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KOEBE SETS FOR UNIVALENT FUNCTIONS WITH TWO PREASSIGNED VALUES

BY

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1. INTRODUCTION

Suppose that $\mathcal{M}$ is a fixed, non-empty class of functions analytic and univalent in the unit disk Δ.

In [6] the following definition of the Koebe set $\mathcal{K}(\mathcal{M})$ of $\mathcal{M}$ was given:

$$\mathcal{K}(\mathcal{M}) = \bigcap_{f \in \mathcal{M}} f(\Delta) \tag{1.1}$$

$\mathcal{K}(\mathcal{M})$ is not necessarily a domain in cases considered below so that the notion Koebe set rather that Koebe domain seems to be more adequate for our purposes.

If $\mathcal{S}$ denotes the class of functions f univalent and normalized in the usual way: $f(0) = 0$, $f'(0) = 1$, then obviously $\mathcal{K}(\mathcal{M})$ is Koebe's one-quarter disk.

In this paper we determine the set $\mathcal{K}(\mathcal{M})$ for various classes of univalent functions subject to Montel's normalization ([9], p. 66):

$$f(0) = a \,, f(z_0) = \mathrm{b} \,. \tag{1.2}$$

Some thirty years ago an analogous problem was investigated by W. W. Rogosinski [10] who gave the solution under an additional assumption of starshapedness of f.

In this paper we give a general and simple method of evaluating $\mathcal{K}(\mathcal{M})$ for univalent functions subject to the normalization (1.2). This enables us to reduce this problem to the following extremal problem which has an independent interest: given a point w and a class $\mathfrak{G} = \mathfrak{G}(a\,, b)$ of simply connected domains Ω in the open plane $\mathcal{E}^2$ such that $a\,, b \in \Omega$, find the supremum $\mu(w\,, \mathfrak{G})$ of Green's function $g(a\,, b\,; \Omega)$ for all $\Omega \in \mathfrak{G}$ such that $w \in \mathcal{E}^2 \setminus \Omega$. This problem is solved here for classes of convex, starlike and close-to-convex domains (Lemmas 2—4). Also the general case of arbitrary simply connected domains is considered (Theorem 1).

As corollaries of Lemmas 1—4 we obtain Theorems 3—5 which yield $\mathcal{K}(\mathcal{M})$ for the relevant classes of functions.

2. BASIC LEMMAS

We now prove the following

Lemma 1. *Suppose that* $\mathfrak{G}$ *is a class of simply connected domains* Ω *containing two fixed, different points* a , b *of the finite plane* $\mathscr{E}^2$. *Suppose that* $\mathfrak{G}$ *has following properties:*

(i) *if for a given, finite* w *there exists in* $\mathfrak{G}$ *a domain omitting* w, *there exists in* $\mathfrak{G}$ *a domain* Ω_w *with the maximal Green's function* $g(a , b ; \Omega)$ *among all* $\Omega \in \mathfrak{G}$ *such that* $w \in \mathscr{E}^2 \setminus \Omega$, *i.e.*

$$g(a , b ; \Omega) \leq g(a, b ; \Omega_w) = \mu(w ; \mathfrak{G}) \tag{2.1}$$

for any $\Omega \in \mathfrak{G}$ *such that* $w \in \mathscr{E}^2 \setminus \Omega$;

(ii) *the set* $\{z : g(a , z ; \Omega_w) > \delta\}$ *belongs to* $\mathfrak{G}$ *for all* $0 < \delta < g(a , b ; \Omega_w)$.

Let $\mathfrak{G}_\gamma = \{\Omega \in \mathfrak{G} : g(a , b ; \Omega) = \gamma\} , \gamma > 0$.

Then $\bigcap_{\Omega \in \mathfrak{G}_\gamma} \Omega = \{w : \mu(w , \mathfrak{G}) < \gamma\}$.

