ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

482

QUASIRATIONAL MAPPINGS ON PARABOLIC RIEMANN SURFACES

BY

ILPO LAINE

HELSINKI 1970
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1971.482

Communicated 11 September 1970 by LAURI MYRBERG

KESKUSKIRJAPAINO
HELSINKI 1970

Preface

I am deeply indebted to Professor Lauri Myrberg for his inspiring advice and encouragement during my work. My gratitude is due to Docent Olli Martio who was kind to read my manuscript.

I also wish to thank Mrs. Leena Kauppinen, Ph. Cand., who revised the language of the manuscript.

Joensuu, September 1970

ILPO LAINE

§ 1. INTRODUCTION

1.1. Summary. In this paper we study analytic mappings $f: R \to S$ on a parabolic Riemann surface R into a parabolic or a compact Riemann surface S.

After some preliminary considerations we discuss the characterization of the most simple analytic mappings in the second paragraph. These quasirational mappings have been treated in some form previously ([4]—[7], [11]—[15], [21]). Our concepts are to some extent based on the paper by L. Myrberg [11]. We give the natural generalizations of many results presented in that paper.

In the third paragraph we concentrate on the relations between quasirationality and different compactifications of the Riemann surfaces R and S. A consequence of our results is a theorem concerning the structure of the ideal boundaries of parabolic surfaces.

In the last paragraph we consider the question about the existence of analytic and specially quasirational mappings.

1.2. Compactifications of Riemann surfaces. The topological concepts and notations we use are mainly those of Kelley [8]. For the set-theoretic difference of the sets A and B we use the notation $A - B$ and for a subset and a proper subset the notations $\subseteq$ and $\subset$. By C we denote the family of continuous bounded real-valued functions and by C_0 the subfamily of C consisting of functions with compact support.

Let us consider a Riemann surface R and a class Q of real-valued continuous functions $q: R \to X$ with $X = \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$. It is well known ([3], p. 97] that there exists a compact space R_Q^* such that any function $q \in Q$ has a continuous extension $q_Q^*: R_Q^* \to X$ and that the extended functions q_Q^* separate the points of $R_Q^* - R$, i.e. for $z_1^* \neq z_2^*$ there exists a function $q \in Q$ such that $q_Q^*(z_1^*) \neq q_Q^*(z_2^*)$. The space R_Q^* is unique up to a homeomorphism. R_Q^* is called the Q-compactification of R or, if no confusion can arise, the compactification of R. The compact space $\Delta_Q^R = R_Q^* - R$ is the ideal boundary of R. To signify that the closure operation is performed in the compactified space R_Q^* we use the notation $\bar{A}^Q$ for the closure of a set $A \subseteq R_Q^*$. The notation $\bar{A}$ means the closure of $A \subseteq R$ performed in R.

Lemma 1. *R_Q^* is a Hausdorff space.*

Proof. We know that there exists a continuous mapping ψ imbedding R in the compact space $X^{Q \cup C_0}$ ([3], p. 97). Because X is Hausdorff, the topological product $X^{Q \cup C_0}$ and its subspace R_Q^* are also Hausdorff spaces ([8], p. 92 and 133).

In this paper we frequently use two specific examples of R_Q^*, namely the compactification of Kérekjártó — Stoilow (R_K^*) and that of Royden (R_D^*). The defining classes Q of continuous functions are respectively the class K of continuous functions with constant values in the components of the complement of a compact set and the class D of continuous Dirichlet functions. By a Dirichlet function $f \in D$ we mean a continuous function $f : R \to \mathbf{R}$ with the following properties: 1° there exists a locally summable differential c, which we denote by df, such that

$$\int_R df \wedge c_0 = - \int_R f \, dc_0$$

for every smooth differential c_0 with compact support in R and 2°

$$\|df\|^2 = \int_R df \wedge df^* < \infty \, .$$

We refer to [3], p. 66, 74 and 78.

Note that the above definition of the K-compactification is equivalent to the purely topological definition in [1], p. 82, because we can easily construct a homeomorphism between these two compactifications. Thus the K-compactification is characterized by the following properties: (i) R_K^* is a locally connected Hausdorff space and (ii) Δ_K^R is totally disconnected and non-separating on R_K^*.

1.3. Polar sets on Riemann surfaces. We state at first

Definition 1. *A set E on a Riemann surface is polar, if on every hyperbolic subregion $G \subseteq R$ there exists a positive superharmonic function v such that $v|G \cap E = \infty$.*

In the following lemma we have collected some familiar properties of polar sets (cf. [3], p. 30—31).

Lemma 2. *(1) Every subset of a polar set is again polar.*
(2) The union of a countable number of polar sets is polar.
(3) The complement of a closed polar set is connected.
(4) A polar set does not contain any continuum.
(5) If S and R are two Riemann surfaces and $E \subset S \subseteq R$, then E is polar on S if and only if E is polar on R.

A region Ω on a Riemann surface R is said to be regular, if $\bar{\Omega}$ is compact, if the relative boundary $\partial\Omega$ consists of a finite number of analytic Jordan curves and if $R - \Omega$ contains only non-compact components. We frequently use the following important lemma ([17], p. 25).

Lemma 3. *On an open Riemann surface R there exists an exhaustion of R by regular regions Ω_i, i.e. $\bar{\Omega}_i \subset \Omega_{i+1}$ and $\bigcup_{i=1}^{\infty} \Omega_i = R$.*

Lemma 4. *A closed set E on a parabolic surface R is polar if and only if $R - E$ is a parabolic Riemann surface.*

Proof. Let us first take a closed polar set $E \subset R$ and assume that $R - E$ is hyperbolic contrary to our assertion. Select then a parametric disc $\Omega_0 \subset R - E$ and form an exhaustion $\{\Omega_i\}$ of $R - E$ by regular regions $\Omega_i \supset \bar{\Omega}_0$. By hyperbolicity of $R - E$ the harmonic measures ω_{Ω_i} defined in $\bar{\Omega}_i - \Omega_0$ converge to a harmonic limit function $\omega = \lim_{\Omega_i \to R-E} \omega_{\Omega_i} \not\equiv 0$ defined in $(R - E) - \Omega_0$ ([1], p. 204—205). Since E is polar there exists in $R - \bar{\Omega}_0$ a positive superharmonic function v_1 such that $v_1|E = \infty$. Let v_2 be a finite potential on $R - \bar{\Omega}_0$ with $\lim_{\zeta \to \Delta_K^R} v_2(\zeta) = \infty$ ([3], p. 90). Selecting $\zeta_0 \in (R - E) - \Omega_0$ such that $v_1(\zeta_0) < \infty$ and a constant $a \in \mathbf{R}$ conveniently the positive superharmonic function $v = a(v_1 + v_2)$ defined in $(R - E) - \bar{\Omega}_0$ satisfies (i) $\liminf_{\zeta \to \zeta' \epsilon E \cup \Delta_K^R \cup \partial\Omega_0} (v(\zeta) - \omega(\zeta)) \geqq 0$ and (ii) $v(\zeta_0) < \omega(\zeta_0)$. This however violates the maximum principle ([3], p. 12).

Assume, on the other hand, that $R - E$ is a parabolic Riemann surface. Let $G \subset R$ be a hyperbolic subregion such that $E \subset G$ and $\Omega_0 = R - \bar{G}$ is a regular region. Both restrictions are immaterial by lemma 2. Let $\{\Omega_i\}$ be an exhaustion of $R - E$ by regular regions $\Omega_i \supset \bar{\Omega}_0$. Define on G a function for every n by

$$v_n(\zeta) = \begin{cases} \omega_{\Omega_n}(\zeta)\,, & \text{when } \zeta \in \Omega_n - \bar{\Omega}_0 \\ 1\,, & \text{when } \zeta \in R - \Omega_n\,. \end{cases}$$

By parabolicity of $R - E$ we can choose $\{\Omega_i\}$ such that $v = \sum_{n=1}^{\infty} v_n$ converges at a given point $\zeta_0 \in R - E$. Then v is superharmonic on G and $v|E = \infty$, hence E is polar.

