and any open arc $A_0 \subset A$, $\zeta \in A_0$. If $m_i(E_w) > 0$, let E'_w be a closed set such that $E'_w \subset E_w$ and $m(E'_w) > 0$. Set

$$E'_\zeta = \{\zeta : f(\zeta) \in E'_w\}\,.$$

Since E'_w is a Borel set, E'_ζ is metrically dense at ζ. Thus, since $E'_\zeta \subset E_\zeta$,

$$m_i(E_\zeta \cap A_0) > 0\,.$$

Now suppose $m_e(E_w) < 2\pi$. Let

$$F_\zeta = \{\zeta : f(\zeta) \in C_w - E_w\}\,.$$

By what was just proven, since $m_i(C_w - E_w) > 0$,

$$m_i(F_\zeta \cap A_0) > 0\,.$$

By Theorem 0, the set of radial limits $f(\zeta)$, $\zeta \in A_0$, covers C_w, and so

$$\begin{aligned} m(A_0) &= m_e(E_\zeta \cap A_0) + m_i(F_\zeta \cap A_0) \\ &> m_e(E_\zeta \cap A_0)\,. \end{aligned}$$

This completes the proof.

§ 3. Functions of class (U)

If a bounded analytic function $f(z)$ in D has radial limits of modulus one at almost every point of C_z (thus $|f(z)| < 1$), then $f(z)$ is said to belong to the class (U). It is apparent how Theorem 1 applies to functions of class (U).

For functions $f(z)$ of class (U) satisfying $f(0) = 0$, Löwner's lemma assumes the following form.

Theorem 2. *Let $f(z)$ be of class (U), $f(0) = 0$. Let E_w be any subset of C_w, and set*

$$E_\zeta = \{\zeta : f(\zeta) \in E_w\} .$$

Then

$$m_i(E_\zeta) \leq m_e(E_w) \quad \text{and} \quad m_i(E_w) \leq m_e(E_\zeta) . \tag{7}$$

If E_w is a Borel set, then

$$m(E_\zeta) = m(E_w) . \tag{8}$$

Proof. Let

$$F_\zeta = \{\zeta : f(\zeta) \in C_w - E_w\} .$$

Since $f(z)$ is of class (U),

$$m(E_\zeta \cup F_\zeta) = 2\pi .$$

By the extension of Löwner's lemma in [5, page 34],

$$m_i(E_\zeta) \leq m_e(E_w) \tag{9}$$

and

$$\begin{aligned} m_i(F_\zeta) &\leq m_e(C_w - E_w) \\ &= 2\pi - m_i(E_w) . \end{aligned}$$

Thus

$$\begin{aligned} m_i(E_w) &\leq 2\pi - m_i(F_\zeta) \\ &= m_e(E_\zeta) . \end{aligned} \tag{10}$$

By (9) and (10), we have (7).

If E_w is a Borel set, then E_ζ is measurable, and (7) implies (8).

§ 4. Limitations on the distribution of Fatou points

The *lower mean metric density* of a measurable set E on C_z at a point $\zeta \in C_z$ is defined to be

$$\delta_-(E, \zeta) = \liminf_{\varepsilon \to 0} \frac{m(E \cap A_\varepsilon)}{2\varepsilon} ,$$

where A_ε is the open arc on C_z of length 2ε and midpoint ζ. It is a fundamental result in measure theory that

$$\delta_-(E, \zeta) = 1$$

at almost every point $\zeta \in E$. Thus, it is evident that given two measurable sets E_1 and E_2 on an open arc A on C_z,

$$m(E_1) > 0, \quad m(E_2) > 0,$$

both E_1 and E_2 can be metrically dense at every point of A, but it cannot be the case that both E_1 and E_2 have positive lower mean metric density at each point of A.

Therefore, in Theorem 1 it is not necessarily the case that E_ζ has positive lower mean metric density at each singularity of $f(z)$ on A. For, if E_w is a Borel set on C_w,

$$0 < m(E_w) < 2\pi,$$

and each $\zeta \in A$ is a singularity of $f(z)$, we arrive at such a conclusion by considering the sets

$$E_1 = E_\zeta \cap A, \quad E_2 = F_\zeta \cap A,$$

where

$$F_\zeta = \{\zeta : f(\zeta) \in C_w - E_w\}.$$

§ 5. An application to normal analytic functions

Let $f(z)$ be meromorphic in D. Then $f(z)$ is said to be normal if and only if the family

$$f\left(\frac{z+a}{1+\bar{a}z}\right), \quad |a| < 1,$$

is normal in the sense of Montel, where convergence is defined in terms of the spherical metric.

