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481

ON GREEN'S INEQUALITIES FOR THE THIRD COEFFICIENT OF BOUNDED UNIVALENT FUNCTIONS

BY

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1. Choosing g and D.

The class of bounded univalent functions is defined as follows. It consists of functions $f(z)$ analytic and univalent in the disc U: $|z| < 1$ and normalized so that

$$(1)\qquad \begin{cases} f(z) = \sum_1^\infty b_\nu z^\nu \,, \; |f(z)| < 1 \,, \\ b_1 = \text{a positive constant} \Rightarrow b_1 \in (0, 1] \,. \end{cases}$$

The corresponding class is denoted by $S(b_1)$. Observe that according to Schwarz's lemma $S(1)$ consists only of the identity mapping z.

To each f there belongs the bounded function

$$(2)\qquad \begin{cases} b_1^{-1} f(z) = \sum_1^\infty a_\nu z^\nu \,, \; a_\nu = \dfrac{b_\nu}{b_1} \,, \\ |b_1^{-1} f(z)| < b_1^{-1} \,. \end{cases}$$

We are going to consider coefficient inequalities for the a_n-coefficients by a method starting from an inequality due to Green's formula (cf. (2) and (3) in [8])

$$(3)\qquad 0 \leqq \iint_D |g'(W)|^2 \, d\sigma = \frac{1}{i} \int_{\partial D} \operatorname{Re} \{g(W)\} \, g'(W) \, dW \,.$$

Here, D is supposed to be a simply connected simple domain in the W-plane. The *generating function* g is to be chosen so that g' is analytic and $\operatorname{Re} g$ harmonic, in D closure, $\bar{D}$.

In [5] was derived an inequality generalizing the Nehari inequality for the $S(b_1)$-functions. The result was sufficiently strong to maximize the coefficient a_3 for the interval $e^{-1} \leqq b_1 < 1$ of the parameter b_1. In [8], the reason for this success was made clear, in checking of the totality of corresponding extremal functions. In the generalized Nehari inequality, the generating function was in accordance with the extremal function, if $b_1 \in [e^{-1}, 1)$. It is the most natural problem to try to find such a generating function g which extends a sharp inequality for the remaining interval $0 < b_1 < e^{-1}$.

In [1], De Temple extends the generalized Nehari inequality by applying variational technique. His result is sufficiently strong to solve the a_3-problem mentioned. In the present paper, only Green's inequality will be utilized to trying to find the corresponding extension. There appears that these two methods mentioned are able to give at least some results equivalent to each other, save the form.

For finding the generating function g correctly we will make use of the information for the extremum function got by aid of Löwner's method, or by using Schiffer's differential equation. On the basis of the results obtained it is evident that when further inequalities are constructed for the a_n-coefficients, it must be most useful repeateadly to follow the guidance, obtainable in particular from Schiffer's differential equation.

Consider the function f maximazing the coefficient $a_3 > 0$ for $b_1 \in (0, e^{-1})$. We get an equation for f by aid of Löwner's method ([6] and [7]) as well as by means of Schiffer's differential equation (in [4], the corresponding problem for a_4 is delt with). The condition in question can be written by aid of the following sequence of equations

$$
(4) \qquad \begin{cases} \log \dfrac{w-1}{w+1} - \dfrac{2w}{1-w^2} = \log z - \dfrac{1}{b_1(1+t)}(z - z^{-1}) \,, \\ w = \sqrt{\dfrac{y+1}{y-t}} \,, \\ y = \frac{1}{2}(f + f^{-1}) \,, \\ t = \frac{1}{2}(d + d^{-1}) \,, \quad \text{where} \quad d > 0 \quad \text{is the branch point of the forked slit of } f(U). \end{cases}
$$

Further, we introduce the connection

$$
W = \frac{w-1}{w+1}
$$

and observe that the left side of the first equation (4) can be written

$$
g(W) = x_0 \log W + x_1(W - W^{-1}) \tag{5}
$$

with $x_0 = 1, x_1 = -\frac{1}{2}$. Therefore, it is the most natural try to choose this $g(W)$ as a generating function in Green's inequality. Here x_0 is kept real, x_1 is complex and d is an omitted value of f. By aid of the connections

$$
y = \tfrac{1}{2}(f + f^{-1}) \,, \tag{6}
$$

$$
w = \sqrt{\frac{y+1}{y-t}} \,, \tag{7}
$$

(8) $$W = \frac{w-1}{w+1},$$

we have to determine a proper domain D.

All the mappings in question are simple. They are illustrated in Figure 1, in the typical general case where $f(\partial K_r)$, $K_r \in U$, cuts the real axes of the f-plane four times.