1.4. Basic concepts of the value distribution theory. The classical theory of Nevanlinna concerning the value distribution of meromorphic functions has been generalized mainly by Sario (see e.g. [18]) to analytic mappings $f: R \to S$ between arbitrary Riemann surfaces. In the theory of Sario

([18], p. 51—60) there is at first formed on S a proximity function $s(\zeta, a)$. This function is uniformly bounded from below. We denote $d\omega = \Delta s\, dS$, where the density Δs is that one introduced in [18], p. 57 and dS means the Euclidean area element in a parametric disc. Then

$$\int_S d\omega = 4\pi$$

([18], p. 58). Let R_0 be a fixed parametric disc and $R_k \supset \bar{R}_0$ a regular region on R. We can define in $\bar{R}_k - R_0$ a harmonic function u such that $u|\partial R_0 = 0$ and $u|\partial R_k = k$. If the real constant k is selected to satisfy $\int_{\partial R_0} du^* = 1$, the function u is uniquely defined. We immediately verify that $R_{k_1} \subseteq R_{k_2}$ implies $k_1 \leqq k_2$. Hence the directed limit

$$\lim_{R_k \to R} k = k_{\max} \leqq +\infty$$

exists. It is well known that $k_{\max} < +\infty$ if and only if R is hyperbolic. We denote by $\nu(f, h, \zeta)$ the number of the ζ-points of the analytic mapping $f: R \to S$ in the region

$$\tilde{R}_h = R_0 \cup \{\zeta \in S | 0 \leqq u(\zeta) < h\}$$

counted with their multiplicities. The basic functions of the value distribution theory are

$$\begin{cases} A(f, k, \zeta) = 4\pi \int_0^k \nu(f, h, \zeta)\, dh \\ B(f, k, \zeta) = \int_{\partial R_k - \partial R_0} s(f(\xi), \zeta)\, du^* \\ C(f, k) = \int_0^k \int_{\tilde{R}_h} d\omega(f(\xi))\, dh\,. \end{cases}$$

The following two relations between these concepts are well known ([18], p. 60 and 65):

Lemma 5.

$$\begin{cases} C(f, k) = A(f, k, \zeta) + B(f, k, \zeta) \\ C(f, k) = \dfrac{1}{4\pi} \int_S A(f, k, \zeta)\, d\omega(\zeta)\,. \end{cases}$$

Let $n(f, \zeta, z)$ be the multiplicity of the root of the equation $f(z) = \zeta$. The number $n(f, \zeta, A)$ of the ζ-points of an analytic mapping $f : R \to S$ in a set $A \subseteq R$ counted with respect to their multiplicities has the representation

$$n(f, \zeta, A) = \sum_{z \in A} n(f, \zeta, z) .$$

We use the corresponding notations $n(f^*, \zeta, z)$ and $n(f^*, \zeta, A)$ also for continuous extensions $f^* : R^*_{Q(R)} \to S^*_{Q(S)}$ of analytic mappings $f : R \to S$. Now $\zeta \in S^*_{Q(S)}$ and $A \subseteq R^*_{Q(R)}$. This will be justified after we have later defined the multiplicity of a ζ-point for the extended mappings.

1.5. Regions of type SO_{HB} and mappings of type Bl. We say that a hyperbolic region $G \subseteq R$ is of type SO_{HB}, if $H_1^G = 1$. Here the notation H_Φ^G means the unique solution of the Dirichlet problem for the boundary function $\Phi : \partial G \to \mathbf{R}$ ([3], p. 21).

Definition 2. *An analytic mapping $f : R \to S$ is of type Bl, if for every point $\zeta \in S$ there exists an open neighbourhood $G \subset S$ such that the components of $f^{-1}(G)$ are of type SO_{HB}.*

Lemma 6. *Every non-constant analytic mapping $f : R \to S$ on a parabolic Riemann surface is of type Bl.*

Proof. Let $\zeta \in S$. Take a parametric disc U such that $\zeta \in U$. The components of $f^{-1}(U)$ are hyperbolic subregions of R. Since every hyperbolic subregion on a parabolic Riemann surface is of type SO_{HB} ([3], p. 31), f is of type Bl.

Remark. If $f : R \to S$ is an analytic mapping on a parabolic Riemann surface, $U \subset S$ is a region and V a component of $f^{-1}(U)$, then also $f|V : V \to U$ is a mapping of type Bl.

We frequently use the following result of Heins ([6], p. 470 and [3], p. 116).

Lemma 7. *If a non-constant analytic mapping $f : R \to S$ is of type Bl, then outside of a polar set* $n(f, \zeta, R) = \max_{\alpha \in S} n(f, \alpha, R)$.

Remark. If $\max_{\alpha \in S} n(f, \alpha, R) < \infty$, then the exceptional polar set is closed.

1.6. Lindelöfian mappings. The following definition makes sense only on hyperbolic surfaces or on hyperbolic subregions of parabolic surfaces.

Definition 3. *A non-constant analytic mapping $f : R \to S$ is Lindelöfian, if for every point $\zeta \in S$*

$$\sum_{f(a) = \zeta} n(f, \zeta, a)\, g(z, a, R) < \infty ,$$

where $z \notin f^{-1}(\zeta)$ *and* $g(z, a, R)$ *is the Green's function of* R *with the pole at* a.

1.7. Analytic functions on Riemann surfaces. In general we speak about analytic mappings $f: R \to S$ between two Riemann surfaces. If in particular $S = S_0 =$ the Riemann sphere, we will emphasize this situation by speaking about analytic functions.

§ 2. CHARACTERIZATION OF QUASIRATIONAL MAPPINGS

2.1. Definition of quasirational mappings. From now on we suppose that R is a parabolic and S either a parabolic or a compact Riemann surface. We are looking for some subclass of analytic mappings $f: R \to S$ which would consist of as simple mappings as possible. Such a subclass is that of quasirational mappings which we define by

Definition 4. *An analytic mapping is quasirational, if it has a continuous extension* $f^* : R_K^* \to S_K^*$.

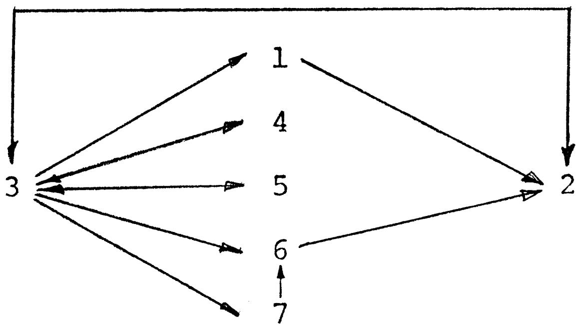
Remark. If $R = S = S_0 - \{\infty\}$, then $R_K^* = S_K^* = S_0$ and the class of quasirational functions on R consists of ordinary rational functions.

2.2. Characteristic properties. The following theorem gives some equivalent properties for quasirationality. These statements contain only partially new results. For the known parts of the theorem we refer to [3]. [4], [5] and [6].

Theorem 1. *For a non-constant analytic mapping* $f: R \to S$ *the following properties are equivalent:*

1. $f: R \to S$ *is quasirational;*
2. $n(f, \zeta, R)$ *is finite in some non-polar set* $E \subset S$;
3. *there exists an integer* N *such that* $n(f, \zeta, R) \leqq N$ *at every point* $\zeta \in S$;
4. $C(f, k) = O(k)$;
5. $f: R \to S$ *is a Dirichlet mapping, i.e. a mapping with a continuous extension* $f_D^* : R_D^* \to S_D^*$;
6. *the restriction* $f|G : G \to S$ *into any hyperbolic subregion* $G \subset R$ *is a Lindelöfian mapping;*
7. $C(f|G, k) = O(1)$.

Proof. The proof of the theorem is arranged according to the adjacent scheme:



a) 2. ⇔ 3. This is a result of Heins ([6], p. 470). We refer here to lemma 7.

b) 3. ⇒ 1. Let $z^* \in \Delta_K^R$ be an arbitrary point of the ideal boundary and $\mathcal{G}(z^*)$ the family of all open connected neighbourhoods $G \subseteq R_K^*$ of z^* with the boundary ∂G compact in R. By lemma 1 S_K^* is a Hausdorff space, hence $\hat{f}(z^*) = \bigcap_{G \in \mathcal{G}(z^*)} \overline{f(G \cap R)}^K$ is either a continuum or a point ([1], p. 8). In the first case we can find out a subcontinuum $F \subseteq \hat{f}(z^*) \cap S$. By property 3, lemma 2 and lemma 7 there exists a point $\zeta_0 \in F$ such that $n(f, \zeta_0, R) = N$. By the remark to lemma 7 we can even find an open neighbourhood U of ζ_0 such that $n(f, \alpha, R) = N$ for every point $\alpha \in U$. Let us denote $f^{-1}(\zeta_0) = \{z_1, \ldots, z_k\}$. By continuity of f we are able to construct disjoint open parametric discs D_i around every z_i such that their closures are compact and that $f(\bar{D}_i) \subseteq U$. Select now a connected open neighbourhood $V \subseteq \bigcap_{i=1}^k f(\bar{D}_i)$ of ζ_0. At once we see that $f^{-1}(V) \cap \bigcup_{i=1}^k \bar{D}_i$ consists of exactly k components and that $n(f, \alpha, f^{-1}(V) \cap \bigcup_{i=1}^k \bar{D}_i) = N$ for all $\alpha \in V$. Hence $n(f, \alpha, R - \bigcup_{i=1}^k \bar{D}_i) = 0$. Since $\zeta_0 \in \overline{f(G \cap R)}^K$ for every $G \in \mathcal{G}(z^*)$, there exists an $\alpha \in V \cap f(G \cap R)$. Selecting $G \subset R_K^* - \bigcup_{i=1}^k \bar{D}_i$ we have a contradiction. Thus $\hat{f}(z^*)$ must be a single point.

