ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

448

DEFINITIONS FOR QUASIREGULAR MAPPINGS

BY

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HELSINKI 1969
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doi:10.5186/aasfm.1969.448

Communicated 9 May 1969 by Olli Lehto and K. I. Virtanen

KESKUSKIRJAPAINO
HELSINKI 1969

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1. Introduction

The theory of quasiconformal mappings in the euclidean n-space has been quite extensively studied during the last ten years. Usually, a quasiconformal mapping is assumed to be a homeomorphism. For $n = 2$, there exists also a theory of non-homeomorphic quasiconformal mappings. See Lehto—Virtanen [13, Kapitel VI] and Künzi [12, Kapitel 5]. These mappings are often called quasiconformal functions (not mappings). Some authors call them pseudoanalytic functions, but this term has been used by Bers in a different sense. We prefer the word *quasiregular*, and do not make any distinction between the words »mapping» and »function». It is fairly easy to generalize several function-theoretic results for 2-dimensional quasiregular mappings, thanks to the following factorization theorem: Every 2-dimensional quasiregular mapping can be represented in the form $g \circ h$ where h is a quasiconformal homeomorphism and g is a complex analytic function.

Higher dimensional quasiregular mappings, under the name »mappings with bounded distortion», have been considered by Rešetnjak since 1966 in a series of important papers [15, 16, 17, 18, 19, 20]. (See also Callender [2].) He uses an analytic definition which will be given in 2.20. He also hints at a geometric definition in [16, p. 629]. The purpose of this paper is to give several equivalent characterizations for quasiregularity. These are based on the linear dilatations and on the capacity of a condenser. The last concept, defined in Section 5, is a generalization of the modulus of a ring domain. Some of these results are new also for $n = 2$. In Section 8, we give some applications. For example, we show that the branch set of a nonconstant quasiregular mapping has measure zero.

2. Preliminary results

2.1. *Notation and terminology.* The real number system is denoted by R^1 and its two-point compactification $R^1 \cup \{\infty, -\infty\}$ by $\dot{R}^1$. We let R^n, $n \geqq 2$, denote the euclidean n-space, and R^{n-1} will be identified with the subspace $x_n = 0$ of R^n. For $x \in R^n$ we write $x = x_1 e_1 + \cdots + x_n e_n$ where $e_1, \ldots, e_n$ are the coordinate unit vectors of R^n. For each set

$A \subset R^n$ we let $\complement A$, $\bar{A}$, ∂A, and $\operatorname{int} A$ denote the complement, closure, boundary, and interior of A, all taken with respect to R^n. Furthermore, $d(A)$ is the diameter of A. Given two sets A and B in R^n, $d(A, B)$ is the distance between A and B, $A \setminus B$ is the set-theoretic difference of A and B, and $A + B$ is the set of all points $a + b$ such that $a \in A$ and $b \in B$. Given $x \in R^n$ and $r > 0$, we let $B^n(x, r)$ denote the open ball $\{y \in R^n \mid |y-x| < r\}$, and $S^{n-1}(x, r)$ the sphere $\partial B^n(x, r)$. We shall also employ the abbreviations

$$B^n(r) = B^n(0, r), \ B^n = B^n(1), \ S^{n-1}(r) = S^{n-1}(0, r), \ S^{n-1} = S^{n-1}(1).$$

The Lebesgue measure of a set $A \subset R^n$ will be written as $m_n(A)$, or simply as $m(A)$ if there is no danger of misunderstanding. $m_n(A)$ is also defined for sets in n-dimensional spheres and linear submanifolds of $R^{n'}$, $n' > n$. The Lebesgue integral of a function f over a set $A \subset R^n$ is written as

$$\int_A f \, dm_n \quad \text{or} \quad \int_A f(x) \, dm_n(x),$$

where the subscript n may again be omitted. We set $\Omega_n = m_n(B^n)$ and $\omega_{n-1} = n\Omega_n = m_{n-1}(S^{n-1})$. The linear measure $\Lambda_1(A)$ of a set $A \subset R^n$ is defined as follows: For $t > 0$ let

$$\Lambda_1^t(A) = \inf \sum_{i=1}^{\infty} d(A_i)$$

over all countable coverings $\{A_1, A_2, \ldots\}$ of A such that $d(A_i) < t$. Then

$$\Lambda_1(A) = \lim_{t \to 0} \Lambda_1^t(A) = \sup_{t > 0} \Lambda_1^t(A).$$

If $A \subset R^n$ is a Borel set, $\operatorname{Bor} A$ denotes the class of all Borel subsets of A.