In the mapping (7), one must take care that the plane $\dfrac{y+1}{y-t}$ is cut along a slit, resulting

$$w = \sqrt{\frac{y+1}{y-t}} = w(z), \ z \in U,$$

to be analytic. The nature of this mapping implies that for $z_1, z_2 \in U$:

(9) $$z_1 \neq z_2 \Rightarrow w(z_1) \neq w(z_2) \text{ and } w(z_1) \neq -w(z_2).$$

This further implies for $W = W(z)$ that

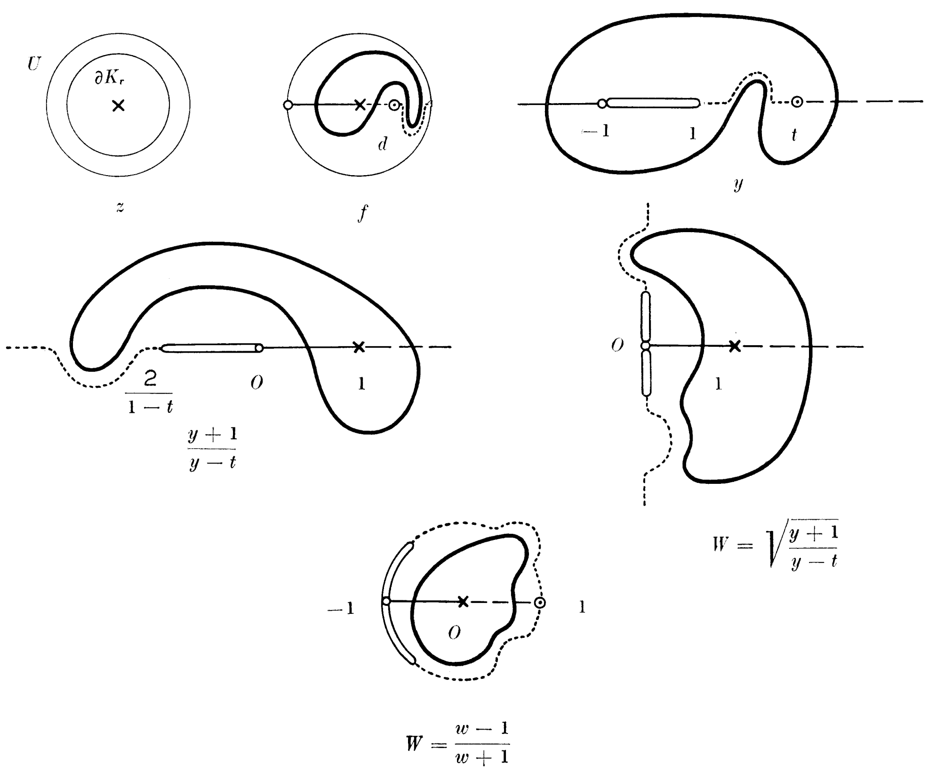

Fig. 1.

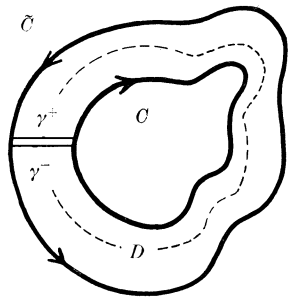

Fig. 2.

$$z_1 \neq z_2 \Rightarrow W(z_1) \neq W(z_2) \text{ and } W(z_1) \neq \frac{1}{\overline{W(z_2)}} . \tag{10}$$

Thus, we observe that $W(z)$ is a Bieberbach — Eilenberg function [2].

The curve $W(\partial K_r) = C$ is a simple analytic curve. Apply to C the inversion $\frac{1}{\overline{W}}$ to get $\tilde{C}$. According to (10) C and $\tilde{C}$ have no common points. The simple ring domain determined by $-C$ and $\tilde{C}$ is cut simply connected by a properly chosen slit (γ^+, γ^-). The domain D thus determined has the boundary

$$\partial D = \tilde{C} \cup \gamma^+ \cup -C \cup \gamma^- .$$

— Here we assume C to be positively orientated. Hence, the Green's inequality (3) assumes the form

$$0 \leqq \frac{1}{i} \int_{\tilde{C}} \operatorname{Re}\{g(Z)\}\, g'(Z)\, dZ - \frac{1}{i} \int_{C} \operatorname{Re}\{g(W)\}\, g'(W)\, dW . \tag{11}$$