Proof. Suppose that w does not belong to $\bigcap_{\Omega \in \mathfrak{G}_\gamma} \Omega$. Then there exists $\Omega_1 \in \mathfrak{G}_\gamma$ such that $w \in \mathscr{E}^2 \setminus \Omega_1$. Now, $g(a , b ; \Omega_1) = \gamma \leq \mu(w)$ by (2.1). Thus $[w \notin \bigcap_{\Omega \in \mathfrak{G}_\gamma} \Omega]$ implies $[\mu(w) \geq \gamma]$, or $[\mu(w) < \gamma]$ implies $[w \in \bigcap_{\Omega \in \mathfrak{G}_\gamma} \Omega]$.

Suppose now that $\mu(w) \geq \gamma$. From (i) it follows that $g(a , b ; \Omega_w) = \mu(w) \geq \gamma$. Consider now $\Omega^\delta = \{z : g(a , z ; \Omega_w) > \delta\}$. Clearly $\Omega^0 = \Omega_w$. Since $\gamma > 0$, we have $g(a , b ; \Omega^\delta) = \gamma$ for suitably chosen δ with $\Omega^\delta \in \mathfrak{G}_\gamma$ by (ii). Since $\Omega^\delta \subset \Omega_w$, we have $w \in \mathscr{E}^2 \setminus \Omega^\delta$ and consequently, $w \notin \bigcap_{\Omega \in \mathfrak{G}_\gamma} \Omega$. This means that conversely, $[w \in \bigcap_{\Omega \in \mathfrak{G}_\gamma} \Omega]$ implies $[\mu(w) < \gamma]$. Our lemma is proved.

We shall be now concerned with the evaluation of the function $\mu(w , \mathfrak{G})$ for various classes of domains. We start with the class of convex domains $\mathfrak{G}^c$.

Lemma 2. *Let* $\mathfrak{G}^c = \mathfrak{G}^c(a , b)$ *be the class of all convex domains* Ω *containing the points* a , b. *If* w *lies outside* $[a , b]$, $\Omega \in \mathfrak{G}^c$ *and* $w \in \mathscr{E}^2 \setminus \Omega$, *then*

$$\mu(w , \mathfrak{G}^c) = \sup_{\Omega \in \mathfrak{G}^c} g(a , b ; \Omega) = g(a , b ; \Omega_w) = \tag{2.2}$$

$$= \log \frac{|w - a| + |w - b|}{|a - b|}$$

The extremal domain is a half-plane Ω_w *whose boundary* l *contains the point* w *and subtends equal angles with segments* $[w , a] , [w , b]$.

Proof. If Ω is a convex domain containing a , b and leaving w

outside, there exists a half-plane H containing Ω and such that $w \in \operatorname{fr} H$. Since $g(a, b; \Omega) \leq g(a, b; H)$, it is sufficient to consider just the half-planes H with $w \in \operatorname{fr} H$. Suppose now that H is the right half-plane, $w = 0$, $a = de^{i\theta}$, $b = he^{i(\alpha+\theta)}$. Here d, h, α are real and fixed $(d, h > 0)$, whereas θ has to be chosen so that $g(de^{i\theta}, he^{i(\alpha+\theta)}; H)$ is a maximum.

We have

$$(2.3) \qquad g(W, de^{i\theta}; H) = -\log |z|, \text{ where}$$
$$z = (W - de^{i\theta})/(W + de^{-i\theta}), \ W = he^{i(\alpha+\theta)}.$$

Hence $\max g(W, de^{i\theta}; H)$ corresponds to

$$(2.4) \qquad \min |z| = \min_{\theta} | he^{i\alpha} - d | \, | he^{i(\alpha+\theta)} + de^{-i\theta} |^{-1} =$$
$$= |a - b| (h + d)^{-1} = |a - b| (|w - a| + |w - b|)^{-1}.$$

The extremal case occurs for $2\theta = -\alpha$, i.e. the normal of $\operatorname{fr} H$ at w bisects the angle $[a, w, b]$. Now, the equality (2.2) follows immediately from (2.3) and (2.4). A simply connected domain is called close-to-convex if it is an image domain of a disk under a close-to-convex mapping, cf. [5]. A necessary and sufficient condition for Ω to be close-to-convex is that $\mathscr{E}^2 \setminus \Omega$ is a union of closed rays not intersecting each other [2], [8]. We say that the rays l_1 and l_2 do not intersect each other if $l_1 \cap l_2$ is either empty, or it reduces to the origin of one of the rays.

