ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

476

ESTIMATES FOR NORMAL MEROMORPHIC FUNCTIONS

BY

Ch. POMMERENKE

HELSINKI 1970
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1971.476

Communicated 13 March 1970 by OLLI LEHTO

KESKUSKIRJAPAINO
HELSINKI 1970

1. Introduction and results

The function $f(z)$ is called *normal* in $D = \{z \in C : |z| < 1\}$ if it is meromorphic and

$$(1.1) \qquad \alpha = \alpha_f = \sup_{|z|<1} (1 - |z|^2) \frac{|f'(z)|}{1 + |f(z)|^2} < \infty .$$

We call α the *order of normality* of $f(z)$. The order is invariant under spherical rotations of $f(z)$. If $f(z) = g(\varphi(z))$ with $|\varphi(z)| < 1$ then $\alpha_f \leqq \alpha_g$. Normal functions were introduced by Lehto and Virtanen [7]. Further important results were obtained by Hayman [4], Bagemihl and Seidel [2], and MacLane [8].

Let the meromorphic function $f(z)$ map D onto the Riemann surface F over the sphere. Around the point $f(z) \in F$, we consider the largest schlicht disk on F. Let $\delta(z) = \delta_f(z)$ denote the angular radius of this disk measured from the center of the sphere. The plane projection of this disk is

$$\left\{w : \left|\frac{w - f(z)}{1 + \overline{f(z)}\, w}\right| < d^*(z)\right\}, \; d^*(z) = \tan \frac{\delta(z)}{2} .$$

Our main tool will be the generalization of Schwarz' lemma due to Ahlfors [1]. We shall prove:

1. Let $\sup \delta(z) < \frac{\pi}{3}$, or let $f(z)$ be locally univalent and $\sup \delta(z) < \frac{\pi}{2}$. Then $f(z)$ is normal. The number $\pi/2$ is best possible.

2. For any normal function of order α,

$$d^*(z) \leqq \frac{(1 - |z|^2)\, |f'(z)|}{1 + |f(z)|^2} \leqq 2 \coth \frac{\pi}{4\alpha} \cdot \sqrt{d^*(z)} .$$

3. If $f(z)$ is normal and analytic in D then

$$(1 - |z|^2)\, |f'(z)| \leqq 2 (\log^+ |f(z)| + \alpha) \cdot \max (|f(z)|, 1) .$$

This is a somewhat more precise form of some results of Hayman [4] [5, Section 6.5].

4. As a consequence we shall obtain a simple proof of Schottky's and Landau's theorem with good quantitative bounds.

2. Ahlfors' Lemma

We shall prove a result of Ahlfors [1] in a somewhat different form; compare also [9].

Ahlfors Lemma. *Let* $u(z) \geqq 0$ *be continuous in* D . *For each* $z_0 \in D$, *let either* $u(z_0) \leqq 1$, *or let there exist a function* $\varphi(z)$ *analytic at* z_0 *such that* $|\varphi(z_0)| < 1$ *and, for small* $|z - z_0|$,

$$v(z) = \frac{(1 - |z|^2)\,|\varphi'(z)|}{1 - |\varphi(z)|^2} \leqq u(z)\,,\ v(z_0) = u(z_0)\,. \tag{2.1}$$

Then $u(z) \leqq 1$ *for* $z \in D$.

Proof. Suppose first that $u(z)$ is continuous in D and $u(z) \to 0$ as $|z| \to 1$. It follows that the supremum is attained in D , say at z_0 . As (2.1) remains unchanged under a bilinear mapping of D onto itself we may assume that $z_0 = 0$. Also we may assume that $\varphi(0) = 0$. It follows from (2.1) that, for small $|z|$,

$$v(z) \leqq u(z) \leqq u(0) = v(0) = |\varphi'(0)|\,. \tag{2.2}$$

We can write $\varphi(z) = a(z + bz^2 + cz^3 + \ldots)$, $a = |\varphi'(0)|$. Since $|1 + w| = 1 + \operatorname{Re} w + (\operatorname{Im} w)^2/2 + o(|w|^3)$ as $w \to 0$ we obtain from (2.1) and (2.2) that

$$v(z) = |a|\,(1 + 2\operatorname{Re} bz + 3\operatorname{Re} cz^2 + 2\,(\operatorname{Im} bz)^2 + (|a|^2 - 1)\,|z|^2 + o(|z|^3)) \leqq |a|$$

as $z \to 0$. It follows first that $b = 0$, hence second that $|a|^2 - 1 \leqq 0$ and therefore that $u(0) = |\varphi'(0)| = |a| \leqq 1$. In the general case, we consider $u^*(z) = u(rz)\,(1 - |z|^2)\,/\,(1 - r^2|z|^2)$ and let $r \to 1 - 0$.