In [1, Theorem 3], Bagemihl and Seidel proved that if the set of Fatou points of a normal analytic function $f(z)$ is of measure zero on a subarc A of C_z, then $f(z)$ has ∞ as a Fatou value at some point of A. In the same paper is shown the following: Given any $\varepsilon > 0$, there exists an analytic function $f(z)$ normal in D such that the set of Fatou points of $f(z)$ has measure less than ε, but ∞ is not a Fatou value of $f(z)$. However, the set of Fatou points of $f(z)$ is an open subset of C_z and the following question remained unanswered. If E_ζ is the set of Fatou points of a normal analytic function $f(z)$ in D and

$$0 < m(E_\zeta \cap A_0) < m(A_0)$$

for each subarc A_0 of an arc A on C_z, need ∞ be a Fatou value of $f(z)$ at a point of A? The following theorem answers this question negatively.

Theorem 3. *Given* $\varepsilon > 0$, *there exists an analytic function* $f(z)$ *normal in* D *for which* E_ζ, *the set of Fatou points of* $f(z)$, *satisfies*

$$0 < m(E_\zeta \cap A) < m(A) \tag{11}$$

for each subarc A *of* C_z *and* $m(E_\zeta) < \varepsilon$, *but* ∞ *is not a Fatou value of* $f(z)$.

Proof. By Bagemihl and Seidel's result [1, Theorem 4], there exists an analytic function $w = g(\tau)$ normal in $|\tau| < 1$ for which E_τ, the set of Fatou points of $g(\tau)$, satisfies

$$0 < m(E_\tau) < \varepsilon ,$$

and ∞ is not a Fatou value of $g(\tau)$. As noted previously, E_τ is an open subset of $C_\tau : |\tau| = 1$. The function $g(\tau)$ is in fact finite and continuous at each point of E_τ and does not have an asymptotic value at any point of $C_\tau - E_\tau$.

Let $\tau = \varphi(z)$, $0 = \varphi(0)$, be a function of class (U) for which every point of C_z is a singularity of $\varphi(z)$. We claim that the function

$$w = f(z) = g(\varphi(z)) ,$$

which is normal in D (see [3, page 57]), has the required properties.

Let

$$E_\zeta = \{\zeta : \varphi(\zeta) \in E_\tau\} \tag{12}$$

and

$$F_\zeta = \{\zeta : \varphi(\zeta) \in C_\tau - E_\tau\} .$$

Since E_τ is a Borel set, both E_ζ and F_ζ are metrically dense at each point of C_z by Theorem 1. Thus (11) holds for E_ζ defined by (12). Also, by Theorem 2,

$$m(E_\zeta) = m(E_\tau) < \varepsilon .$$

It only remains to prove that E_ζ is the set of Fatou points of $f(z)$. Since $g(\tau)$ is finite and continuous at each point of E_τ, this will also show that ∞ is not a Fatou value of $f(z)$. Finally, we need only verify that E_ζ is the set of points on C_z at which $f(z)$ has a radial limit since, by a theorem of Lehto and Virtanen [3, Theorem 2], ζ is a Fatou point of $f(z)$ if and only if $f(z)$ has a radial limit at ζ.

If $\zeta \in E_\zeta$, then the radial limit $\varphi(\zeta) \in E_\tau$. Since $g(\tau)$ is continuous at each point of E_τ, the radial limit $f(\zeta) = g(\varphi(\zeta))$ exists.

If $\zeta \notin E_\zeta$, then there are two possibilities: Either $\zeta \in F_\zeta$ or $\zeta \notin E_\zeta \cup F_\zeta$. If $\zeta \in F_\zeta$, then $\varphi(\zeta)$ exists, but $\varphi(\zeta) \notin E_\tau$. As noted previously, $w = g(\tau)$ does not have an asymptotic value at any point of $C_\tau - E_\tau$. Thus $f(z)$ does not have a radial limit at ζ. Finally, suppose $\zeta \notin E_\zeta \cup F_\zeta$. Then

$$\lim_{r \to 1} \varphi(r\zeta)$$

does not exist. The set of limit points of $\tau = \varphi(r\zeta)$ as $r \to 1$ is a continuum S in $|\tau| \leq 1$ containing more than one point. If

$$\lim_{r \to 1} f(r\zeta) = \lim_{r \to 1} f(\varphi(r\zeta)) = c \tag{13}$$

were to exist, then $g(\tau) = c$ for each $\tau \in S$, $|\tau| < 1$. Thus S lies on C_τ, for otherwise $g(\tau) \equiv c$ by the identity theorem. But if S lies on C_τ, then $\varphi(z)$ maps the radius at ζ onto an arc γ_τ in $|\tau| < 1$ that converges to the subarc S of C_τ, and

$$\lim_{\substack{|\tau| \to 1 \\ \tau \in \gamma_\tau}} g(\tau) = c\ .$$

Then, by a theorem of Bagemihl and Seidel [1, Theorem 1], $g(\tau) \equiv c$, which is not the case. Hence the limit (13) cannot exist, and ζ is not a Fatou point of $f(z)$.

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