A neighborhood of a point x or a set A is an open set containing x or A. A domain is an open connected non-empty set. The notation $f: G \to R^n$ includes the assumptions that G is a domain in R^n and that f is continuous. If $f: G \to R^n$, $A \subset G$ and $y \in R^n$, we let $N(y, f, A)$ be the number (possibly infinite) of points in $A \cap f^{-1}(y)$. We set $N(f, A) = \sup N(y, f, A)$ over $y \in R^n$.

Given a domain $G \subset R^n$, we let $J(G)$ denote the family of all domains D such that $\bar{D}$ is a compact subset of G. If $f: G \to R^n$, $D \in J(G)$, and $y \in \complement f \partial D$, then $\mu(y, f, D)$ is the degree (topological index) of the triple (y, f, D) [14, p. 125]. Suppose that $x \in G$ has a connected neighborhood $D \in J(G)$ such that $\bar{D} \cap f^{-1}(f(x)) = \{x\}$. Then $\mu(f(x), f, D)$ is independent of the choice of D, and is denoted by $i(x, f)$.

The i^{th} partial derivative of a mapping $f: G \to R^n$ is denoted by $\partial_i f$. If all partial derivatives of f exist at a point $x \in G$, the formal derivative of f at x is the linear mapping $f'(x): R^n \to R^n$, defined by $f'(x)e_i = \partial_i f(x)$, $1 \leqq i \leqq n$. If f is differentiable at x, then $f'(x)$ is the derivative of f, that is,

$$f(x + h) = f(x) + f'(x)h + |h|\varepsilon(x, h)$$

where $\varepsilon(x, h) \to 0$ as $h \to 0$. The Jacobian of f at x ls denoted by $J(x, f)$.

If $T: R^n \to R^n$ is a linear mapping, we set

$$|T| = \max_{|x|=1} |Tx|, \quad l(T) = \min_{|x|=1} |Tx|.$$

If U is an open set in R^n, we let $C^p(U)$ denote the class of all p times continuously differentiable functions $u: U \to R^1$, and $C_0^p(U)$ the class of all $u \in C^p(U)$ whose support spt u is a compact subset of U. A function in $C_0^p(U)$ will be identified in a natural way with a function in $C_0^p(R^n)$, which vanishes in $\complement U$.

Suppose that $f: G \to R^n$. The branch set B_f of f is the set of all points of G at which f fails to be a local homeomorphism. f is open if the image of every open set in G is open in R^n. f is light if for every $y \in R^n$, $f^{-1}(y)$ is totally disconnected. f is discrete if for every $y \in R^n$, $f^{-1}(y)$ is discrete, that is, consists of isolated points. f is sense-preserving if $\mu(y, f, D) > 0$ whenever $D \in J(G)$ and $y \in fD \setminus f\partial D$. f is sense-reversing if $\mu(y, f, D) < 0$ for all such triples (y, f, D). f satisfies the condition (N) if the image of every set of measure zero has measure zero.

A continuum is a compact connected non-empty set. A ring is a domain $A \subset R^n$ such that $\hat{R}^n \setminus A$ has exactly two components, where $\hat{R}^n$ is the one point compactification of R^n.

2.2. *Quasiadditive set functions.* Let U be an open set in R^n. A mapping φ : Bor $U \to \dot{R}^1$ is said to be a *q-quasiadditive* set function, $q \geqq 1$, if the following conditions are satisfied for all Borel sets in U:

(1) $\varphi(A) \geqq 0$.
(2) $A \subset B$ implies $\varphi(A) \leqq \varphi(B)$.
(3) $\varphi(A) < \infty$ if A is compact.
(4) If $A_1, \ldots, A_k$ are disjoint and if $A_i \subset A$, then

$$\sum_{i=1}^{k} \varphi(A_i) \leqq q\varphi(A).$$

From (4) it follows that the same inequality is true for an infinite sequence of disjoint Borel sets $A_1, A_2, \ldots$.