Define now a mapping $f^* : R_K^* \to S_K^*$ by

$$f^*(z) = \begin{cases} f(z), & \text{when } z \in R \\ \hat{f}(z), & \text{when } z \in \Delta_K^R . \end{cases}$$

We have only to prove that this mapping is continuous. Let $E \subseteq S_K^*$ be an open set. The points $z \in R \cap f^{*-1}(E)$ are trivially interior points of $f^{*-1}(E)$. For an ideal boundary point z we denote $f^*(z) = \zeta$. There exists at least one $G \in \mathcal{G}(z)$ such that $\overline{f(G \cap R)}^K \subseteq E$. To prove this assume conversely that $\overline{f(G \cap R)}^K - E \neq \emptyset$ for every $G \in \mathcal{G}(z)$. If $G_i \in \mathcal{G}(z)$ for $i = 1, \ldots, n$, then

$$\bigcap_{i=1}^{n} (\overline{f(G_i \cap R)}^K - E) = \bigcap_{i=1}^{n} \overline{f(G_i \cap R)}^K - E \supseteq f(R \cap \bigcap_{i=1}^{n} G_i) - E \neq \emptyset ,$$

because $\bigcap_{i=1}^{n} G_i$ contains a component $G_0 \in \mathcal{G}(z)$. The finite intersection property implies

$$\emptyset \neq \bigcap_{G \in \mathcal{G}(z)} (\overline{f(G \cap R)}^K - E) = \bigcap_{G \in \mathcal{G}(z)} \overline{f(G \cap R)}^K - E = \hat{f}(z) - E = \zeta - E = \emptyset .$$

If now $z^* \in G \cap \Delta_K^R$, then $G \in \mathcal{G}(z^*)$ and so

$$f^*(z^*) = \bigcap_{G^* \in \mathcal{G}(z^*)} \overline{f(G^* \cap R)}^K \subseteq \overline{f(G \cap R)}^K \subseteq E .$$

So

$$f^*(G) = f^*(G \cap R) \cup f^*(G \cap \Delta_K^R) \subseteq E$$

and also the points $z \in \Delta_K^R \cap f^{*-1}(E)$ are interior points.

c) 1. $\Rightarrow$ 2. Let $z_0^* \in \Delta_K^R$. If $z^* \in \Delta_K^R$ and $z^* \neq z_0^*$, there exists a compact set $Q \subset R$ and a real function $k \in K$ such that k has constant values in the components of $R - Q$ and $k^*(z^*) \neq k^*(z_0^*)$ (see p. 5). Without any restriction we can assume that $Q \in \{\bar{R}_n\}$, where $\{\bar{R}_n\}$ is an exhaustion of R by regular regions. Because there is at most a finite number of components in $R - \bar{R}_n$, it is possible to use only one function $k_n \in K$ for $\bar{R}_n$ with different values in the components of $R - \bar{R}_n$. Thus for every $\bar{R}_n$ at most a finite number of points $z^* \in \Delta_K^R$ can be separated from z_0^*. Because $\{\bar{R}_n\}$ is countable, Δ_K^R is a countable set. Naturally $f^*(\Delta_K^R)$ is countable too and so $f^*(\Delta_K^R) \neq S_K^*$.

Since $f^*(\Delta_K^R)$ is compact, there exists a compact non-polar set $E \subset S$ with $E \cap f^*(\Delta_K^R) = \emptyset$, hence

$$f^{*-1}(E) \cap \Delta_K^R \subseteq f^{*-1}(E) \cap f^{*-1}(f^*(\Delta_K^R)) = f^{*-1}(E \cap f^*(\Delta_K^R)) = \emptyset .$$

Because R_K^* is a Hausdorff space, the disjoint closed sets Δ_K^R and $f^{*-1}(E)$ are compact and separated. This means that for every $\zeta \in E$ we have $n(f, \zeta, R) = n(f, \zeta, f^{*-1}(E)) < \infty$.

d) 3. $\Rightarrow$ 4. If $n(f, \zeta, R) \leq N < \infty$, then also $\nu(f, h, \zeta) \leq N$ for every point $\zeta \in S$. By lemma 5

$$C(f, k) = \int_0^k \int_S \nu(f, h, \zeta) d\omega(\zeta) dh \leq N \int_0^k dh \int_S d\omega(\zeta) = 4\pi N \int_0^k dh = O(k) .$$

e) 4. $\Rightarrow$ 3. Because $B(f, k, \zeta)$ is bounded below ([18], ch. II), we have

$$0 \leq A(f, k, \zeta) = C(f, k) - B(f, k, \zeta) \leq C(f, k) + O(1)$$

for every $\zeta \in S$. If $C(f, k) = O(k)$, then $A(f, k, \zeta) = O(k)$. Now

$$A(f, 2k, \zeta) = A(f, k, \zeta) + 4\pi \int_k^{2k} \nu(f, h, \zeta) dh \geqq A(f, k, \zeta) + 4\pi\nu(f, k, \zeta)k$$

and so

$$\nu(f, k, \zeta) \leqq \frac{A(f, 2k, \zeta) - A(f, k, \zeta)}{4\pi k} \leqq \frac{O(k)}{4\pi k} = O(1) .$$

This is equivalent to the property 3, because $n(f, \zeta, R) = \sup_k \nu(f, k, \zeta)$.

f) 3. $\Leftrightarrow$ 5. Every Dirichlet mapping which is of type B1 has a finite valence ([3], p. 118). The converse follows from [3], p. 110.

g) 3. $\Rightarrow$ 6. Consider the mapping $f: R \to S$. If $G \subset R$ is an arbitrary hyperbolic subregion and if $n(f, \zeta, R) \leqq N < \infty$, then

$$\sum_{f(a)=\zeta} n(f|G, \zeta, a) g(z, a, G) \leqq N \max_{f(a)=\zeta} g(z, a, G) < \infty ,$$

if $z \in G - f^{-1}(\zeta)$. Hence $f|G: G \to S$ is a Lindelöfian mapping.

h) 6. $\Rightarrow$ 2. In the remainder of the proof we essentially show the equivalency of the properties 6 and 7. This has been mentioned by Heins ([5], p. 379) and Fuller ([4], p. 914).

Let us select two parametric discs D_1 and D_2 such that $\bar{D}_1 \subset D_2$ and that $\bar{D}_2$ is compact in R. $Q = R - \bar{D}_1$ is a hyperbolic subregion, hence

$$\sum_{f(a)=\zeta} n(f|Q, \zeta, a) g(z, a, Q)$$

converges, if $z \in Q - f^{-1}(\zeta)$. Additionally we can assume that $\zeta \notin f(\partial D_2)$. Hence $\inf_{z \in \partial D_2} g(z, a, Q) = d > 0$. By parabolicity of R $g(z, a, Q) \geqq d$ for $z \in R - \bar{D}_2$. This implies that $n(f, \zeta, R - \bar{D}_2) < \infty$ for every ζ and so $n(f, \zeta, R) \leqq n(f, \zeta, R - \bar{D}_2) + n(f, \zeta, \bar{D}_2) < \infty$.

i) 3. $\Rightarrow$ 7. In an arbitrary hyperbolic subregion $G \subset R$ we have $\nu(f|G, k, \zeta) \leqq n(f|G, \zeta, G) \leqq n(f, \zeta, R) \leqq N < \infty$. Hence

$$C(f|G, k) = \int_0^k \int_S \nu(f|G, h, \zeta) d\omega(\zeta) dh \leqq N \int_0^k \int_S d\omega(\zeta) dh = 4\pi k N = O(1) ,$$

because $\lim_{G_k \to G} k = k_{\max} < \infty$ by the hyperbolicity of G.