We shall now evaluate the expression $\mu(w, \mathfrak{G}^L)$ for the class $\mathfrak{G}^L$ of close-to-convex domains.

Lemma 3. *Let $\mathfrak{G}^L = \mathfrak{G}^L(a, b)$ be the class of close-to-convex domains Ω containing the points a, b. If $\Omega \in \mathfrak{G}^L$ and $w \in \mathscr{E}^2 \setminus \Omega$ then*

$$(2.5) \qquad \mu(w, \mathfrak{G}^L) = \sup_{\Omega \in \mathfrak{G}^L} g(a, b; \Omega) = g(a, b; \Omega_w) =$$
$$= \frac{1}{2} \log \frac{R_1 + R_2 + 2\sqrt{R_1 R_2}}{R_1 + R_2 - [(R_1 + R_2)^2 - |a - b|^2]^{1/2}}$$

where $R_1 = |w - a|$, $R_2 = |w - b|$.

The extremal domain Ω_w is the open plane $\mathscr{E}^2$ slit along a ray l_w emanating from w which subtends equal angles with segments $[w, a]$, $[w, b]$ and does not intersect the segment $[a, b]$.

Proof. If $\Omega \in \mathfrak{G}^L$ and $w \in \mathscr{E}^2 \setminus \Omega$, it follows from the geometrical definition of Ω that there exists in $\mathscr{E}^2 \setminus \Omega$ a ray l containing the point w. On the other hand, $\Omega \subset \mathscr{E}^2 \setminus l \in \mathfrak{G}^L$ and consequently, $g(a, b; \Omega) \leq g(a, b; \mathscr{E}^2 \setminus l)$. Hence we may restrict ourselves to the domains $\Omega = \mathscr{E}^2 \setminus l$. By shifting l along itself so that w becomes its origin, we increase $g(a, b)$. Thus we may assume that $\Omega = \mathcal{E}^2 \setminus l_w$ and l_w is a ray emanating from w. We can take l_w as the negative real axis and rotate a, b round the origin, i.e. we may take $w = 0$, $a = de^{i\Theta}$, $b = he^{i(\alpha+\Theta)}$. After a transformation $t = \sqrt{w}$ we obtain the case already considered in Lemma 2. In view of the conformal invariance of Green's function we have by (2.2):

$$\mu(w, \mathfrak{G}^L) = \log \frac{\sqrt{d} + \sqrt{h}}{|\sqrt{d}e^{-i\alpha/4} - \sqrt{h}e^{i\alpha/4}|} =$$
$$= \frac{1}{2} \log \frac{d + h + 2\sqrt{hd}}{d + h - 2\sqrt{hd}\cos \alpha/2}.$$

In case of a maximum we may obviously assume that the ray l_w does not intersect the segment $[a, b]$ which means that $0 \leq \alpha \leq \pi$.

Now, $|a - b|^2 = d^2 + h^2 - 2hd \cos \alpha = (d + h)^2 - 2hd(1 + \cos \alpha)$; hence $\sqrt{(d + h)^2 - |a - b|^2} = 2\sqrt{hd} \cos \alpha/2$ and finally

$$\mu(w, \mathfrak{G}^L) = \frac{1}{2} \log \frac{h + d + 2\sqrt{hd}}{h + d - \sqrt{(h + d)^2 - |a - b|^2}}.$$

With $d = R_1$, $h = R_2$ we obtain the desired result. We can prove easily in an analogous manner