The *upper* and *lower derivatives* of a q-quasiadditive set function φ at a point $x \in U$ are defined as follows:

$$\overline{\varphi}'(x) = \lim_{h \to 0} \sup_{d(Q) < h} \frac{\varphi(Q)}{m(Q)} ,$$

$$\underline{\varphi}'(x) = \lim_{h \to 0} \inf_{d(Q) < h} \frac{\varphi(Q)}{m(Q)} ,$$

where Q runs through all open cubes and open balls such that $x \in Q \subset U$.

2.3. **Lemma.** *Suppose that φ is a q-quasiadditive set function in an open set U . Then*

(1) *$\overline{\varphi}'$ and $\underline{\varphi}'$ are Borel functions.*

(2) $\overline{\varphi}'(x) \leqq q\underline{\varphi}'(x) < \infty$ *a.e.*

(3) *For each open set $V \subset U$,*

$$\int_V \underline{\varphi}' \, dm \leqq q\varphi(V) .$$

The proof for $q = 1$ is given in [14, pp. 204—209]. The proof for the general case is completely analogous. The definition of [14] for the derivatives $\overline{\varphi}'$, $\underline{\varphi}'$ is slightly different from ours, because the sets Q are in [14] assumed to be cubes. However, this makes no difference in the proof of 2.3, since the Vitali covering theorem holds for the family of all closed cubes and closed balls.

2.4. *Normal domains.* Given a mapping $f : G \to R^n$, a domain $D \in J(G)$ is said to be a *normal domain* of f if $f\partial D = \partial f D$. A *normal neighborhood* of a point $x \in G$ is a normal domain D such that $D \cap f^{-1}(f(x)) = \{x\}$.

We shall use the concept of a normal domain only for open mappings. In this case, we have always $\partial f D \subset f \partial D$, and the condition $f\partial D \subset \partial f D$ means that f defines a closed mapping $D \to fD$. The definition of Whyburn [28] for a normal domain is slightly more restrictive since he also demands that f defines an open mapping $\bar{D} \to f\bar{D}$. It is not difficult to show that, for discrete open mappings, the domains $U(x, f, r)$, which will be defined in 2.8 and used throughout the paper, satisfy this additional condition, but we shall not use this fact.

If D is a normal domain of an open mapping f , then $fD \cap f\partial D = \emptyset$. Hence, $\mu(y, f, D)$ is constant for $y \in fD$. This constant will be denoted by $\mu(f, D)$.

2.5. **Lemma.** *Suppose that* $f : G \to R^n$ *is open, that* $U \subset R^n$ *is a domain, and that* D *is a component of* $f^{-1}U$ *such that* $D \in J(G)$. *Then* D *is a normal domain,* $fD = U$, *and* $U \in J(fG)$.

Proof. Since f is open, $\partial fD \subset f\partial D$. Next assume $y \in f\partial D$. Then $y = f(x)$ for some $x \in \partial D$. Now $x \notin f^{-1}U$, since otherwise D would not be a component of $f^{-1}U$. Thus, $y \notin U \supset fD$. Hence $y \in f\bar{D} \setminus fD = \overline{fD} \setminus fD = \partial fD$. Consequently, D is a normal domain. Furthermore, $f\partial D \cap U = \emptyset$. Thus $fD = U \cap f\bar{D}$ is both closed and open in U , whence $fD = U$. Finally, $\bar{U} = f\bar{D}$ is a compact subset of fG , i.e., $U \in J(fG)$.