j) 7. $\Rightarrow$ 6. Let $G \subset R$ be an arbitrary hyperbolic subregion. Consider an arbitrary point $\zeta \in S$ and a point $z \in G - f^{-1}(\zeta)$. Select a fixed parametric disc G_0 around z such that $n(f, \zeta, \bar{G}_0) = 0$ and let $\{G_n\}$ be an exhaustion of G by regular regions $\supset \bar{G}_0$. Construct on $\bar{G}_n - G_0$ a harmonic function u_n with constant values on $\partial(\bar{G}_n - G_0)$ such that $u_n|\partial G_0 = 0$ and $u_n|\partial G_n = k_n$. The constant k_n is selected to satisfy $\int\limits_{\partial G_0} du_n^* = 1$. Define a constant μ_n such that

$$\mu_n \cdot \max_{a \in \partial G_0} g(a\,,z\,,G_n) = k_n = (k_n - u_n(a))|\partial G_0\,.$$

For every sufficiently large value of n we have

$$\mu_n = \frac{k_n}{\max\limits_{a \in \partial G_0} g(a\,,z\,,G_n)} \geqq \frac{k_{\max}}{2\max\limits_{a \in \partial G_0} g\,(a, z\,,G)} = \mu > 0\,,$$

where $k_{\max} = \lim\limits_{G_n \to G} k_n$. By the maximum principle we have in $G_n - \bar{G}_0$

$$g(a\,,z\,,G_n) \leqq \frac{1}{\mu_n}\,(k_n - u_n(a)) \leqq \frac{1}{\mu}\,(k_n - u_n(a))\,.$$

Because $C(f|G\,,k_n) = O(1)$ and $B(f|G\,,k_n\,,\zeta)$ is bounded below, we have with the standard meaning for h presented in the value distribution theory:

$$0 \leqq \sum_{f(a)=\zeta} n(f|G\,,\zeta\,,a)g(a\,,z\,,G_n) \leqq \frac{1}{\mu} \sum_{f(a)=\zeta} n(f|G\,,\zeta\,,a)(k_n - u_n(a))$$

$$= \frac{1}{\mu}\int\limits_{k_n}^{0} h d\nu(f|G\,,h\,,\zeta) = \frac{1}{\mu}\int\limits_{0}^{k_n} \nu(f|G\,,h\,,\zeta)dh = \frac{1}{\mu}A(f|G\,,k_n\,,\zeta)$$

$$= \frac{1}{\mu}C(f|G\,,k_n) - \frac{1}{\mu}B(f|G\,,k_n\,,\zeta) \leqq \frac{1}{\mu}C(f|G\,,k_n) + O(1) = O(1)\,.$$

Hence

$$\sum_{f(a)=\zeta} n(f|G\,,\zeta\,,a)g(z\,,a\,,G) = \lim_{n\to\infty} \sum_{f(a)=\zeta} n(f|G\,,\zeta\,,a)g(a\,,z\,,G_n) \leqq M < \infty\,.$$

§ 3. QUASIRATIONALITY AND DIFFERENT COMPACTIFICATIONS

3.1. A preliminary lemma. In theorem 1 we saw that quasirationality is equivalent to the concept of »to be a Dirichlet mapping». A rather general question is the following one: If $R^*_{Q(R)}$ and $S^*_{Q(S)}$ are some compactifications of R and S, what conditions would imply that a quasirational mapping $f: R \to S$ is extendable to a continuous mapping $f^*: R^*_{Q(R)} \to S^*_{Q(S)}$, and conversely, if an extension of this kind is possible, what are the conditions to ensure the quasirationality of f. We present two theorems in this direction. However, the contents of our theorems are rather narrow, since they further imply a theorem which strongly restricts the structure of the ideal boundaries of parabolic Riemann surfaces.

The following lemma is a direct consequence of [3], p. 99.

Lemma 8. *Let $Q_1 \subseteq Q_2$ be two classes of real-valued continuous functions on R. Then it is possible to extend the identity mapping $i : R \to R$ continuously to $i^* : R^*_{Q_2} \to R^*_{Q_1}$.*

We immediately obtain two corollaries.

Corollary 1. $i^*(\Delta^R_{Q_2}) \subseteq \Delta^R_{Q_1}$.

Corollary 2. *If $f : R \to S$ is continuously extendable to $f^*_{12} : R^*_{Q_1} \to S^*_{Q_2}$, if $Q_1 \subseteq Q_4$ and if $Q_3 \subseteq Q_2$, then there also exists a continuous extension $f^*_{43} : R^*_{Q_4} \to S^*_{Q_3}$.*

3.2. Quasirationality and continuous extendability.

Theorem 2. *If an analytic mapping $f : R \to S$ is continuously extendable to a mapping $f^* : R^*_{Q(R)} \to S^*_{Q(S)}$, if $K \subseteq Q(S)$ and if $Q(R) \subseteq D$, then f is quasirational.*

Proof. Note first that if f is constant the theorem is trivial, so we exclude this case. By the preceding corollary 2 there exists a continuous extension $f^*_{DK} : R^*_D \to S^*_K$. We follow the method of the proof of theorem 10.8 in [3], p. 118. Let us select three open parametric discs $V_2 \subset V_1 \subset G \subset S$ such that $\bar{V}_2 \subset V_1$, $\bar{V}_1 \subset G$ and $\bar{G}^{Q(S)} \subset S$. Let ψ be a continuous function on S such that $\psi | \bar{V}_2 = 1$, $\psi | (S - V_1) = 0$ and ψ is harmonic in $V_1 - \bar{V}_2$. Then ψ is continuously extendable to $\psi^* : S^*_K \to X$ and so $h^* = \psi^* \circ f^*_{DK}$ is a continuous real-valued function on R^*_D. A theorem of Stone ([3], p. 5) implies that the class of the extensions to R^*_D of all continuous Dirichlet functions on R is dense in $C(R^*_D)$, hence there exists a function $\varphi \in D$ with $|\varphi - h^* | R| < 1/3$. We immediately see that $\varphi_0 = 3 \sup (\inf (\varphi , 2/3) , 1/3) - 1$ defines a continuous bounded Dirichlet function which is $= 1$ in $f^{-1}(\bar{V}_2)$ and $= 0$ in $f^{-1}(S - V_1)$. Denote by G_i the components of $f^{-1}(G)$ which are all of type SO_{HB}, and by G'_i the sets $f^{-1}(V_1 - \bar{V}_2) \cap G_i$. Basic properties of Dirichlet functions ([3]) imply that

$$\|dh^*\|_{G_i} = \|dh^*\|_{G'_i} \leqq \|d\varphi_0\|_{G_i}$$

for all i. Because $h^* | G_i = \psi \circ (f | G_i)$ and $f | G_i : G_i \to G$ is of type B1, we have

$$\|d\psi\|^2_G \max_{\zeta \in G} n(f , \zeta , R) = \sum_i \|d\psi\|^2_G \max_{\zeta \in G} n(f , \zeta , G_i)$$
$$= \sum_i \|dh^*\|^2_{G_i} \leqq \sum_i \|d\varphi_0\|^2_{G_i} \leqq \|d\varphi_0\|^2 < \infty .$$

Thus

$$\max_{\zeta \in G} n(f, \zeta, R) \leqq \frac{\|d\varphi_0\|^2}{\|d\psi\|_G^2} < \infty ,$$

which implies quasirationality by theorem 1.

As a preparation for the following theorem we need

Definition 5. *A point x in a topological space X has a fundamental system $\mathcal{V}(x)$ of neighbourhoods, if for every neighbourhood U of x there exists a $V \in \mathcal{V}(x)$ such that $\bar{V} \subset U$.*

We immediately see that it is possible to construct a fundamental system of neighbourhoods consisting of open sets only.

Theorem 3. *Let R and S be two parabolic Riemann surfaces and $R^*_{Q(R)}$, $S^*_{Q(S)}$ their compactifications. Every quasirational mapping $f: R \to S$ has a continuous extension $f^*: R^*_{Q(R)} \to S^*_{Q(S)}$, if either*

(i) $Q(S) \subseteq K$;

(ii) $K \subseteq Q(R)$

or

(1) $K \subseteq Q(R)$;

(2) all points $\zeta \in S^*_{Q(S)}$ have a countable fundamental system of neighbourhoods;

(3) there exists no continuum in $\Delta^S_{Q(S)}$;

(4) $K \subseteq Q(S)$.

Proof. The first case is trivial by corollary 2. To prove the second one let $f: R \to S$ be a non-constant quasirational mapping. By corollary 2 the generality is not restricted if we assume that $Q(R) = K$ and $K \subset Q(S)$. Since R^*_K is locally connected, every point $z \in R^*_K$ has a countable fundamental system of connected open neighbourhoods.