3. Conditions for normality

Theorem 1. *Let* $f(z)$ *be meromorphic in* D *and*

$$\delta(z) \leqq \beta < \frac{\pi}{3}\ (z \in D)\,.$$

Then

$$\frac{(1 - |z|^2)\,|f'(z)|}{1 + |f(z)|^2} \leqq \frac{2}{\sqrt{\lambda}}\,\frac{\sqrt{d^*(z)}\,(\lambda - d^*(z))}{1 + d^*(z)^2}\quad (z \in D) \tag{3.1}$$

where $\lambda = \tan\dfrac{\beta}{2} \cdot (2\cos\beta + 1)\,/\,(2\cos\beta - 1)$. *Hence* $f(z)$ *is normal of order*

$$\alpha \leqq \frac{2 \sin \beta}{\sqrt{4 \cos^2\beta - 1}} < \infty . \tag{3.2}$$

Proof. We may assume that $\delta(z) < \beta$ (otherwise we consider $\beta + \varepsilon < \pi/3$ and let $\varepsilon \to 0$). For each $z_0 \in D$ there exists c such that, for small $|z - z_0|$,

$$d^*(z) \leqq |w(z)| , d^*(z_0) = |w(z_0)| , w(z) = \frac{c - f(z)}{1 + \bar{c} f(z)} . \tag{3.3}$$

If z_0 is a multiple point ($f'(z_0) = 0$ or multiple pole) we take $c = f(z_0)$, and we have $d^*(z) = |w(z)|$ for small $|z - z_0|$.

We have $\lambda = \tau(3 - \tau^2) / (1 - 3\tau^2)$ where $\tau = \tan \beta/2 < 1/\sqrt{3}$, hence $d^*(z) < \tau < \lambda$. Therefore we can define

$$u(z) = \frac{(1 - |z|^2) |f'(z)|}{1 + |f(z)|^2} \cdot \frac{\sqrt{\lambda}(1 + d^*(z)^2)}{2\sqrt{d^*(z)}\,(\lambda - d^*(z))} . \tag{3.4}$$

This function remains continuous at the multiple points. For $z_0 \in D$ we define

$$\varphi(z) = \sqrt{\lambda}\, \frac{\sqrt{w(z)} - \sqrt{w(z_0)}}{\lambda - \sqrt{\overline{w(z_0)}\, w(z)}}$$

for small $|z - z_0|$. Computation shows that

$$v(z) = \frac{(1 - |z|^2) |\varphi'(z)|}{1 - |\varphi(z)|^2} = \frac{(1 - |z|^2) |f'(z)|}{1 + |f(z)|^2} \cdot \frac{\sqrt{\lambda}\,(1 + |w(z)|^2)}{2\sqrt{|w(z)|}\,(\lambda - |w(z)|)} . \tag{3.5}$$

We have

$$\frac{d}{dt} \frac{1 + t^2}{\sqrt{t}(\lambda - t)} = \frac{- t^3 + 3\lambda t^2 + 3t - \lambda}{2t^{3/2}\,(\lambda - t)^2} .$$

The numerator increases for $0 < t < \lambda$ and vanishes for $t = \tau = \tan \beta/2$. Therefore $(1 + t^2) / \sqrt{t}\,(\lambda - t)$ decreases for $0 < t < \tau$. By assumption $\delta(z_0) < \beta$, hence $|w(z_0)| = d^*(z_0) < \tau$. Thus it follows from (3.3), (3.4), and (3.5) that, for small $|z - z_0|$,

$$v(z) \leqq u(z) , v(z_0) = u(z_0) .$$

Thus we can apply Ahlfors' lemma to obtain $u(z) \leqq 1$ for $z \in D$. This proves (3.1), and (3.2) is an immediate consequence.

Example 1. We consider the Weierstrass p-function that satisfies

$$p'(z)^2 = 4\left(p(z)^3 - \frac{1}{3\sqrt{3}}\right)$$

and define

$$f(z) = p\left(\frac{1+z}{1-z}\right).$$

By choosing a sequence (z_k) with $z_k \to 1$, $f(z_k) = 0$ we immediately obtain from the differential equation that $f(z)$ is not normal. Here $\sup_{|z|<1} \delta(z) = 2 \arctan \frac{1}{\sqrt{2}} \approx 71^\circ$.