2.6. **Lemma.** *Suppose that* $f : G \to R^n$ *is open and that* D *is a normal domain of* f . *If* E *is either a domain or a continuum in* fD , *then* f *maps every component of* $D \cap f^{-1}E$ *onto* E . *Furthermore, if* F *is a compact subset of* fD , *then* $D \cap f^{-1}F$ *is compact.*

Proof. The case where E is a domain follows from 2.5. If $E \subset fD$ is compact, then $D \cap f^{-1}E = \bar{D} \cap f^{-1}E$ is compact. Moreover, f defines an open mapping $D \cap f^{-1}E \to E$ ([27, (7.2), p. 147]). If E is a continuum, every component of $D \cap f^{-1}E$ is mapped onto E by [27, (7.5), p. 148].

2.7 **Lemma.** (*Path lifting*). *Suppose that* $f : G \to R^n$ *is light and open and that* $D \subset G$ *is a normal domain. Suppose also that* $\beta : [a , b] \to fD$ *is a path, that* $a \leqq t_0 \leqq b$, *and that* $x_0 \in D$ *such that* $f(x_0) = \beta(t_0)$. *Then there is a path* $\alpha : [a , b] \to D$ *such that* $\alpha(t_0) = x_0$ *and* $f \circ \alpha = \beta$.

Proof. Considering the restrictions of β to $[a , t_0]$ and $[t_0 , b]$ separately, we may assume that t_0 is an end point, say $t_0 = a$. Set $I = [a , b]$, $J' = \beta I$, and $J = D \cap f^{-1}J'$. Then J and J' are compact, and f defines an open mapping $J \to J'$. Define $g : J \times I \to J' \times I$ by $g(x , t) = (f(x) , t)$ and $\beta_1 : I \to J' \times I$ by $\beta_1(t) = (\beta(t) , t)$. Then g is a light open mapping, and $\beta_1 I$ is an arc. By a result of Whyburn [27, (2.1), p. 186] there is an arc $J_0 \subset J \times I$ such that $(x_0 , a) \in J$ and such that g maps J_0 homeomorphically onto $\beta_1 I$. Set $\alpha = P \circ (g|J_0)^{-1} \circ \beta_1$ where $P : J \times I \to J$ is the projection. Then $\alpha(t_0) = x_0$ and $f \circ \alpha = \beta$.

2.8. *Notation.* If $f : G \to R^n$, $x \in G$, and $r > 0$, then the x-component of $f^{-1}B^n(f(x) , r)$ is denoted by $U(x , f , r)$.

2.9. **Lemma.** *Suppose that* $f : G \to R^n$ *is discrete and open. Then* $\lim_{r \to 0} d(U(x , f , r)) = 0$ *for every* $x \in G$. *If* $U(x , f , r) \in J(G)$, *then* $U(x , f , r)$

is a normal domain and $fU(x, f, r) = B^n(f(x), r) \in J(f(G))$. *Furthermore, for every point* $x \in G$ *there is a positive number* σ_x *such that the following conditions are satisifed for* $0 < r \leqq \sigma_x$:

(1) $U(x, f, r)$ *is a normal neighbourhood of* x.

(2) $U(x, f, r) = U(x, f, \sigma_x) \cap f^{-1}B^n(f(x), r)$.

(3) $\partial U(x, f, r) = U(x, f, \sigma_x) \cap f^{-1}S^{n-1}(f(x), r)$ *if* $r < \sigma_x$.

(4) $\mathbf{C}U(x, f, r)$ *is connected.*

(5) $\mathbf{C}\bar{U}(x, f, r)$ *is connected.*

(6) *If* $0 < r < s \leqq \sigma_x$, *then* $\bar{U}(x, f, r) \subset U(x, f, s)$, *and* $U(x, f, s) \setminus \bar{U}(x, f, r)$ *is a ring.*

Proof. Given $x \in G$ and $\varepsilon > 0$, choose a neighborhood W of x such that $W \in J(G)$, $d(W) < \varepsilon$, and $\bar{W} \cap f^{-1}(f(x)) = \{x\}$. Then $U(x, f, r) \subset W$ for $0 < r < d(f(x), f\partial W)$. Hence $d(U(x, f, r)) \to 0$ as $r \to 0$. If $U(x, f, r) \in J(G)$, it follows from 2.5 that $U(x, f, r)$ is a normal domain and that $fU(x, f, r) = B^n(f(x), r) \in J(fG)$.