Lemma 4. *Let* $\mathfrak{G}^* = \mathfrak{G}^*(a, b)$ *be the class of all domains* Ω *starlike with respect to* a *and containing* b. *If* w *lies outside* $[a, b]$, $\Omega \in \mathfrak{G}^*$ *and* $w \in \mathcal{E}^2 \setminus \Omega$ *then*

$$(2.6) \qquad \mu(w, \mathfrak{G}^*) = \sup \{g(a, b; \Omega) : \Omega \in \mathfrak{G}^*, w \in \mathcal{E}^2 \setminus \Omega\} =$$
$$= \log \left[\frac{R_1 + R_2}{|a - b|} + \sqrt{\left(\frac{R_1 + R_2}{|a - b|}\right)^2 - 1}\right] = \operatorname{Ar} \cosh \frac{R_1 + R_2}{|a - b|}.$$

The extremal domain Ω_w *is the open plane* $\mathcal{E}^2$ *slit along a ray* l_w *emanating from* w *whose prolongation contains the point* a.

Due to symmetry of μ with respect to R_1, R_2 we have also

$$\mu[w, \mathfrak{G}^*(a, b)] = \mu[w, \mathfrak{G}^*(b, a)].$$

3. AN EXTREMAL PROBLEM FOR SIMPLY CONNECTED DOMAINS

We shall be now concerned with a counterpart of Lemmas 2—4 for general simply connected domains. We prove the following

Theorem 1. *Let* $\mathfrak{G} = \mathfrak{G}(a, b)$ *be the class of all simply connected domains* Ω *containing the points* a, b. *If* $\lambda(\tau)$ *is the modular function and* τ_0 *is the unique solution of the equation*

$$\lambda(\tau) = (b - a)/(w - a) \tag{3.1}$$

contained in the fundamental domain B of $\lambda(\tau)$ then the maximal value $\mu(w, \mathfrak{G})$ of Green's function $g(a, b; \Omega)$ for $\Omega \in \mathfrak{G}$ such that $w \in \mathscr{E}^2 \setminus \Omega$ satisfies

$$\mu(w, \mathfrak{G}) = -\log \nu^{-1}(\tfrac{1}{2} \operatorname{im} \tau_0), \tag{3.2}$$

where

$$\nu(r) = \tfrac{1}{4} K(\sqrt{1 - r^2})/K(r) \tag{3.3}$$

is the modulus of $\Delta \setminus [0, r]$. The extremal domain Ω_w for which the upper bound (3.2) is attained is a slit domain $\mathscr{E}^2 \setminus C_w$, the slit C_w being the image of the segment $[0, \frac{1}{2}]$ under the $\wp$-function of Weierstrass with periods $1, \tau_0$.

Proof. Suppose that $\Omega \in \mathfrak{G}$ and $w \in \mathscr{E}^2 \setminus \Omega$. After a suitable translation we may achieve $a + b + w = 0$. Consider the family Γ' of all closed, rectifiable curves situated in Ω and separating a, b from $\operatorname{fr} \Omega$. It is well known that the modulus $\operatorname{mod} \Gamma'$ of the family Γ' satisfies

$$\operatorname{mod} \Gamma' = \nu(e^{-g}) \tag{3.4}$$

where $g = g(a, b; \Omega)$, cf. [3].

Consider now the family Γ of all closed, rectifiable curves separating a, b from w and such that the curves of both families Γ', Γ are homotopic with respect to $\mathscr{E}^2 \setminus \{a; b; w\}$. Let $\wp$ be the elliptic function of Weierstrass with periods $1, \tau$ $(\operatorname{im} \tau > 0)$ which are chosen so that $w = e_1 = \wp(\frac{1}{2})$, $a = e_2 = \wp(\frac{1}{2}\tau)$, $b = e_3 = \wp(\frac{1}{2}(1 + \tau))$. The corresponding value τ is a solution of the equation

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2} = \frac{b - a}{w - a}, \tag{3.5}$$

λ being the elliptic modular function.