Let $\mathcal{G}(z^*)$ be this infinite family of open connected neighbourhoods of a point $z^* \in \Delta^R_K$. Because f is continuous and $G \cap R$ connected for every $G \in \mathcal{G}(z^*)$, the set

$$\hat{f}(z^*) = \bigcap_{G \in \mathcal{G}(z^*)} \overline{f(G \cap R)}^{Q(S)}$$

is a non-void, compact and connected set and so either a continuum or a point.

To prove that $\hat{f}(z^*)$ is a point, let us select an arbitrary point $\zeta \in \hat{f}(z^*)$. Note first that $f(G \cap R) \cap U(\zeta) \neq \emptyset$ for every open neighbourhood $U(\zeta)$ of ζ and for every $G \in \mathcal{G}(z^*)$. Indeed, $\zeta \in \hat{f}(z^*) \cap U(\zeta) \subseteq \overline{f(G \cap R)}^{Q(S)} \cap U(\zeta) \neq \emptyset$ which immediately implies the desired property. This fact and the existence of a countable fundamental system of neighbourhoods in R^*_K and $S^*_{Q(S)}$ enable us to construct two sequences of distinct points $\{z_i\}$ in R and $\{\zeta_i\}$ in S with $z_i \to z^*$, $\zeta_i \to \zeta$ and $f(z_i) = \zeta_i$. By quasirationality there exists a unique $\zeta_K \in S_K^*$ with $f(z_i) \to \zeta_K$. From a certain value of i we have for the same reason

$$i_S^*(\zeta_i) = \zeta_i = f(z_i) \in U(\zeta_K)\,,$$

where $i_S^* : S_{Q(S)}^* \to S_K^*$ is the continuously extended identity mapping $i : S \to S$ and $U(\zeta_K)$ is a given neighbourhood of ζ_K in S_K^*. By continuity of i_S^* this is possible only if $i_S^*(\zeta) = \zeta_K$. Thus we have $i_S^*(\hat{f}(z^*)) = \{\zeta_K\}$. If $\zeta_K \in S$, then $\hat{f}(z^*) = \{\zeta_K\}$ and if $\zeta_K \in \Delta_K^S$, then $\hat{f}(z^*) \subseteq \Delta_{Q(S)}^S$. By property (3) $\hat{f}(z^*)$ must be a point. If we define a mapping $f^* : R_K^* \to S_{Q(S)}^*$ with

$$f^*(z) = \begin{cases} f(z)\,, & \text{when } z \in R \\ \hat{f}(z)\,, & \text{when } z \in \Delta_K^R\,, \end{cases}$$

we have the required extension.

To prove the continuity of $f^*(z)$ we can reproduce the continuity proof in b) of the proof of the theorem 1 with $S_{Q(S)}^*$ in place of S_K^*.

Remark 1. The assumptions (2) and (3) in theorem 3 are essential. For instance in R_D^* there exists ideal boundary points without any countable fundamental system of neighbourhoods ([3], p. 103). Seibert ([20], p. 7) on the other hand has constructed examples of parabolic Riemann surfaces with an ideal boundary homeomorphic to the unit circle.

Remark 2. There remains an interesting question about the relations between continuous extensions of non-quasirational mappings $f : R \to S$ and different compactifications of R and S. We know that for R_D^*, S_D^* these mappings are not continuously extendable (theorem 1). On the other hand there exists continuous extensions of all analytic mappings to $f_W^* : R_W^* \to S_W^*$ and $f_C^* : R_C^* \to S_C^*$ for the Wiener and Čech compactifications ([3], p. 111) and [8], p. 153). Specially the question about the boundary behaviour of non-quasirational analytic mappings with respect to compactifications S_Q^* with $D \subset Q \subset W$ is open.

3.3. On the structure of the ideal boundaries.

Theorem 4. *Let $f : R \to S$ be a non-constant quasirational mapping. Under the conditions (1)—(4) of theorem 3 the extended mapping $f^* : R_{Q(R)}^* \to S_{Q(S)}^*$ is surjective.*

Proof. Let $\zeta_0 \in S_{Q(S)}^*$ be arbitrary. By lemma 7 f covers S except possibly a polar set E. $S - E$ is dense in $S_{Q(S)}^*$ and (2) holds, so we can construct two sequences of distinct points $\{z_i\}$ in R and $\{\zeta_i\}$ in $S - E$ with $\zeta_i \to \zeta_0$ and $f(z_i) = \zeta_i$. The point set $\{z_i\}$ has at least one cluster point z_0 in $R_{Q(R)}^*$. By continuity of f^* we must have $f^*(z_0) = \zeta_0$.

Theorem 5. *If $K \subseteq Q$, if every point $z \in R_Q^*$ has a countable fundamental system of neighbourhoods and if Δ_Q^R does not contain any continuum, then $R_Q^* = R_K^*$ for every parabolic Riemann surface.*

Proof. Note that the identity of two compactifications is to be understood in the sense that they are homeomorphic.

Take the identity mapping $i : R \to R$ which is quasirational. By lemma 8 and theorem 3 there are continuous extensions $i_1^* : R_Q^* \to R_K^*$ and $i_2^* : R_K^* \to R_Q^*$. By theorem 4 these extensions are surjective and so $i_1^* \circ i_2^* : R_K^* \to R_K^*$ is a continuous surjection on R_K^*, whose restriction to R is the identity mapping. By continuity $i_1^*(i_2^*(z)) = z$ for all $z \in R_K^*$. Elementary algebraic considerations show that i_1^* and i_2^* are homeomorphisms, thus $R_Q^* = R_K^*$.

3.4. N-valency of quasirational mappings. At first we define the multiplicity of the extended mappings at the ideal boundary points. Our definition coincides with the usual definition of multiplicity for $z \in R$. We denote by $\mathcal{G}(z)$ the family of all open connected neighbourhoods of $z \in R_{Q(R)}^*$.

Definition 6. *For an analytic non-constant mapping $f : R \to S$ with a continuous extension $f^* : R_{Q(R)}^* \to S_{Q(S)}^*$ we denote for every $G \in \mathcal{G}(z)$*

$$N(f , G) = \max_{\zeta \in S} n(f , \zeta , G \cap R) .$$

The multiplicity of f^* at $z \in R_{Q(R)}^*$ is defined by

$$n(f^* , f^*(z) , z) = \min_{G \in \mathcal{G}(z)} N(f , G) .$$

A trivial consequence of this definition is

Lemma 9. *There always exists a neighbourhood $G_0 \in \mathcal{G}(z)$ with the property $n(f^* , f^*(z) , z) = N(f , G)$ for all $G \subseteq G_0$ with $G \in \mathcal{G}(z)$.*

Let us note that $\mathcal{G}(z)$ is always a non-void family and hence the above definition is applicable without any restrictions to the compactifications $R_{Q(R)}^*$ and $S_{Q(S)}^*$ alone, if we just have the continuous extension f^*.

Lemma 10. *Let $f : R \to S$ be a non-constant quasirational mapping and $f^* : R_K^* \to S_K^*$ its continuous extension. If $\zeta_0 \in S_K^*$, if $G \in \mathcal{G}(\zeta_0)$ and if V is a component of $f^{*-1}(G)$, then the restricted mapping $f^*|V : G \to G$ is surjective.*

Proof. Elementarily we can verify that $R \cap f^{*-1}(G) = f^{-1}(G \cap S)$, hence $f^*|V \cap R : V \cap R \to G \cap S$ is a mapping of type Bl of a component of $f^{-1}(G \cap S)$ into $G \cap S$. By lemma 7 $f^*|V \cap R$ covers $G \cap S$ except possibly a polar set E. We can construct two sequences of distinct points $\{z_i\} \subset V \cap R \subseteq V$ and $\{\zeta_i\} \subset G \cap S - E \subseteq G$ with $\zeta_i \to \zeta_0$ and $f^*(z_i) = \zeta_i$. The point set $\{z_i\}$ has at least one cluster point z_0 in the compact set $\bar{V}^K$. By continuity of f^* we have $f^*(z_0) = \zeta_0$. Immediately we see that z_0 is an interior point: $z_0 \in V$.