The first part of the proof implies that x has a normal neighborhood D. We choose σ_x such that $0 < \sigma_x < d(f(x), f\partial D)$ and such that $U(x, f, \sigma_x) \subset B^n(x, t) \subset D$ for some $t > 0$. To verify the properties (1)—(6) we may assume that $0 < r < \sigma_x$. Set $U = U(x, f, r)$, $U_0 = U(x, f, \sigma_x)$, and $V = fU = B^n(f(x), r)$. The condition (1) is clear by what was proved above. Since $U_0 \cap f^{-1}(f(x)) = \{x\}$, (2) follows from 2.6. Suppose next that $z \in U_0 \cap f^{-1}\partial V$. Since f is open, every neighborhood of z meets $f^{-1}V$. By (2), this implies $z \in \bar{U}$. Thus $z \in \partial U$, whence $U_0 \cap f^{-1}\partial V \subset \partial U$. On the other hand, $f\partial U = \partial fU = \partial V$ implies $\partial U \subset U_0 \cap f^{-1}\partial V$, and (3) is proved.

Since $U \subset B^n(x, t) \subset D$, there is exactly one component E of $\mathbf{C}U$ which meets $\mathbf{C}D$. We show that $E = \mathbf{C}U$. Set $F = \mathbf{C}U \setminus E$. Since $D \cap f^{-1}(f(x)) = \{x\}$, it follows from 2.6 that $U = D \cap f^{-1}V$. Hence fF does not meet V. Since f is open, $\partial fF \subset f\partial F \subset f\partial U = \partial V$. Since fF is bounded, $fF \subset \partial V$. Since f is open, $\operatorname{int} F = \emptyset$. Setting $U_1 = \mathbf{C}E$ we thus have $\bar{U}_1 = \bar{U}$. Hence $fU_1 \subset \operatorname{int} f\bar{U} = V$, which implies $fF \subset V$. This proves $F = \emptyset$. Thus, $\mathbf{C}U = E$ is connected.

If $x, y \in \mathbf{C}\bar{U}$, there is r_1 such that $r < r_1 < \sigma_x$ and such that $x, y \in \mathbf{C}U(x, f, r_1) \subset \mathbf{C}\bar{U}$. Hence (5) follows from (4).

The relation $\bar{U}(x, f, r) \subset U(x, f, s)$ follows from the last statement of 2.6. By (4), the components of the complement of $A = U(x, f, s) \setminus \bar{U}(x, f, r)$ are $\bar{U}(x, f, r)$ and $\mathbf{C}U(x, f, s)$. By (5) and the Phragmén-Brouwer theorem [11, p. 359], A is connected. Thus A is a ring.

2.10. **Corollary.** *If* $f: G \to R^n$ *is discrete and open, then every point in* G *has arbitrarily small normal neighborhoods.*

2.11. **Lemma.** *If* $f : G \to R^n$ *is discrete and open,* $\dim B_f \leqq n-2$.

This important result was proved by Černavskii [3, 4]. Another proof is given in [25]. It implies that $G \setminus B_f$ is connected. Hence $i(x, f)$ has a constant value, either $+1$ or -1 , in $G \setminus B_f$. In the first case f is sense-preserving, and in the second case sense-reversing. For convenience, we shall restrict ourselves in this paper to sense-preserving mappings. This is obviously an unessential restriction.

2.12. **Lemma.** *Suppose that* $f : G \to R^n$ *is sense-preserving, discrete, and open.*

(1) *If* $D \in J(G)$, *then* $N(y, f, D) \leqq \mu(y, f, D)$ *for all* $y \in \complement f\partial D$, *and* $N(y, f, D) = \mu(y, f, D)$ *for* $y \in \complement f(\partial D \cup (D \cap B_f))$.
(2) *If* D *is a normal domain,* $N(f, D) = \mu(f, D)$.
(3) *If* $A \subset G$ *is compact,* $N(f, A) < \infty$.
(4) *Every point* $x \in G$ *has a neighborhood* V *such that if* U *is a neighborhood of* x *and if* $U \subset V$, *then* $N(f, U) = i(x, f)$.
(5) $x \in B_f$ *if and only if* $i(x, f) \geqq 2$.