The equation (3.5) has a countable number of solutions τ_k. There is also a countable number of homotopy classes Γ_k of closed curves separating a, b from w. If τ_k is a suitably chosen solution of (3.5) then

$$\operatorname{mod} \Gamma_k = \tfrac{1}{2} \operatorname{im} \tau_k, \tag{3.6}$$

cf. e.g. [1], p. 56.

All the solutions of (3.5) are congruent to each other with respect to the subgroup M_0 of the modular transformations $\tau' = (a\tau + b)(c\tau + d)^{-1}$ with $a \equiv d \equiv 1 \pmod 2$, $c \equiv b \equiv 0 \pmod 2$, $ad - bc = 1$.

Let B be the fundamental region of λ w.r.t. M_0, i.e.

$$\text{(3.7)} \qquad \operatorname{int} B = \{\tau : (\operatorname{im} \tau > 0) \wedge (|\operatorname{re} \tau| < 1) \wedge \wedge (|\tau - \tfrac{1}{2}| > \tfrac{1}{2}) \wedge (|\tau + \tfrac{1}{2}| > \tfrac{1}{2})\} .$$

To get B we add that part of $\operatorname{fr} B$ where $\operatorname{re} \tau \leq 0$ and $\operatorname{im} \tau > 0$. There exists a unique solution τ_0 of (3.5) contained in B, cf. [4], p. 176.

The subgroup M_0 is generated by the transformations

$$T_0 = \tau + 2 , T_1 = \tau/(1 - 2\tau) ,$$

cf. ibid., p. 176. The transformation T_0^k (or $T_0^{-k} = (T_0^{-1})^k$) gives for a suitably chosen integer k a point τ with $|\operatorname{re} \tau| \leq 1$ and does not change $\operatorname{im} \tau$.

Hence we may consider only those Γ_k which correspond to $|\operatorname{re} \tau_k| \leq 1$. Suppose now that τ lies in the strip $|\operatorname{re} \tau| \leq 1$ outside B, i.e. τ satisfies one of the inequalities $|\tau \mp \frac{1}{2}| \leq \frac{1}{2}$. Then the point $\tau' = \tau(1 \mp 2\tau)^{-1}$ lies in B, whereas $\operatorname{im} \tau' = |1 \mp 2\tau|^{-2} \operatorname{im} \tau \geq \operatorname{im} \tau$. Thus among all τ_k which satisfy (3.5) the point τ_0 with maximal imaginary part can be taken as the unique solution of (3.5) contained in B. We have $\Gamma' \subset \Gamma = \Gamma_k$ for some k, hence by (3.4) and (3.6)

$$\nu(e^{-g}) = \operatorname{mod} \Gamma' \leq \operatorname{mod} \Gamma \leq \max_k \operatorname{mod} \Gamma_k = \tfrac{1}{2} \operatorname{im} \tau_0$$

which implies $e^{-g} \geq \nu^{-1}(\frac{1}{2} \operatorname{im} \tau_0)$, or

$$\text{(3.8)} \qquad g = g(a , b ; \Omega) \leq - \log \nu^{-1}(\tfrac{1}{2} \operatorname{im} \tau_0) ,$$

for any simply connected domain Ω with $w \in \mathscr{E}^2 \setminus \Omega$. We now construct an extremal domain Ω_w for which the sign of equality in (3.8) is attained.

Given the points a , b , w with $a + b + w = 0$ (which may be achieved after a suitable translation), we find the solution $\tau_0 = s_0 + it_0 \in B$ of the equation (3.5).

