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ON THE MAXIMAL DILATATION OF QUASICONFORMAL EXTENSIONS

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On The maximal dilatation of quasiconformal extensions

The following theorem appears on page 100 of the book *Quasikonforme Abbildungen* by O. Lehto and K. I. Virtanen, Springer-Verlag (see also [4]).

Theorem. *Let f be a K-quasiconformal mapping of a plane domain D, and let F be a compact set in D. Then there exists a quasiconformal mapping $\tilde{f}$ of the entire plane which is equal to f in F. Furthermore the maximal dilatation of $\tilde{f}$ depends only on K, D, and F.*

The proof given is indirect and no information on the maximal dilatation of $\tilde{f}$ is available. In this note we provide a constructive proof along with some crude estimates which show directly the independence of the maximal dilatation.

The proof of the above theorem (see [4]) reduces to the following

Theorem 1. *Set $A_i = \{1 < |z| < R_i\}$, $i = 1, 2$, and $A_\varepsilon = \{1+\varepsilon \leq |z| \leq R_1-\varepsilon\}$. Suppose f is a sense preserving homeomorphism of the inner and outer contours of A_1 onto the inner and outer contours of A_2 respectively which for some $\varepsilon > 0$ can be extended to a K-quasiconformal mapping of $A_1 - A_\varepsilon$ into A_2. Then f can be extended to a quasiconformal mapping $\tilde{f}$ of A_1 onto A_2.*

This theorem has also found recent application in [5]. The proof given there is also indirect, however a construction of $\tilde{f}$ is outlined, and together with certain estimates for quasisymmetric functions developed by the author [3], we will obtain an upper estimate of the maximal dilatation of $\tilde{f}$ which depends only on K, R_1, R_2 and ε. Since we are only interested in showing that such an estimate exists, we will make no attempt to collect our various estimates into one grand formula.

We begin by representing the universal covering surface of A_i, $i = 1, 2$, by the upper half plane $\{y > 0\}$, $z = x+iy$. Then $a_i z$ for some $a_i > 1$ is the generator of the cyclic group of cover transformations. Now f can be lifted to a sense preserving homeomorphism f^* of each interval $\{x < 0\}$, $\{x > 0\}$ on the real axis onto itself (each interval lies over one pair of corresponding contours of A_1 and A_2) so that $f^*(\pm 1) = \pm 1$ and

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$$f^*(a_1^n x) = a_2^n f^*(x) , \tag{1}$$

for each integer n. If we define $f^*(0) = 0$, f^* is then a homeomorphism of the entire real line onto itself. Our procedure is as follows. First we show f^* is locally quasisymmetric on the right and left half x-axis, then quasisymmetric there, finally k-quasisymmetric on the entire axis, k depending only on K, R_i and ε. Then we use the Beurling-Ahlfors mapping [1] to obtain a k^2-quasiconformal extension f of f^* which maps $\{y > 0\}$ onto itself. Noting from its explicit formula that $\hat{f}$ also satisfies (1) with x replaced by z, we can project $\hat{f}$ to the desired mapping $\tilde{f}$ of A_1 onto A_2.

From the symmetry of the problem, we need, for the moment, only consider f^* for $x > 0$. The hypotheses of the theorem imply that f^* can be extended to a K-quasiconformal mapping g of a sector $0 \leq \arg z \leq \alpha$ onto a domain in the upper half plane bordering the positive real axis. Here $\alpha \leq \pi$ depends only on R_1 and ε. By reflection in the real axes, we can assume g maps the sector G: $-\alpha \leq \arg z \leq \alpha$ onto a domain G' symmetric in the real axis and containing the positive real axis in its interior.

Fix $x > 0$. In the image plane let $r_1 < r = d(f^*(x), \partial G')$, where d denotes euclidean distance and ∂ means boundary. Let C' and C_1' denote the circles $|z - f^*(x)| = r$ and r_1 respectively, and let C and C_1 be the corresponding inverse images under g. Choose q on C_1 such that $|x - q| = d(x, C_1)$. Then $|x - q| < d(x, \partial G)$ and by Theorem 11 of [2]

$$\frac{|x - q|}{d(x, \partial G)} \geq \Theta_K^{-1}(r_1/r) ,$$

where $\Theta_K(t)$, $0 < t < 1$, is a certain strictly increasing continuous function with $\Theta_K(0+) = 0$, $\Theta_K(1-) = +\infty$. If we let $r_1 \to r$, then $|x - q| \to d(x, C)$ and we get

$$d(x, C) \geq \Theta_K^{-1}(1) d(x, \partial G) = \gamma x ,$$

where $\gamma < 1$ is given by

$$\gamma = \begin{cases} \Theta_K^{-1}(1) (\sin \alpha) & \text{if} \quad 0 < \alpha \leq \pi/2 \\ \Theta_K^{-1}(1) & \text{if} \quad \pi/2 \leq \alpha \leq \pi . \end{cases} \tag{2}$$

It now follows from Mori's Lemma [6, page 60] that

$$e^{-\pi K} \leq \frac{f^*(x+t) - f^*(x)}{f^*(x) - f^*(x-t)} \leq e^{\pi K} , \tag{3}$$

for $0 < t \leq \gamma x$, which proves f^* is locally quasisymmetric.