We call the meromorphic function $f(z)$ *locally univalent* if there are no multiple poles and if $f'(z) \neq 0$ for $|z| < 1$.

Theorem 2. *Let $f(z)$ be meromorphic and locally univalent in D, and let*

$$\delta(z) \leqq \beta < \frac{\pi}{2} \quad (z \in D).$$

Then

$$\frac{(1-|z|^2)\,|f'(z)|}{1+|f(z)|^2} \leqq \frac{2d^*(z)\,(\lambda - \log d^*(z))}{1+d^*(z)^2} \quad (z \in D)$$

where $\lambda = 1 / \cos\beta + \log \tan \beta/2$. Hence $f(z)$ is normal of order

$$\alpha \leqq \tan\beta < \infty .$$

Proof. Because $d^*(z) \neq 0$ we can define

$$u(z) = \frac{(1-|z|^2)\,|f'(z)|}{1+|f(z)|^2}\,\frac{1+d^*(z)^2}{2d^*(z)\,(\lambda - \log d^*(z))},$$

(3.6) $\quad \varphi(z) = [\log w(z) - \log w(z_0)] \,/\, [2\,\lambda - \log \overline{w(z_0)} - \log w(z)]$.

Then

$$v(z) = \frac{(1-|z|^2)\,|f'(z)|}{1+|f(z)|^2}\,\frac{1+|w(z)|^2}{2|w(z)|\,(\lambda - \log|w(z)|)}.$$

Since $(1+t^2)\,/\,t\,(\lambda - \log t)$ decreases for $0 < t < \tau = \tan\,\beta\,/\,2$ the assertion follows as in the proof of Theorem 1.

Example 2. The function

$$f(z) = \exp\left(i\,\frac{1+z}{1-z}\right)$$

is not normal. Yet $\sup_{|z|<1} \delta(z) = \frac{\pi}{2}$. Hence the number $\pi\,/\,2$ in Theorem 2 cannot be replaced by a smaller number.

4. The size of schlicht disks

Theorem 3. *Let* $f(z)$ *be normal of order* α . *Then*

$$d^*(z) \leqq \frac{(1-|z|^2)\,|f'(z)|}{1+|f(z)|^2} \leqq 2\,\frac{e^{\pi/2\alpha}+1}{e^{\pi/2\alpha}-1}\sqrt{d^*(z)} \quad (z \in D)\,.$$

The left-hand inequality holds for any meromorphic function. This inequality is essentially due to Seidel and Walsh [10]. It is best possible as $f(z) = \alpha z$ shows.

Proof. Let

$$g(z) = \frac{f(\varphi(z)) - f(z_0)}{1 + \overline{f(z_0)}\, f(\varphi(z))}\,, \quad \varphi(z) = \frac{z + \bar z_0}{1 + \bar z_0\, z}\,.$$

Then $g(z)$ is normal in D with $\alpha_f = \alpha_g$, and satisfies $g(0) = 0$,

$$\frac{(1-|z_0|^2)\,|f'(z_0)|}{1+|f(z_0)|^2} = |g'(0)|\,,\ \delta_g(0) = \delta_f(z_0)\,.$$

Then $d_g^*(0) = \tan \delta_g(0) / 2$ becomes the radius $d(0)$ of the largest schlicht disk around 0 on the Riemann image surface $g(D)$, in the plane metric. Thus we have to prove that

$$\frac{r^2}{4}\,|g'(0)|^2 \leqq d(0) \leqq |g'(0)| \tag{4.1}$$

where

$$r = (e^{\pi/2\alpha} - 1) / (e^{\pi/2\alpha} + 1) = \tanh \frac{\pi}{4\alpha}\,.$$

The right-hand inequality follows easily from Schwarz' lemma, even for an arbitrary meromorphic function [10, p. 133].