The function

$$u = \exp 2 \pi i \zeta = \exp 2 \pi i(\xi + i\eta)$$

maps the parallelogram $P = [0, 1, 1 + \tau_0 , \tau_0]$ whose sides $[0 , \tau_0]$, $[1 , 1 + \tau_0]$ are identified onto the annulus $A = \{u : \exp(-2 \pi t_0) < |u| < 1\}$. The points $\zeta = \frac{1}{2} ; \frac{1}{2} \tau_0 ; \frac{1}{2}(1 + \tau_0)$ correspond to $u = -1$; $\exp(\pi i \tau_0)$; $-\exp(\pi i \tau_0)$, resp. We take now $r \in (0 , 1)$ such that

$$\text{(3.9)} \qquad \nu(r) = \tfrac{1}{2} \operatorname{im} \tau_0 = \tfrac{1}{2\pi} \log e^{\pi t_0}$$

and map the ring domain $\Delta \setminus [0 , r]$ conformally onto the annulus $A_1 = \{u : \exp(- \pi t_0) < |u| < 1\}$ so that $z = 0 , r$ correspond to $u = \exp (\pi i \tau_0)$, $- \exp(\pi i \tau_0)$, resp. The points of A_1 correspond to the lower half of P in the ζ-plane. If we identify in $\Delta \setminus [0 , r]$ the opposite edges of the slit $[0, r]$ which corresponds to the identification of points on $[\frac{1}{2}\tau_0, 1 + \frac{1}{2}\tau_0]$ symmetric with respect to $\frac{1}{2}(1+\tau_0)$ then the resulting transformation

$$(3.10) \qquad z \to \zeta \to W = \wp(\zeta; 1, \tau_0) = \wp(\zeta)$$

maps $1:1$ conformally the unit disk Δ onto the W-plane slit along the arc C_w where C_w is the image of $[0, 1]$ under $\wp(\zeta)$. Obviously the images of $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ under $\wp$ are identical and equal C_w.

The fact that $\wp$ is an even elliptic function of order 2 implies the univalence of the resulting mapping, also cf. [7] p. 47. The points on $|z| = 1$ corresponding to $u = 1, -1$ and $\zeta = 0, \frac{1}{2}$, resp. give the end-points of C_w in the W-plane, i.e. the points $\wp(0) = \infty$, $\wp(\frac{1}{2}) = w$, whereas
$z = 0 \leftrightarrow u = \exp(\pi i \tau_0) \leftrightarrow \zeta = \frac{1}{2}\tau_0 \leftrightarrow W = \wp(\frac{1}{2}\tau_0) = a$,
$z = r \leftrightarrow u = -\exp(\pi i \tau_0) \leftrightarrow \zeta = \frac{1}{2}(1+\tau_0) \leftrightarrow W = \wp(\frac{1}{2}(1+\tau_0)) = b$.

The family of closed curves situated in Δ and separating $0, r$ from $\mathrm{fr}\,\Delta$ has the modulus $\nu(r) = \frac{1}{2}\,\mathrm{im}\,\tau_0$ according to (3.9). On the other hand, by the conformal invariance of Green's function

$$(3.11) \qquad g(a, b; \mathscr{E}^2 \setminus C_w) = g(0, r; \Delta) = -\log r.$$

From (3.9) and (3.11) the equality

$$(3.12) \qquad g(a, b; \mathscr{E}^2 \setminus C_w) = -\log \nu^{-1}(\tfrac{1}{2}\,\mathrm{im}\,\tau_0)$$

follows by eliminating r.

Theorem 1 is proved.

A related extremal problem was investigated by Schiffer [11] who solved it in a different way by variational methods; also cf. [12].

4. THE DETERMINATION OF KOEBE SETS

Given $z_0 \in \Delta$ consider the class $\mathcal{M} = \mathcal{M}(z_0)$ of functions analytic and univalent in the unit disk Δ which satisfy the conditions

$$(4.1) \qquad f(0) = 0, \quad f(z_0) = 1.$$

If $f \in \mathcal{M}$ then $\Omega = f(\Delta) \in \mathfrak{G}(0, 1)$ where $\mathfrak{G}(0, 1)$ is the class of all simply connected domains containing $0, 1$. With each $f \in \mathcal{M}$ we can associate a domain $\Omega = f(\Delta) \in \mathfrak{G}_\gamma$ where $\gamma = -\log |z_0|$ which is an obvious consequence of the conformal invariance of Green's function.