Next we must show there exists $k \geq 1$ such that

$$1/k \leq \frac{f^*(x+t) - f^*(x)}{f^*(x) - f^*(x-t)} \leq k ,$$

for all $0 < t \leq x$. By multiplying numerator and denominator by an appropriate power of a_2 and using (1) we can assume $1 \leq x < a_1$. If $t \leq \gamma x$, take $k = e^{\pi K}$. Otherwise $\gamma x \leq t \leq x$ and

$$(4) \quad \frac{f^*(x+t) - f^*(x)}{f^*(x) - f^*(x-t)} \leq \frac{f^*(2x) - f^*(1)}{f^*(x) - f^*(x-\gamma x)} \leq \frac{f^*(2a_1) - 1}{f^*(x) - f^*(x-\gamma x)} \leq \frac{a_2^m - 1}{f^*(x) - f^*(x-\gamma)} ,$$

where $m > 1$ is the unique integer such that $a_1^{m-1} < 2a_1 \leq a_1^m$. Similarly

$$(5) \quad \frac{f^*(x+t) - f^*(x)}{f^*(x) - f^*(x-t)} \geq \frac{f^*(x+\gamma x) - f^*(x)}{f^*(a_1) - f^*(0)} \geq \frac{f^*(x+\gamma) - f^*(x)}{a_2} .$$

In order to obtain a lower bound for $f^*(x+\gamma) - f^*(x)$ independent of f^*, as x varies from 1 to a_1, we let $\gamma_1 \leq \gamma/2$ be the largest number such that $(a_1-1)/\gamma_1 = n$, $n \geq 2$ an integer, and then partition the interval $1 \leq x \leq a_1$ into n equal parts, $1 = x_0 < x_1 < \ldots < x_n = a_1$, each segment having length γ_1. Then if $x_{s-1} < x \leq x_s$, $f^*(x+\gamma) - f^*(x) \geq f^*(x+2\gamma_1) - f^*(x) \geq f^*(x_{s+1}) - f^*(x_s)$. By multiplying and dividing by $f^*(x_j) - f^*(x_{j-1})$, $j = 0, \ldots, s$, and using (3) we get

$$(6) \quad f^*(x+\gamma) - f^*(x) \geq e^{-s\pi K} [f^*(x_1) - f^*(x_0)] \geq e^{-n\pi K} [f^*(x_1) - 1] .$$

To obtain a lower estimate on $f^*(x_1) - 1$, we use an obvious iteration on (3). The result is

$$(7) \quad f^*(x_1) - 1 \geq \frac{a_2 - 1}{1 + e^{\pi K} + \ldots + e^{(n-1)\pi K}} ,$$

which together with (6), (5) and (4) prove that f^* is k-quasisymmetric on $\{x > 0\}$, and therefore by symmetry on $\{x < 0\}$, where k depends only on R_1, R_2, K and ε.

Finally suppose $x = 0$ and $t > 0$ is arbitrary. As before, by multiplying numerator and denominator by an appropriate power of a_2, we can assume $1 \leq t < a_1$ and hence we obtain the simple estimate

$$(8) \quad 1/a_2 = 1/f^*(a_1) \leq \frac{f^*(t)}{-f^*(-t)} \leq f^*(a_1) = a_2 .$$

By Theorem 3 of [3] it follows that f^* is $a_2(1+k+k^2)$-quasisymmetric on the entire line, which concludes the proof.

As an application of this theorem, suppose $w(z)$ is a K-quasiconformal mapping of the unit disc D onto itself. Fix $0 < \varrho < 1$ arbitrarily. We wish to find a K_0-quasiconformal mapping $\tilde{w}$ of the entire plane which agrees with w on $D_\varrho : |z| \leq \varrho$, and is, say, the identity for $|z| \geq 1$, where K_0 depends only on K and ϱ. Except for the contingency regarding K_0, the existence of such a mapping is guaranteed by the theorem quoted in the introduction. Denote by A' the image under w of the annulus $A_1 : \varrho \leq |z| \leq 1$. Map A' conformally by ζ onto an annulus $A_2 : \varrho' \leq |z| \leq 1$. Define the homeomorphism f of the inner and outer contours of A_1 onto the corresponding contours of A_2 by $f = \zeta \circ w$ for $|z| = \varrho$ and $f = \zeta \circ$ identity for $|z| = 1$. Then clearly f can be extended to a K-quasiconformal mapping of $A_1 - A_\varepsilon$ into A_2 where A_ε is a certain annulus $\varrho + \varepsilon \leq |z| \leq 1 - \varepsilon$, ε depending only on K and ϱ. It follows from Theorem 1 that f can be extended to a K_0-quasiconformal mapping ξ of A_1 onto A_2. Hence the desired K_0-quasiconformal mapping $\tilde{w}$ is given by w for $|z| \leq \varrho$, $\zeta^{-1} \circ \xi$ for $\varrho \leq |z| \leq 1$ and the identity for $|z| \geq 1$.