Lemma 11. *Let $f : R \to S$ be a non-constant quasirational mapping. Then $f^* : R_K^* \to S_K^*$ is an open mapping.*

Proof. Let $G \subseteq R_K^*$ be an open set and $z \in G$. Denote $f^*(z) = \zeta$ and let U be an open connected neighbourhood of ζ. The number of the components in $f^{*-1}(U)$ is bounded by $N = \max_{\alpha \in S} n(f, \alpha, R)$, since Δ_K^S and Δ_K^R are non-separating and the number of the components in $f^{-1}(U \cap S)$ is finite ([3], p. 118). Let us select U such that $f^{*-1}(U)$ has a maximal number of components and denote by V that component containing z. If $U' \subset U$ is another open connected neighbourhood of ζ, then $V' = V \cap f^{*-1}(U')$ is connected by the above-mentioned maximality. Let $\{U_i\}$ be a sequence of open connected neighbourhoods of ζ such that $U_0 = U$, $\overline{U_{i+1}}^K \subset U_i$ for all values of i and $\bigcap_{i=0}^{\infty} U_i = \{\zeta\}$. By continuity of f^* we have

$$f^{*-1}(\zeta) \cap \bar{V}^K = \bigcap_{i=0}^{\infty} \overline{V \cap f^{*-1}(U_i)}^K .$$

Since every $V_i = V \cap f^{*-1}(U_i)$ is connected, $f^{*-1}(\zeta) \cap \bar{V}^K$ is either a continuum or a point. Because $f^{*-1}(\zeta) \subseteq \{z_1, \ldots, z_k\} \cup \Delta_K^R$, where $\{z_1, \ldots, z_k\} = f^{-1}(\zeta)$ is a finite set, the first case is impossible. Thus z is the only ζ-point of f^* in $\bar{V}^K$.

If there exists a U_i such that $V_i \subseteq G$, then $U_i \subseteq f^*(G)$ and ζ is an interior point of $f^*(G)$. Otherwise $V_i - G \neq \emptyset$ for every i. This enables us to construct a sequence $\{z_i\} \subset V$ such that $z_i \in V_i - G$, $f^*(z_i) = \zeta_i \in U_i$ and $\zeta_i \to \zeta$. The set $\{z_i\}$ has a cluster point $z_0 \neq z$ in $\bar{V}^K$ and by continuity $f^*(z_0) = \zeta$, a contradiction. The lemma follows.

Theorem 6. *Let $f : R \to S$ be a non-constant quasirational mapping and $f_Q^* : R_Q^* \to S_K^*$ its continuous extension. If f_Q^* is an open mapping, then*

$$n(f_Q^*, \zeta, R_Q^*) < \infty$$

for all $\zeta \in S_K^*$.

Proof. Let us assume that there exists a point $\zeta \in S_K^*$ with $n(f_Q^*, \zeta, R_Q^*) = \infty$. The multiplicity of all ζ-points is $\leqq N$, thus the set $E = \{z \in R_Q^* | f_Q^*(z) = \zeta\}$ is infinite. The compactness of R_Q^* implies the existence of a cluster point z_0 of the set E. By continuity of f_Q^* we have $f_Q^*(z_0) = \zeta$. Take a neighbourhood $G_0 \in \mathcal{G}(z_0)$ and let $z_1, \ldots, z_{N+1}$ be disjoint points in $E \cap (G_0 - \{z_0\})$, where $N = \max_{\alpha \in S} n(f, \alpha, R)$, and $G_i \subset G_0$ disjoint open neighbourhoods of these points. Since f_Q^* is open, there exists an open connected neighbourhood $U \subseteq \bigcap_{i=1}^{N+1} f_Q^*(G_i)$ of ζ and a component G_i' of $f^{-1}(U \cap S)$ in every G_i. The restricted mappings $f|G_i'$ are of type Bl, so there exists at least one point $\xi \in \bigcap_{i=1}^{N+1} f(G_i')$. Thus we have a contradiction

$$n(f, \xi, R) \geqq \sum_{i=1}^{N+1} n(f|G_i', \xi, G_i') \geqq N + 1 .$$

Remark. Let S be a compact Riemann surface. In the next paragraph we show that there exists a parabolic Riemann surface R and a quasirational mapping $f: R \to S - \{\zeta_0\}$ such that $n(f, \zeta, R) = N$ for all $\zeta \in S - \{\zeta_0\}$. Now $(S - \{\zeta_0\})_K^* = S$ and so by lemma 8 there exists a continuous extension $f_D^*: R_D^* \to S$. Clearly $f_D^*(\varDelta_D^R) = \{\zeta_0\}$. Since the power of $\varDelta_D^R$ is at least that of a continuum ([3], p. 103), we have $n(f_D^*, \zeta_0, R_D^*) = \infty$. Thus a continuous extension of a non-constant quasirational mapping is not necessarily open.

Theorem 7. *Let $f: R \to S$ be a non-constant quasirational mapping and $f_Q^*: R_Q^* \to S_K^*$ its continuous extension with $K \subseteq Q$ on R. If the extension $i_{QK}^*: R_Q^* \to R_K^*$ of the identity mapping is open, then*

$$n(f_Q^*, \zeta, R_Q^*) \geqq N = \max_{\alpha \in S} n(f, \alpha, R)$$

for all $\zeta \in S_K^*$. The equality holds for all $\zeta \in S_K^*$ if and only if $R_Q^* = R_K^*$.

Proof. Consider first the case of the K-compactification. Let $\zeta \in S_K^*$. By lemma 11 and theorem 6 $f_K^{*-1}(\zeta)$ is a finite set: $f_K^{*-1}(\zeta) = \{z_1, \ldots, z_k\}$. Let us select disjoint open connected neighbourhoods G_i for every z_i such that $n(f_K^*, \zeta, z_i) = \max_{\alpha \in S} n(f, \alpha, G_i \cap R)$ for every G_i. By lemma 11 the set $U' = \bigcap_{i=1}^{k} f_K^*(G_i)$ is an open set such that $\zeta \in U'$. Let U be that component of U' containing ζ. Every G_i contains an open component V_i of $f_K^{*-1}(U)$. Since the restricted mappings $f_K^*|V_i \cap R$ are of type Bl, there exists a point $\xi \in U \cap S$ such that $n(f, \xi, R) = N$ and $n(f, \xi, V_i \cap R) = \max_{\alpha \in S} n(f, \alpha, V_i \cap R)$ for $i = 1, \ldots, k$. Thus

$$n(f_K^*, \zeta, R_K^*) = \sum_{i=1}^{k} n(f_K^*, \zeta, V_i) = \sum_{i=1}^{k} \max_{\alpha \in S} n(f, \alpha, V_i \cap R)$$

$$= \sum_{i=1}^{k} n(f, \xi, V_i \cap R) = n(f, \xi, R) = N .$$

In the general case we know that the extension $i_{QK}^*: R_Q^* \to R_K^*$ of the identity mapping $i: R \to R$ is continuous and surjective, hence

$$n(i_{QK}^*, z, R_Q^*) \geqq 1$$

for every $z \in R_K^*$. Further all values of i_{QK}^* are of multiplicity one. Consider any points $\zeta \in S_K^*$, $z \in R_K^*$ and $z' \in R_Q^*$ such that $i_{QK}^*(z') = z$ and $f_K^*(z) = \zeta$. Because $f_Q^* = f_K^* \circ i_{QK}^*$, then $f_Q^*(z') = \zeta$. Let $G' \in \mathcal{G}(z')$ be selected to satisfy $n(f_Q^*, \zeta, z') = \max_{\alpha \in S} n(f, \alpha, G' \cap R)$. By corollary 1 to lemma 8 we have $i_{QK}^*(G') \cap R = i_{QK}^*(G' \cap R)$, thus $\max_{\alpha \in S} n(f, \alpha, G' \cap R) = \max_{\alpha \in S} n(f, \alpha, i_{QK}(G') \cap R)$. Since $i_{QK}^*(G')$ is an open connected neighbourhood of z, definition 6 yields

$$n(f_Q^*, \zeta, z') \geqq n(f_K^*, \zeta, z),$$

thus

$$n(f_Q^*, \zeta, R_Q^*) \geqq \sum_{z \in f_K^{*-1}(\zeta)} n(f_K^*, \zeta, z) n(i_{QK}^*, z, R_Q^*).$$

From the above formulae we deduce the inequality

$$n(f_Q^*, \zeta, R_Q^*) \geqq n(f_K^*, \zeta, R_K^*).$$

If now for some continuous extension $f_Q^* : R_Q^* \to S_K^*$ the equality holds for all $\zeta \in S_K^*$, then

$$N = n(f_Q^*, \zeta, R_Q^*) \geqq \sum_{z \in f_K^{*-1}(\zeta)} n(f_K^*, \zeta, z) n(i_{QK}^*, z, R_Q^*)$$

$$\geqq \sum_{z \in f_K^{*-1}(\zeta)} n(f_K^*, \zeta, z) = n(f_K^*, \zeta, R_K^*) = N.$$

Thus $n(i_{QK}^*, z, R_Q^*) = 1$ for all $z \in f_K^{*-1}(\zeta)$. This equation is valid for all $z \in R_K^*$. Hence $i_{QK}^* : R_Q^* \to R_K^*$ is a continuous bijection. Since R_Q^* and R_K^* are compact Hausdorff spaces, i_{QK}^* is a homeomorphism.