Apply the inversion $W = \frac{1}{\overline{Z}}$ to the first integral

$$\int_{\tilde{C}} \operatorname{Re}\{g(Z)\}\, g'(Z) dZ = \int_{\tilde{C}} \operatorname{Re}\{g(Z)\} \left[\frac{x_0}{Z} + x_1(1 + Z^{-2})\right] dZ$$

$$= \int_{-C} - \operatorname{Re}\{g(W)\}\, [x_0\, W + x_1(1 + W^2)]\, \frac{-dW}{W^2}$$

$$= - \int_{C} \operatorname{Re}\{g(W)\}\, g'(W)\, dW .$$

Thus, (11) is reduced to the form

$$0 \geqq \frac{2}{i} \int_{C} \operatorname{Re}\{g(W)\} g'(W)\, dW . \tag{12}$$

2. An application to coefficient a_3.

Consider the function $f(z)$ normalized by rotation so that $a_3 \geqq 0$. Suppose that

$$f(z) \neq d = \text{complex}. \tag{13}$$

Clearly, there exist $S(b_1)$-functions for which the omitted values d are not real, when some normalization, e.g. $a_3 \geqq 0$, is fixed.

Now, replace $f(z)$ by the rotated function $\tau^{-1}f(\tau z)$, $|\tau| = 1$, which turns the image domain round the origin by the angle $-\arg \tau$. Choose τ so that

$$\arg \tau = \arg d \; ; \; d = |d| \, \tau \, . \tag{14}$$

Hence, $\tau^{-1}f(\tau z) \neq |d| > 0$. Apply the inequality (12) to the function $\tau^{-1}f(\tau z)$.

We will utilize the following development, valied in U:

$$\tag{15} \begin{cases} w = \sqrt{\dfrac{y+1}{y-t}} = 1 + \sum\limits_1^\infty c_\nu z^\nu \, , \; t = \frac{1}{2}\left(|d| + |d|^{-1}\right) ; \\ c_1 \; = \; (1 + t) \, b_1 \, , \\ c_2 \; = \; c_1(\tau a_2 + \dfrac{3\,t - 1}{2} b_1) \, , \\ c_3 \; = \; c_1 \left[\tau^2 a_3 + (3t - 1) \, b_1 \, \tau a_2 + (\frac{5}{2} t^2 - t - \frac{1}{2}) \, b_1^2\right] . \end{cases}$$

Sinse $g(W(z))$ is analytic in $0 < |z| < 1$, the following development holds for these values of z:

$$\tag{16} \begin{cases} g(W(z)) \; = \; x_0 \log z - \dfrac{2x_1}{c_1} z^{-1} + \sum\limits_{n=0}^\infty C_n z^n \, ; \\ C_0 \; = \; x_0 \log \dfrac{c_1}{2} + \dfrac{2x_1}{c_1}\left(\dfrac{c_2}{c_1} - \dfrac{c_1}{2}\right) \\ \quad \; = \; x_0 \log \dfrac{c_1}{2} + \dfrac{2x_1}{c_1}\left(\tau a_2 + (t - 1) b_1\right) , \\ C_1 \; = \; x_1 \left[\dfrac{c_1}{2} - \dfrac{2}{c_1}\left(\dfrac{c_2^2}{c_1^2} - \dfrac{c_3}{c_1}\right)\right] + x_0 \left(\dfrac{c_2}{c_1} - \dfrac{c_1}{2}\right) \\ \quad \; = \; \dfrac{2x_1}{c_1}\left[\tau^2 a_3 + (3t - 1) b_1 \tau a_2 + (\frac{5}{2} t^2 - t - \frac{1}{2}) b_1^2 \right. \\ \qquad \left. - \left(\tau a_2 + \dfrac{3t - 1}{2} b_1\right)^2 + \dfrac{c_1^2}{4}\right] + x_0(\tau a_2 + (t - 1) \, b_1) \, . \end{cases}$$

The form of development (16) is exactly the same as that encountered in connection with the generalized Nehari inequality (cf. [5], (52)). Therefore the result (59) of [5] can directly be utilized to give

$$\sum_{n=1}^{\infty} n|C|^2 \leqq \left|\frac{2x_1}{c_1}\right|^2 - 2\, x_0 \operatorname{Re} C_0 \,. \tag{17}$$

We will make use of the following consequence

$$|C_1|^2 \leqq \left|\frac{2x_1}{c_1}\right|^2 - 2\, x_0 \operatorname{Re} C_0 \tag{18}$$

of (17). Further, x_0 will be so chosen so that $\operatorname{Re} C_0 = 0$. Hence, we have

$$\begin{cases} |C_1| \leqq \left|\dfrac{2x_1}{c_1}\right| , \ \operatorname{Re} C_0 = 0 \, ; \\ \text{equality iff } C_2 = C_3 = \ldots = 0 \, . \end{cases} \tag{19}$$