Conversely, if $\Omega \in \mathfrak{G}_\gamma$ then Ω can be mapped on Δ conformally so that $\varphi(0) = 0$, $\varphi(1) = \zeta_0$, where $-\log|\zeta_0| = \gamma$. Hence $z_0 = e^{i\beta}\zeta_0$ for suitably chosen β and $\varphi^{-1}(e^{-i\beta} z) \in M$.

Thus

(4.2) $$\bigcap_{f \in \mathcal{M}(z_0)} f(\Delta) = \bigcap_{\Omega \in \mathfrak{G}_\gamma} \Omega$$

where $\gamma = - \log |z_0|$.

The above considerations as well as Lemma 1 yield

Theorem 2. *The Koebe set* $\mathcal{K}(\mathcal{M}) = \bigcap_{f \in \mathcal{M}} f(\Delta)$ *for the class* $\mathcal{M} = \mathcal{M}(z_0)$ *of functions analytic and univalent in the unit disk* Δ *subject to the normalization* (4.1) *is the image set of* $B \cap H(r)$ *under the mapping* $w = 1/\lambda(\tau)$, *where* $r = |z_0|$, $H(r) = \{\tau : \text{im}\ \tau < 2\, \nu(r)\}$, ν *being defined by* (3.3); λ *is the elliptic modular function and* B *is its fundamental region.*

The set $\mathcal{K}(\mathcal{M})$ *is symmetric with respect to the point* $w = \frac{1}{2}$ *and is a simply connected Jordan domain for* $0 < r < 2^{-1/2}$. *For* $2^{-1/2} \leq r < 1$ *the set* $\mathcal{K}(\mathcal{M})$ *is a union of two congruent, disjoint, simply connected Jordan domains.*

Proof. According to our previous remarks $\mathcal{K}(\mathcal{M}) = \bigcap_{\Omega \in \mathfrak{G}_\gamma} \Omega$ with $\gamma = - \log r$. In view of Lemma 1 and Theorem 1 $\bigcap_{\Omega \in \mathfrak{G}_\gamma} \Omega = \{ w : \mu(w , \mathfrak{G}) < \gamma\} = \{w : \text{im}\ \tau_0(w) < 2\, \nu(r)\}$. Thus $\tau_0(w) \in B \cap H(r)$. Moreover, $\tau_0(w)$ satisfies (3.5) with $a = 0,\ b = 1$, i.e. $\lambda(\tau_0) = 1/w$, or $w = 1/\lambda(\tau_0)$. This shows that $w \in \mathcal{K}(\mathcal{M})$, iff $w = 1/\lambda(\tau_0)$ with $\tau_0 \in B \cap H(r)$.

We now prove the symmetry property.

To this end it is sufficient to show that $1 - w_0 \in \mathcal{K}(\mathcal{M})$ as soon as $w_0 \in \mathcal{K}(\mathcal{M})$.

Suppose that $\tau_0 \in B \cap H(r)$ satisfies $w_0 = [\lambda(\tau_0)]^{-1}$. Obviously one of the points $\tau_0 \mp 1$, say τ_1 , also belongs to $B \cap H(r)$. Since $\lambda(\tau_0 \mp 1) = \lambda(\tau_0)/[\lambda(\tau_0) - 1] = \lambda(\tau_1)$, we have $w_1 = [\lambda(\tau_1)]^{-1} = [\lambda(\tau_0)]^{-1} [\lambda(\tau_0) - 1] = 1 - w_0 \in \mathcal{K}(\mathcal{M})$.