§ 4. EXISTENCE THEOREMS

4.1. Exceptional sets of analytic mappings. Let R be a parabolic and S a parabolic or a compact Riemann surface and $f : R \to S$ an analytic mapping. It is a known result of Heins that $n(f, \zeta, R) < N = \max_{\alpha \in S} n(f, \alpha, R) \leqq \infty$ at most in a polar set K (lemma 7). This set K can be represented as a union $K = \bigcup_{i=0}^{N-1} K_i$ of closed polar sets $K_i = \{\zeta \in S | n(f, \zeta, R) = i < N\}$. By a theorem of Matsumoto ([10], p. 143) it seems to be probable that the above result is maximal. Indeed, modifying the construction method of Matsumoto we have the following theorem.

Theorem 8. *Let* $K = \bigcup_{i=0}^{\infty} K_i$ *be formed out of disjoint closed polar sets* K_i *on a parabolic (or a compact) Riemann surface* S. *Then it is possible to construct a parabolic Riemann surface* R *and an analytic mapping* $f: R \to S$ *such that*

$$\begin{cases} n(f, \zeta, R) = \infty, & \text{when } \zeta \in S - K \\ n(f, \zeta, R) = p, & \text{when } \zeta \in K_p \text{ for } p = 0, 1, 2, \ldots \end{cases}$$

Proof. Every region $S_n = S - \bigcup_{i=0}^{n} K_i$ can be considered as a parabolic surface by lemma 4. We define an exhaustion of S_n by regular regions $S_{n,k}$ on every surface S_n during the following proof. The boundaries $\partial S_{n,k}$ consist of a finite number of closed analytic curves and every component of $S_n - S_{n,k}$ is non-compact. Let us denote by $\omega_{n,k}$ the harmonic measure of $\partial S_{n,k}$ with respect to the open set $S_{n,k} - \overline{S_{n,k-1}}$. By parabolicity we can form the exhaustion $\{S_{0,k}\}$ to satisfy

$$\begin{cases} D(\omega_{0,1}) \leqq \dfrac{1}{2} \\ D(\omega_{0,k}) \leqq \dfrac{1}{k}, & \text{when } k \geqq 2 \end{cases}$$

([18], p. 181), where $D(\omega_{i,j})$ means the Dirichlet integral of $\omega_{i,j}$ over $S_{i,j} - \overline{S_{i,j-1}}$. In the set $S_{0,1} - \overline{S_{0,0}}$ we select a compact arc L_0 with $L_0 \cap (K_0 \cup K_1) = \emptyset$. In forming $\{S_{1,k}\}$ we suppose that $L_0 \subset S_{1,1} - \overline{S_{1,0}}$ and that

$$\begin{cases} D(\omega_{1,2}) \leqq \dfrac{1}{8} \\ D(\omega_{1,k}) \leqq \dfrac{1}{k^2}, & \text{when } k \geqq 3. \end{cases}$$

If necessary, we further modify L_0 by taking a sufficiently small part of it and denoting that part again by L_0, just making it satisfy

$$\begin{cases} D(\omega'_{0,1}) \leqq 2\, D(\omega_{0,1}) \\ D(\omega'_{1,2}) \leqq 2\, D(\omega_{1,2}), \end{cases}$$

where $\omega'_{0,1}$ (resp. $\omega'_{1,2}$) denotes the harmonic measure of $\partial S_{0,1} \cup L_0$ (resp. $\partial S_{1,1} \cup L_0$) with respect to $(S_{0,1} - \overline{S_{0,0}}) - L_0$ (resp. $(S_{1,1} - \overline{S_{1,0}}) - L_0$). In the set $S_{1,2} - \overline{S_{1,1}}$ we select a compact arc L_1 such that

$L_1 \cap \bigcup_{i=0}^{2} K_i = \emptyset$. Generally, we form $\{S_{n,k}\}$ such that $L_{n-1} \subset S_{n,n} - \overline{S_{n,n-1}}$ and that

$$\begin{cases} D(\omega_{n,n+1}) \leqq \dfrac{1}{2(n+1)^{n+1}} \\ D(\omega_{n,k}) \quad \leqq \dfrac{1}{k^{n+1}}\,, \quad \text{when} \quad k \geqq n+2\,. \end{cases}$$

Additionally we assume the arc L_{n-1} to be so small that

$$\begin{cases} D(\omega'_{n-1,n}) \leqq 2\, D(\omega_{n-1,n}) \\ D(\omega'_{n,n+1}) \leqq 2\, D(\omega_{n,n+1}) \end{cases}$$

with a clear meaning of $\omega'_{i,j}$. In $S_{n,n+1} - \overline{S_{n,n}}$ we select again a compact arc L_n with $L_n \cap \bigcup_{i=1}^{n+1} K_i = \emptyset$.

Now we connect the surfaces S_i together in the following way: S_0 and S_1 will be connected crosswise along the arc L_0 with the resulting surface $\hat{R}_1$, $\hat{R}_1$ and S_2 along L_1 resulting $\hat{R}_2, \ldots, \hat{R}_n$ and S_{n+1} along L_n resulting $\hat{R}_{n+1}$ and so on. It is clear that the limit surface $R = \lim_{n\to\infty} \hat{R}_n$ satisfies the conditions of the theorem, if we prove it to be parabolic. The analytic mapping $f: R \to S$ is nothing else than the covering mapping.

To prove the parabolicity we define an exhaustion $\{R_k\}$ of R as follows: R_0 corresponds to the set $S_{0,0}$ on the sheet S_0, R_1 the sets $S_{0,1}$ and $S_{1,1}$ on the surface $\hat{R}_1, \ldots, R_n$ the sets $S_{0,n}, S_{1,n}, \ldots, S_{n,n}$ on the surface $\hat{R}_n, \ldots$ and so on. The set $R_1 - \bar{R}_0$ then corresponds to $S_{1,1}$ and $S_{0,1} - \overline{S_{0,0}}$ connected along the arc L_0. Let $\bar{\omega}_1$ be the harmonic measure of ∂R_1 with respect to $R_1 - \bar{R}_0$ and define

$$u_1(z) = \begin{cases} \omega'_{0,1}(z)\,, & \text{when} \quad z \in (S_{0,0} - \overline{S_{0,1}}) - L_0 \\ 1 & \text{elsewhere.} \end{cases}$$

The functions u_1 and $\bar{\omega}_1$ have the same boundary values and u_1 is piecewise continuously differentiable. Hence the Dirichlet principle is applicable and we have

$$D(\bar{\omega}_1) \leqq D_{R_1-\bar{R}_0}(u_1) = D(\omega'_{0,1}) \leqq 2\, D(\omega_{0,1}) \leqq 1\,.$$

The set $R_2 - \bar{R}_1$ on R_2 corresponds to $S_{0,2} - \overline{S_{0,1}}$, $S_{1,2} - \overline{S_{1,1}}$ and $S_{2,2}$, the latter two connected along the arc L_1. Defining

$$u_2(z) = \begin{cases} \omega'_{1,2}(z)\,, & \text{when } z \in (S_{1,2} - \overline{S_{1,1}}) - L_1 \\ \omega_{0,2}\,, & \text{when } z \in S_{0,2} - \overline{S_{0,1}} \\ 1 & \text{elsewhere}\,, \end{cases}$$

we have again by Dirichlet principle

$$D(\bar{\omega}_2) \leqq D_{R_2 - \bar{R}_1}(u_2) = D(\omega_{0,2}) + D(\omega'_{1,2}) \leqq D(\omega_{0,2}) + 2\,D(\omega_{1,2}) \leqq 1\,,$$

where $\bar{\omega}_2$ is the harmonic measure of ∂R_2 with respect to $R_2 - \bar{R}_1$. Generally, $R_n - \overline{R_{n-1}}$ on $\hat{R}_n$ corresponds to the sets $S_{0,n} - \overline{S_{0,n-1}}$, $S_{1,n} - \overline{S_{1,n-1}}\,, \ldots , S_{n-1,n} - \overline{S_{n-1,n-1}}\,, S_{n,n}\,,$ the latter two connected along the arc L_{n-1}. Defining

$$u_n(z) = \begin{cases} \omega'_{n-1,n}(z)\,, & \text{when } z \in (S_{n-1,n} - \overline{S_{n-1,n-1}}) - L_{n-1} \\ \omega_{i,n}(z)\,, & \text{when } z \in S_{i,n} - \overline{S_{i,n-1}} \quad (i = 0\,, \ldots , n-2) \\ 1 & \text{elsewhere.} \end{cases}$$

We have with a clear meaning of $\bar{\omega}_n$

$$D(\bar{\omega}_n) \leqq D_{R_n - \bar{R}_{n-1}}(u_n) = \sum_{i=0}^{n-2} D(\omega_{i,n}) + D(\omega'_{n-1,n}) \leqq$$

$$\leqq \sum_{i=0}^{n-2} D(\omega_{i,n}) + 2\,D(\omega_{n-1,n}) \leqq \sum_{i=0}^{n-2} \frac{1}{n^{i+1}} + \frac{1}{n^n} \leqq \frac{2}{n}\,.$$

Thus we have for the exhaustion $\{R_k\}$

$$\sum_{i=1}^{\infty} \frac{1}{D(\bar{\omega}_i)} \geqq 1 + \sum_{i=2}^{\infty} \frac{1}{D(\bar{\omega}_i)} \geqq 1 + \sum_{i=2}^{\infty} \frac{i}{2} = \infty\,.$$

By a criterion of Noshiro ([16], p. 76) R is parabolic.