Introduce the notation

$$a_2 + (t-1) b_1 \tau^{-1} = X \tag{20}$$

and consider the condition

$$0 = 2\operatorname{Re} C_0 = 2x_0 \log \frac{c_1}{2} + \frac{2}{c_1} (x_1 \tau\, X + \bar{x}_1\, \tau^{-1}\, \bar{X}) \, ;$$

$$x_0 = - \frac{x_1 \tau\, X + \bar{x}_1\, \tau^{-1}\, \bar{X}}{c_1 \log \dfrac{c_1}{2}} ,$$

$$x_0 X\, \tau = - \frac{x_1 \tau^2 X^2 + \bar{x}_1\, |X|^2}{c_1 \log \dfrac{c_1}{2}} . \tag{21}$$

Observe that $\dfrac{c_1}{2} < 1$, and hence formula (21) is available. This is seen by starting from the Koebe constant for the $S(b_1)$-functions:

$$K(b_1) = \frac{b_1}{2 - b_1 + 2\sqrt{1 - b_1}} \leqq |d| \leqq 1 \,,$$

which gives for $t = \frac{1}{2}(|d| + |d|^{-1})$

$$1 \leqq t \leqq \tfrac{1}{2}[K(b_1) + K(b_1)^{-1}] = \frac{2}{b_1} - 1 \, ; \tag{22}$$

$$b_1 \leqq \frac{c_1}{2} = \frac{1+t}{2}\, b_1 \leqq 1 \, . \tag{23}$$

Equality on the right holds only for the radial slit mapping, which can be excluded.

In (21), take especially $\arg x_1 = -\arg \tau$, which means that

$$\bar{x}_1 = x_1 \tau^2 \tag{24}$$

and hence

$$x_0 X \tau = -\frac{2x_1}{c_1} \frac{X^2 + |X|^2}{2 \log \frac{c_1}{2}} \tau^2 . \tag{25}$$

Taking this into consideration we obtain for C_1:

$$\begin{aligned} \tau^{-2} C_1 = \frac{2x_1}{c_1} \Bigg[& a_3 + (3t - 1)\, b_1 \tau^{-1} a_2 + (\tfrac{5}{2} t^2 - t - \tfrac{1}{2}) b_1^2 \tau^{-2} \\ & - \left(a_2 + \frac{3t-1}{2} b_1 \tau^{-1} \right)^2 + \frac{c_1^2}{4} \tau^{-2} - \frac{X^2 + |X|^2}{2 \log \frac{c_1}{2}} \Bigg] \\ = \frac{2x_1}{c_1} \Bigg[& a_3 + 2(t-1)\, b_1 \tau^{-1} X - \left(\frac{t^2}{2} - 3t + \tfrac{3}{2} \right) b_1^2 \tau^{-2} - X^2 - \frac{X^2 + |X|^2}{2 \log \frac{c_1}{2}} \Bigg]_0 . \end{aligned} \tag{26}$$

Inequality (19) thus gives

$$|[\ ]_0| \leqq 1 \Rightarrow \operatorname{Re} [\ ]_0 \leqq 1 , \tag{27}$$

which according to (26) can be written in the form

$$\begin{aligned} a_3 + 2(t-1)\, b_1 \operatorname{Re}\{\tau^{-1} X\} - \left(\frac{t^2}{2} - 3t + \tfrac{3}{2} \right) b_1^2 \operatorname{Re}\{\tau^{-2}\} \\ - (\operatorname{Re} X)^2 + (\operatorname{Im} X)^2 - \frac{(\operatorname{Re} X)^2}{\log \frac{c_1}{2}} \leqq 1 . \end{aligned}$$

Our result is thus the following inequality for a_3:

$$\begin{cases} 0 \leqq a_3 \leqq 1 + \left(\frac{t^2}{2} - 3t + \frac{3}{2} \right) b_1^2 \operatorname{Re}\{\tau^{-2}\} \\ - 2(t-1) b_1 \operatorname{Re}\{\tau^{-1} X\} + \left(1 + \frac{1}{\log \frac{c_1}{2}} \right) (\operatorname{Re} X)^2 ; \\ \text{equality iff } \operatorname{Im} X = 0 . \end{cases} \tag{28}$$

The right side depends on two complex numbers, a_2 and d, connected to f. It is clear that estimation of a_3 by aid of (28) can not be succesful without sufficient information of these numbers and their mutual relationships.

Finally, observe that (28) includes our former estimation. For each $S(b_1)$-function one is allowed to choose $d = 1$. Hence, (28) yelds in this case

$$\begin{cases} a_3 - (1 - b_1^2) \leqq \left(1 - \dfrac{1}{\log b_1^{-1}}\right) (\mathrm{Re}\, a_2)^2 \,, \\ \text{equality iff } \mathrm{Im}\, a_2 = 0 \,. \end{cases} \tag{29}$$

This is the condition (79) of [5] obtained earlier from the generalized Nehari inequality.

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