As a second application of the method of proof of Theorem 1, we prove the following

Theorem 2. *Suppose f is a sense preserving self homeomorphism of $|z| = 1$ which for some $\varepsilon > 0$ can be extended to a K-quasiconformal mapping of $1 - \varepsilon < |z| \leq 1$ into the unit disc. Then f can be extended to a quasiconformal self mapping $\tilde{f}$ of the unit disc with the origin mapping to itself. Furthermore the maximal dilatation of $\tilde{f}$ depends only on K and ε.*

This time we represent the universal covering surface of the punctured unit disc, $0 < |z| < 1$, by the upper half plane $\{y > 0\}$, $z = x + iy$, in such a way that $z + 1$ is the generator of the cyclic group of cover transformations. Then f can be lifted to a sense preserving self homeomorphism f^* of the entire real axis such that $f^*(0) = 0$ and

$$f^*(x+n) = f^*(x) + n \, , \tag{9}$$

for each integer n. We need only show f^* is k-quasisymmetric, k depending on K and ε, for then if $\hat{f}(x, y)$ is the Beurling-Ahlfors k^2-quasiconformal extension of f^* to the upper half plane, we see from the explicit formula for $\hat{f}$ that

$$\hat{f}(x+n, y) = \hat{f}(x, y) + n \, .$$

Hence $\hat{f}$ can be projected to give a k^2-quasiconformal self mapping of the punctured unit disc. Since the origin is a removable singularity, this mapping is the desired $\tilde{f}$.

The hypotheses of the theorem imply that f^* can be extended to a

K-quasiconformal mapping of a horizontal strip $0 \leq y < \alpha$ onto a domain in the upper half plane bordering the entire real axis. Here α depends only on ε. Using Gehring's distortion theorem and Mori's lemma exactly as before, we conclude that (3) holds for all x and $0 < t \leq \gamma = \Theta_K^{-1}(1)\alpha$, i.e., f^* is locally quasisymmetric.

To show f^* is globally quasisymmetric, suppose x and $t > 0$ are given. Because of (9) we can assume $0 \leq x \leq 1$. First consider the case $t \geq 2$. Then for some integer $n \geq 2$, $n \leq t < n+1$. Hence $n \leq x+t \leq n+2$ and $-n-1 \leq x-t \leq 1-n$. Consequently

$$\frac{f^*(x+t) - f^*(x)}{f^*(x) - f^*(x-t)} \leq \frac{f^*(n+2) - f^*(0)}{f^*(0) - f^*(1-n)} = \frac{n+2}{n-1} \leq 4 .$$

Similarly this ratio is $\geq 1/4$. Suppose next that $\gamma \leq t \leq 2$ (if $\gamma > 2$, we are done). Choose $\gamma_1 \leq \gamma$ such that $2/\gamma_1 = p$, $p \geq 1$ an integer. Then for some integer m, $1 \leq m < p$, $m\gamma_1 \leq t < (m+1)\gamma_1$. An iteration of the identity

$$\frac{f^*(x+mt) - f^*(x)}{f^*(x) - f^*(x-t)} = \left[1 + \frac{f^*(x+mt) - f^*(x+t)}{f^*(x+t) - f^*(x)} \right] \left[\frac{f^*(x+t) - f^*(x)}{f^*(x) - f^*(x-t)} \right] ,$$

together with (3), which is valied for $t \leq \gamma_1 \leq \gamma$, yields

$$\frac{f^*(x+t) - f^*(x)}{f^*(x) - f^*(x-\gamma_1)} \leq \frac{f^*(x+(m+1)\gamma_1) - f^*(x)}{f^*(x) - f^*(x-\gamma_1)} \leq \beta + \beta^2 + \ldots + \beta^{m+1} ,$$

where $\beta = e^{\pi K}$. Hence

$$1/\beta \leq \frac{f^*(x+\gamma_1) - f^*(x)}{f^*(x) - f^*(x-\gamma_1)} \leq \frac{f^*(x+t) - f^*(x)}{f^*(x) - f^*(x-\gamma_1)} \leq \beta + \ldots + \beta^p .$$

The same inequalities are valid for $[f^*(x)-f^*(x-t)]/[f^*(x+\gamma_1)-f^*(x)]$. Taking ratios we finally get

$$\frac{1}{\beta^3 + \ldots + \beta^{p+2}} \leq \frac{f^*(x+t) - f^*(x)}{f^*(x) - f^*(x-t)} \leq \beta^3 + \ldots + \beta^{p+2} ,$$

and the proof is complete.

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