4.2. Existence of quasirational mappings. The general question about the existence of a quasirational mapping between two given Riemann surfaces R and S seems to be difficult. However, there exist non-trivial quasirational mappings. Truncating the construction in the preceding proof we have for quasirational mappings the corresponding

Theorem 9. *Let* $K = \bigcup_{i=0}^{N-1} K_i$ $(N < \infty)$ *be formed out of disjoint closed polar sets* K_i *on a parabolic (or a compact) Riemann surface* S. *Then it is possible to construct a parabolic Riemann surface* R *and a quasirational mapping* $f: R \to S$ *such that*

$$\begin{cases} n(f\,, \zeta\,, R) = N\,, & \textit{when } \zeta \in S - K \\ n(f\,, \zeta\,, R) = p\,, & \textit{when } \zeta \in K_p \textit{ for } p = 0\,, 1\,, \ldots , N-1\,. \end{cases}$$

Proof. We begin by reproducing the first steps of the procedure in the preceding theorem. Let us truncate the procedure after a finite number of steps: $R = \hat{R}_{N-1}$. The arc L_{N-1} is the last one we define. The covering mapping $f: R \to S$ is the desired one. The parabolicity of R is now very easy to see. Consider $S - K$ which is parabolic. Let $g: T \to S - K$ be the restriction of f into $T = f^{-1}(S - K)$. Now (T, g) is a complete covering surface of $S - K$ ([17], p. 49). Since g is finite-sheeted, then T is parabolic ([17], p. 96). As an immediate consequence R is parabolic.

Theorem 10. *Let R be a parabolic Riemann surface of a finite genus r. Then there exists a quasirational function $f: R \to S_0$ with $n(f^*, \zeta, R_K^*) = r + 1$.*

Proof. The surface R can be embedded into a compact surface of genus r ([2], p. 420). This compact surface is conformally equivalent to a $(r + 1)$-sheeted covering surface of S_0 ([19], p. 275). The theorem follows.

The existence of a quasirational function $f: R \to S_0$ gives us information about hyperbolic subsurfaces G of R. Using conventional notations we say that $G \in O_L$, if there exists no non-constant Lindelöfian function on it and that $G \in U_{\mathrm{HD}}$, if the Kuramochi ideal boundary of G contains at least one point with a positive harmonic measure ([3], p. 169).

Theorem 11. *Let R be a parabolic Riemann surface on which there exists a non-constant quasirational function $f: R \to S_0$. Then every hyperbolic subregion $G \subset R$ is not in $U_{HD} \cup O_L$.*

Proof. a) By theorem 1 we see that there exists an analytic Lindelöfian function on G, thus $G \notin O_L$.

b) Again by theorem 1 G is a finite-sheeted covering surface over S_0. This implies $G \notin U_{HD}$ by [9], p. 88.

University of Helsinki and
University of Joensuu
Finland

References

[1] AHLFORS, L. and SARIO, L.: Riemann surfaces. - Princeton University Press (1960).

[2] BOCHNER, S.: Fortsetzung Riemannscher Flächen. - Math. Ann. 98 (1927).

[3] CONSTANTINESCU, C. and CORNEA, A.: Ideale Ränder Riemannscher Flächen. - Springer-Verlag (1963).

[4] FULLER, D.: Mappings of bounded characteristic into arbitrary Riemann surfaces. - Pacific J. Math. 14 (1964).

[5] HEINS, M.: Functions of bounded characteristic and Lindelöfian maps. - Internat. Congr. Math. Edinburgh (1958).

[6] —»— On the Lindelöf principle. - Ann. of Math. 61 (1955).

[7] —»— Lindelöfian maps. - Ann. of Math. 62 (1955).

[8] KELLEY, J.: General Topology. - v. Nostrand Co. (1955).

[9] MATSUMOTO, K.: A condition for each point of the Kuramochi boundary to be of harmonic measure zero. - Lecture Notes in Mathematics 58 (1968).

[10] —»— Exceptional values of meromorphic functions in a neighbourhood of the set of singularities. - J. Sci. Hiroshima Univ. Ser. A 24 (1960).

[11] MYRBERG, L.: Über quasirationale Funktionen auf parabolischen Riemannschen Flächen. - Ann. Acad. Sci. Fenn. Ser. A I 404 (1967).

[12—14] —»— Über meromorphe Funktionen auf endlich vielblättrigen Riemannschen Flächen.
I. - Ann. Acad. Sci. Fenn. Ser. A I 286 (1960).
II. - Ibid. 301 (1961).
III. - Ibid. 311 (1961).

[15] —»— Über meromorphe Funktionen auf nullberandeten Riemannschen Flächen. - Ann. Acad. Sci. Fenn. Ser. A I 312 (1962).

[16] NOSHIRO, K.: Open Riemann surface with null boundary. - Nagoya Math. J. 3 (1951).

[17] PFLUGER, A.: Theorie der Riemannschen Flächen. - Springer-Verlag (1957).

[18] SARIO, L. and NOSHIRO, K.: Value distribution theory. - v. Nostrand Co. (1966).

[19] SPRINGER, G.: Introduction to Riemann surfaces. - Addison-Wesley (1957).

[20] SEIBERT, P.: Typus und topologische Randstruktur einfachzusammenhängender Riemannscher Flächen. - Ann. Acad. Sci. Fenn. Ser. A I 250/34 (1958).

[21] TSUJI, M.: On a Riemann surface, which is conformally equivalent to a Riemann surface with a finite spherical area. - Comment. Math. Univ. St. Paul. VI 1 (1957).

Printed November 1970

CORRECTION TO "QUASIRATIONAL MAPPINGS ON PARABOLIC RIEMANN SURFACES"

by Ilpo Laine

In 4.1. before theorem 8 we incorrectly said that the sets K_i are closed. Of course, K is a union $K = \bigcup_{i=0}^{N-1} K_i$ of polar sets $K_i = \{\zeta \in S \mid n(f, \zeta, R) = i < N\}$, where $\bigcup_{i=0}^{n} K_i$ is closed for $0 \leqq n < N$.

Theorems 8 and 9 have henceforth the following more general form:

Theorem 8. *Let* $K = \bigcup_{i=0}^{\infty} K_i$ *be formed out of disjoint polar sets* K_i *on a parabolic (or a compact) Riemann surface* S. *If the sets* $\bigcup_{i=0}^{n} K_i$ *are closed for* $0 \leqq n < \infty$, *then it is possible to construct a parabolic Riemann surface* R *and an analytic mapping* $f: R \to S$ *such that*

$$\begin{cases} n(f, \zeta, R) = \infty, & \text{when } \zeta \in S - K \\ n(f, \zeta, R) = p, & \text{when } \zeta \in K_p \text{ for } p = 0, 1, 2, \ldots \end{cases}$$

Theorem 9. *Let* $K = \bigcup_{i=0}^{N-1} K_i \; (N < \infty)$ *be formed out of disjoint polar sets* K_i *on a parabolic (or a compact) Riemann surface* S. *If the sets* $\bigcup_{i=0}^{n} K_i$ *are closed for* $0 \leqq n \leqq N - 1$, *then it is possible to construct a parabolic Riemann surface* R *and a quasirational mapping* $f: R \to S$ *such that*

$$\begin{cases} n(f, \zeta, R) = N, & \text{when } \zeta \in S - K \\ n(f, \zeta, R) = p, & \text{when } \zeta \in K_p \text{ for } p = 0, 1, \ldots, N - 1. \end{cases}$$