Proof. (1) Let $y \in \complement f\partial D$ and let $D \cap f^{-1}(y) = \{x_1, \ldots, x_k\}$. Then

$$\mu(y, f, D) = \sum_{j=1}^{k} i(x_j, f) .$$

Since f is sense-preserving, $i(x_j, f) \geqq 1$. Thus $\mu(y, f, D) \geqq k = N(y, f, D)$. If $y \in \complement f(\partial D \cup (D \cap B_f))$, every $i(x_j, f) = 1$ in the above sum, and we have $\mu(y, f, D) = N(y, f, D)$.

(2) By [5, 2.2], $\dim fB_f \leqq n-2 < n$. Hence there is a point $y \in fD \setminus fB_f$. By (1), $\mu(f, D) = \mu(y, f, D) = N(y, f, D) \leqq N(f, D)$. On the other hand, (1) implies that $N(z, f, D) \leqq \mu(f, D)$ for all $z \in \complement f\partial D$. Hence, $\mu(f, D) = N(f, D)$.

(3) By 2.9, A can be covered by a finite number of normal domains $D_1, \ldots, D_k$. Using (2) we obtain

$$N(f, A) \leqq \sum_{i=1}^{k} N(f, D_i) = \sum_{i=1}^{k} \mu(f, D_i) < \infty .$$

(4) By 2.9, x has a normal neighborhood V . If $U \subset V$ is a neighborhood of x , there is a normal neighborhood V_1 of x such that $V_1 \subset U$. Then (2) implies $i(x, f) = N(f, V_1) \leqq N(f, U) \leqq N(f, V) = i(x, f)$.

(5) follows from (4).

2.13. *Remark.* Since a light sense-preserving mapping is discrete and open [22, p. 333], we could replace the words »sense-preserving, discrete, and open» by »sense-preserving and light» throughout the paper. However, we shall not do this, because it is essential that our mappings are discrete and open, while sense-preservation is assumed mainly for the sake of convenience.

2.14. **Lemma.** *Suppose that* $f : G \to R^n$ *is sense-preserving, discrete, and open, and that* f *is differentiable at* $x_0 \in G$. *Then* $J(x_0, f) \geqq 0$. *If* $x_0 \in B_f$, $J(x_0, f) = 0$. *If* A *is a Borel set in* G *and if* f *is differentiable a.e. in* A , *then*

$$\int_A J(x, f)\, dm(x) \leqq \int_{R^n} N(y, f, A)\, dm(y) . \tag{2.15}$$

Proof. If $J(x_0, f) \neq 0$, then $i(x_0, f) = \operatorname{sgn} J(x_0, f)$ by [14, (68), p. 332]. Since f is sense-preserving, $J(x_0, f) \geqq 0$. By 2.12, $i(x, f) \geqq 2$ for $x \in B_f$. Thus $J(x_0, f) = 0$ if $x_0 \in B_f$. The inequality (2.15) can be derived from general integral inequalities (see [14, p. 260]), but it can also be proved directly as follows. We express $A \setminus B_f$ as a union of disjoint Borel sets $A_1, A_2, \ldots$ such that each A_i is contained in a domain $D_i \subset D$ in which f is injective. Since (2.15) is well known to be true for homeomorphisms and since $J(x, f) = 0$ a.e. in $A \cap B_f$, we obtain

$$\int_A J(x, f)\, dm(x) = \int_{A \setminus B_f} J(x, f)\, dm(x) = \sum \int_{A_i} J(x, f)\, dm(x)$$

$$\leqq \sum \int_{R^n} N(y, f, A_i) dm(y) = \int_{R^n} \sum N(y, f, A_i)\, dm(y) \leqq \int_{R^n} N(y, f, A)\, dm(y) .$$