Let $|z| < 1$ and $S = [0\,, z]$. Then

$$\arctan\ |f(z)| = \int_0^{|f(z)|} \frac{dt}{1+t^2} \leqq \int_{f(S)} \frac{|dw|}{1+|w|^2}$$

$$= \int_S \frac{|g'(\zeta)|}{1+|g(\zeta)|^2}\,|d\zeta| \leqq \alpha \int_0^{|z|} \frac{d\varrho}{1-\varrho^2} = \frac{\alpha}{2}\log\frac{1+|z|}{1-|z|}\,.$$

It follows that

$$|g(z)| < 1 \text{ for } |z| < r\,. \tag{4.2}$$

Therefore the function $h(\zeta) = g(r\zeta) = rg'(0)\zeta + \ldots$ is analytic in $|\zeta| < 1$ and satisfies $|h(\zeta)| < 1$. Hence $g(\zeta)$ maps a certain neighborhood of 0 one-to-one onto a disk around 0 of radius at least $r^2|g'(0)|^2/4$. Hence $d(0) \geqq r^2|g'(0)|^2/4$, and (4.1) is proved.

Remark. The proof shows that any normal function is univalent in the disk

$$|z| < \frac{1}{4} \tanh^2 \frac{\pi}{4\alpha} \cdot \frac{|f'(0)|}{1 + |f(0)|^2}.$$

5. Normal analytic functions

We shall now study normal functions without poles.

Lemma. *Let $f(z)$ be analytic in $|z| < 1$ and let*

$$(5.1) \qquad (1 - |z|^2)\,|f'(z)| \leqq M \quad \textit{whenever} \quad |f(z)| \leqq 1 .$$

Then

$$(5.2) \qquad (1 - |z|^2)\,|f'(z)| \leqq |f(z)|\,(2 \log |f(z)| + M) \quad \textit{whenever} \quad |f(z)| \geqq 1 .$$

Proof. We shall apply Ahlfors' lemma to the function

$$u(z) = \begin{cases} \dfrac{(1 - |z|^2)\,|f'(z)|}{|f(z)|\,(2 \log |f(z)| + M)} & \text{if } |f(z)| \geqq 1 \\ \dfrac{1}{M}(1 - |z|^2)\,|f'(z)| & \text{if } |f(z)| \leqq 1 . \end{cases}$$

This function is continuous in $|z| < 1$.

Let $|z_0| < 1$. Suppose that $u(z_0) > 1$. Then (5.1) implies that $|f(z_0)| > 1$. We put (compare (3.6))

$$\varphi(z) = \frac{\log f(z) - b}{M + \bar{b} + \log f(z)}, \; b = \log f(z_0)$$

where $\operatorname{Re} b > 0$. Then $\varphi(z)$ is analytic near z_0. Computation shows that $v(z) = u(z)$ for small $|z - z_0|$. Hence Ahlfors' lemma implies that $u(z) \leqq 1$ for $z \in D$, and (5.2) follows.

Theorem 4. *Let $f(z)$ be analytic and normal in D of order α. Then*

$$(5.3) \quad (1 - |z|^2)|f'(z)| \leqq 2\,(\log^+ |f(z)| + \alpha) \max(|f(z)|, 1) \quad (z \in D) .$$

Hence

$$(5.4) \qquad \log^+ |f(z)| \leqq \frac{1 + |z|}{1 - |z|} \log^+ |f(0)| + \frac{2\alpha|z|}{1 - |z|} \quad (z \in D) .$$

These results were proved by Hayman [4, Theorem 2] [5, Section 6.5] without the explicit dependence on α. The factor 2 is best possible.

Proof. If follows from (1.1) that $(1 - |z|^2)|f'(z)| \leqq 2\alpha$ whenever $|f(z)| \leqq 1$. Hence we can apply the lemma with $M = 2\alpha$. We immediately obtain (5.3). It follows that, for each ϑ,

$$\frac{\partial}{\partial r} \log[\log^+|f(re^{i\vartheta})| + \alpha] \leqq \frac{2}{1 - r^2}$$

for all except a countable number of values r. This inequality implies (5.4).

6. Landau's and Schottky's theorem

Theorem 5. *Let $f(z)$ be analytic in D and $f(z) \neq 0, 1$. Then $f(z)$ is normal of order $\alpha \leqq 4\sqrt{2}$. Hence, for $|z| < 1$,*

$$(1 - |z|^2)|f'(z)| \leqq 2\,|f(z)|\,(|\log|f(z)|| + 4\sqrt{2}), \tag{6.1}$$

$$\log^+|f(z)| \leqq \frac{1 + |z|}{1 - |z|} \log^+|f(0)| + \frac{8\sqrt{2}|z|}{1 - |z|}. \tag{6.2}$$

Explicit bounds in Schottky's theorem were obtained by Landau, Valiron, Ostrowski, Pfluger, Ahlfors [1], Hayman, and Jenkins. Hayman [3] showed that

$$\log^+|f(z)| \leqq \frac{1 + |z|}{1 - |z|} (\log^+|f(0)| + \pi)$$

where π cannot be replaced by a smaller constant. Our inequality (6.2) is not quite as good except for small $|z|$.