Suppose now that $\nu(r) > \frac{1}{4}$, or $r < 2^{-1/2}$. Then the image of $B \cap M(r)$ is a Jordan domain whose boundary has the parametric representation

(4.3) $$w = [\lambda(t + 2i\, \nu(r))]^{-1} ,\ - 1 \leq t \leq 1 .$$

If $\nu(r) \leq \frac{1}{4}$, or $r \geq 2^{-1/2}$, then the set $B \cap [- 1 + 2\, i\, \nu\, (r) , 1 + 2\, i\, \nu\, (r)]$ is a union of three segments. If τ , τ_1 are the end-points of the intermediate segment then $[\tau_1 = \tau/(1 - 2\, \tau)] \in M_0$ and hence $\lambda(\tau_1) = \lambda(\tau)$ which means that the image under $1/\lambda$ of $[\tau , \tau_1]$ is a closed Jordan curve. Similarly the images of the remaining two segments set up a congruent Jordan curve. Theorem 2 is proved.

A slightly more general case of functions with normalization (1.2) reduces to the case just considered by the transformation $W = (w - a)/(b - a)$.

We now apply Lemma 1 in a similar way as before with $\mathfrak{G} = \mathfrak{G}^c$, $\mathfrak{G}$, $\mathfrak{G}^L$ resp. and obtain in view of Lemma 2—4 the following theorems.

Theorem 3. *Let* $\mathcal{M}^c(z_0, a, b)$ *be the class of all convex mappings* f *of the unit disk* Δ *subject to the normalization* (1.2). *The Koebe set* $\mathcal{K}[\mathcal{M}^c(z_0, a, b)]$ *is the ellipse*

$$\{w : |w - a| + |w - b| < |a - b|/|z_0|\}.$$

Theorem 4 [10]. *Let* $\mathcal{M}^*(z_0, a, b)$ *be the class of all mappings* f *of the unit disk* Δ *subject to the normalization* (1.2) *and starlike with respect to* a. *The Koebe set* $\mathcal{K}[\mathcal{M}^*(z_0, a, b)]$ *is the ellipse*

$$\{w : |w - a| + |w - b| < \tfrac{1}{2}|a - b|(|z_0| + |z_0|^{-1})\}.$$

Theorem 5. *Let* $\mathcal{M}^L(z_0, a, b)$ *be the class of all close-to-convex mappings of the unit disk* Δ *subject to the normalization* (1.2). *The Koebe set* $\mathcal{K}[\mathcal{M}^L(z_0, a, b)]$ *has the form*

$$\left\{w : \frac{R_1 + R_2 + 2\sqrt{R_1 R_2}}{R_1 + R_2 - [(R_1 + R_2)^2 - |a - b|^2]^{1/2}} < |z_0|^{-2}\right\}$$

where $R_1 = |w - a|$, $R_2 = |w - b|$.

Obviously $\mathcal{K}(\mathcal{M}^L)$ has $w_0 = \frac{1}{2}(a + b)$ as a centre of symmetry. Moreover, $w_0 \in \mathcal{K}(\mathcal{M}^L)$ iff $|z_0| < 2^{-1/2}$. It is easily verified that for $|z_0| < 2^{-1/2}$ the set $\mathcal{K}(\mathcal{M}^L)$ is a Jordan domain and for $|z_0| \geq 2^{-1/2}$ it is a union of two disjoint Jordan domains containing a and b, resp.

We conclude with an interesting consequence of Theorem 2. Since the reflections with respect to the real axis and the straight line through $0, z_0$ yield again a mapping of the class $\mathcal{M}$, we see that the real axis, as well as the line $\operatorname{re} w = 1/2$ are lines of symmetry of $\mathcal{K}(\mathcal{M})$. This implies that $[0, 1] \subset \mathcal{K}(\mathcal{M})$, if $0 < |z_0| < 2^{-1/2}$. Hence we deduce the following COROLLARY. *If* f *is regular and univalent in the unit disk* $\triangle$ *and* $z_1, z_2 \in \triangle$ *are such that the hyperbolic distance* $h(z_1, z_2) < \operatorname{ar\,tanh}(2^{-1/2})$ *then the image domain* $f(\triangle)$ *contains the straight line segment with end points* $f(z_1), f(z_2)$. *The constant* $\operatorname{ar\,tanh}(2^{-1/2})$ *is best possible.*

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