2.16. *ACL-mappings.* Let R_i^{n-1} be the subspace of R^n defined by $x_i = 0$, and let $P_i : R^n \to R_i^{n-1}$ be the orthogonal projection. Suppose that U is an open set in R^n. A mapping $g : U \to R^m$ is said to be ACL if g is continuous and if for each open n-interval Q such that $\bar{Q} \subset U$, g is absolutely continuous on almost every line segment in $\bar{Q}$, parallel to the coordinate axes. More precisely, if E_i is the set of all points $z \in P_i Q$ such that g is not absolutely continuous on $\bar{Q} \cap P_i^{-1}(z)$, then $m_{n-1}(E_i) = 0$, $1 \leqq i \leqq n$. An ACL-mapping has partial derivatives a.e. If these are locally L^p-integrable, $p \geqq 1, g$ is said to be ACL^p.

Suppose that $f : G \to R^n$ is a discrete open mapping and that $Q \in J(G)$ is an n-interval. For each Borel set $A \subset P_i Q$ we set $\varphi_i(A, Q) = m(f(Q \cap P_i^{-1} A))$. Then $A \mapsto \varphi_i(A, Q)$ is a q-quasiadditive set function, where $q = N(f, Q) < \infty$ by 2.12. Hence, by 2.3, its upper derivative $\bar{\varphi}_i'(z, Q)$ is finite for almost every $z \in P_i Q$.

2.17. **Lemma.** *Suppose that* $f: G \to R^n$ *is discrete and open and that for every domain* $D \in J(G)$ *there is a finite constant* C_D *such that*

$$(2.18) \qquad (\sum_{j=1}^{k} d(f\Delta_j))^n \leqq C_D \bar{\varphi}_i'(z, Q) (\sum_{j=1}^{k} m_1(\Delta_j))^{n-1}$$

whenever Q *is an open* n*-interval in* D, $1 \leqq i \leqq n$, $z \in P_i Q$, *and* $\Delta_1, \ldots, \Delta_k$ *are disjoint closed subintervals of* $Q \cap P_i^{-1}(z)$. *Then* f *is* ACL^n.

Proof. This lemma is a generalization of a result of Agard [1]. The following proof is essentially due to him.

Let $Q \in J(G)$ be an open n-interval. A simple limiting process shows that (2.18) is true whenever $\Delta_1, \ldots, \Delta_k$ are non-overlapping subintervals of $\bar{Q} \cap P_i^{-1}(z)$. Thus f is ACL. To prove that f is ACL^n, it suffices to show, by symmetry, that $|\partial_n f|^n$ is integrable over Q.

Choose an integer j_0 such that $0 < 1/j_0 < d(Q, \partial G)$ and set

$$g(x) = |\partial_n f(x)|, \; g_j(x) = \frac{j}{2} \int_{-1/j}^{1/j} g(x+te_n)\, dt .$$

Then $g_j(x)$ is defined for almost every $x \in Q$ and for all $j \geqq j_0$. We first show that $g_j \to g$ a.e. in Q. It is well known that g is measurable, in fact, g is a Borel function (Saks [21, p. 170]). Hence, the function $(x, t) \mapsto g(x+te_n)$ is measurable in $Q \times (-1/j, 1/j)$. By Fubini's theorem, this implies that g_j is measurable. Write $Q = Q_0 \times J$, where $Q_0 = P_n Q$ is an open $(n-1)$-interval and $J = (a, b)$ is an open 1-interval. Then almost every $z \in Q_0$ has the property that $t \mapsto f(z, t)$ is absolutely continuous for $t \in (a-1/j_0, b+1/j_0)$. For such z, Lebesgue's theorem implies that $g_j(z, t) \to g(z, t)$ for almost every $t \in J$. From Fubini's theorem it follows that $\liminf g_j(x) = g(x) = \limsup g_j(x)$ a.e. in Q. Thus $g_j \to g$ a.e. in Q.