Inequality (6.1) is equivalent to the estimate

$$|a_1| \leqq 2|a_0|\,(|\log|a_0|| + 4\sqrt{2}) \quad (4\sqrt{2} < 5.66).$$

Jenkins [6] proved a slightly weaker estimate with 5.94 instead of 5.66. The best known upper bound 4.76 is due to Lai [11]. It is not possible to replace 5.66 by 4.37.

Proof. Let $g(z)$ map D onto the universal covering surface of the plane punctured in 0, 1, ∞ such that $g(0) = f(0)$. Then $f(z)$ is subordinate to $g(z)$, that is, there exists a function $\varphi(z)$ analytic in D with $|\varphi(z)| \leqq |z|$ such that $f(z) = g(\varphi(z))$. Hence

$$\begin{aligned}(1 - |z|^2)|f'(z)| &= \frac{(1 - |z|^2)\,|\varphi'(z)|}{1 - |\varphi(z)|^2} \cdot (1 - |\varphi(z)|^2)\,|g'(\varphi(z))| \\ &\leqq (1 - |\varphi(z)|^2)\,|g'(\varphi(z))|.\end{aligned}$$

Therefore the orders satisfy $\alpha_f \leqq \alpha_g$, and it is sufficient to prove (6.1) and (6.2) for $g(z)$.

The function $g(z)$ is locally univalent. The function $h(z) = g(z)^{1/4}$ is also analytic and locally univalent in D, and $h(z) \neq 0, \infty, \pm 1, \pm i$. It follows from elementary geometric considerations that $\delta_h(z) \leqq$ arc sin $\frac{1}{3}\sqrt{6}$. Hence Theorem 2 shows that $\alpha_h \leqq \sqrt{2}$. Therefore

$$\frac{(1-|z|^2)\,|g'(z)|}{1+|g(z)|^2} = \frac{4\,(1-|z|^2)\,|h'(z)|\,|h(z)|^3}{1+|h(z)|^8} \leqq 4\,\frac{(1-|z|^2)\,|h'(z)|}{1+|h(z)|^2}\,.$$

Consequently $\alpha_g \leqq 4\alpha_h \leqq 4\sqrt{2}$. We obtain (6.1) and (6.2) from Theorem 4. For $|f(z)| \leqq 1$, we apply Theorem 4 to $1/f(z)$. If we change the proof slightly we do not have to assume that the twice-punctured plane has a covering surface of hyperbolic type.

Technische Universität Berlin

References

1. Ahlfors, L. V. An extension of Schwarz's Lemma - Trans. Amer. Math. Soc. *43* (1938), 359–364.
2. Bagemihl, F. and Seidel, W. Koebe arcs and Fatou points of normal functions - Comm. Math. Helv. *36* (1961), 9–18.
3. Hayman, W. K. Some remarks on Schottky's theorem - Proc. Cambridge Philos. Soc. *43* (1947), 442–454.
4. —»— Uniformly normal families, Lectures on functions of a complex variable - Univ. of Michigan Press, Ann Arbor 1955, 199–212.
5. —»— Meromorphic functions - Oxford Mathematical Monographs, Oxford 1964.
6. Jenkins, J. A. On explicit bounds in Landau's theorem - Canad. J. Math. *8* (1956), 423–425.
7. Lehto, O. and Virtanen, K. I. Boundary behaviour and normal meromorphic functions - Acta Math. *97* (1957), 47–65.
8. MacLane, G. R. Asymptotic values of holomorphic functions - Rice Univ. Studies *49* no. 1 (1963).
9. Pommerenke, Ch. On Bloch functions - J. London Math. Soc. (2) *2* (1970), 689–695.
10. Seidel, W. and Walsh, J. L. On the derivatives of functions analytic in the unit circle and their radii of univalence and p-valence - Trans. Amer. Math. Soc. *52* (1942), 128–216.
11. Lai, W. T. Über den Satz von Landau - Sci. Record N. S. *4* (1960), 339–342.

Printed November 1970