Again by Fubini's theorem, almost every $u \in J$ has the property that $g_j(z, u) \to g(z, u)$ for almost every $z \in Q_0$. Consider such u, and set

$$F_j(E) = \varphi_n(E, Q_0 \times (u-1/j, u+1/j))$$

for all Borel sets $E \subset Q_0$ and for $j \geqq j_0$. Then the set functions F_j are q-quasiadditive for $q = N(f, D)$ where $D = Q_0 \times (a-1/j_0, b+1/j_0)$. If $\bar{F}_j'(z) < \infty$, it follows from (2.18) that the function $t \mapsto f(z, t)$ is absolutely continuous on $[u-1/j, u+1/j]$ and that its total variation is not greater than $(C_D \bar{F}_j'(z)(2/j)^{n-1})^{1/n}$. Consequently,

$$g_j(z, u)^n = \left(\frac{j}{2} \int_{u-1/j}^{u+1/j} |\partial_n f(z, t)|\, dt \right)^n \leqq C_D \bar{F}_j'(z) j/2 .$$

Integrating over $z \in Q_0$ and using 2.3 we obtain

(2.19)

$$\int_{Q_0} g_j(z\,,\,u)^n dm_{n-1}(z) \leqq \tfrac{1}{2}\, C_D q^2 j F_j(Q_0) = \tfrac{1}{2}\, C_D q^2 j m(f(Q_0 \times (u-1/j,\, u+1/j)))\,.$$

For each Borel set $E \subset J$ set $\psi(E) = m(f(Q_0 \times E))$. Then ψ is a q-quasiadditive set function in J. As $j \to \infty$, (2.19) implies by Fatou's lemma

$$\int_{Q_0} g(z\,,\,u)^n\, dm_{n-1}(z) \leqq C_D q^2 \bar{\psi}'(u)\,.$$

Integrating over $u \in J$ and using 2.3 we obtain

$$\int_{Q} g^n\, dm \leqq C_D q^4 \psi(J) = C_D q^4 m(fQ) < \infty\,.$$

Thus g^n is integrable over Q.

2.20. *Quasiregular mappings.* A mapping $f : G \to R^n$ is said to be *quasiregular* if f is ACL^n and if there exists a constant $K \geqq 1$ such that

$$|f'(x)|^n \leqq KJ(x\,,f) \tag{2.21}$$

a.e. in G. The smallest $K \geqq 1$ for which this inequality is true is called the *outer dilatation* of f and denoted by $K_O(f)$. If f is quasiregular, then the smallest $K \geqq 1$ for which the inequality

$$J(x\,,f) \leqq Kl(f'(x))^n \tag{2.22}$$

holds a.e. in G is called the *inner dilatation* of f and denoted by $K_I(f)$. The *maximal dilatation* of f is the number $K(f) = \max\,(K_I(f)\,,\,K_O(f))$. If $K(f) \leqq K$, f is said to be *K-quasiregular.* If f is not quasiregular, we set $K_O(f) = K_I(f) = K(f) = \infty$.

The above definition is a natural generalization of the analytic definition for quasiconformal mappings [24]. A sense-preserving mapping is K-quasiconformal if and only if it is a K-quasiregular homeomorphism.

It is not true that every ACL^n-mapping which satisfies (2.22) is quasiregular. For example, the projection $f(x) = x_1 e_1$ satisfies (2.22) with $K = 1$, since $J(x\,,f) = l(f'(x)) = 0$ everywhere.

The above definition has been used by Callender [2] and Rešetnjak [15] in a slightly different form. In their definition (2.21) is replaced by

$$\Big(\sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i f_j(x)^2\Big)^{n/2} \leqq n^{n/2} KJ(x\,,f)\,, \tag{2.23}$$

where K is sometimes replaced by K^n or by $K^{n/2}$. Let $K_R(f)$ be the smallest $K \geqq 1$ for which (2.23) holds a.e. Then it is easy to show that

$$K_R(f) \leqq K_O(f) \leqq n^{n/2} K_R(f) \tag{2.24}$$

and that these inequalities are the best possible. Furthermore, we have the inequalities

$$K_O(f) \leqq K_I(f)^{n-1}\,,\; K_I(f) \leqq K_O(f)^{n-1}\,, \tag{2.25}$$

which also are the best possible.

2.26. **Lemma.** *Let $f: G \to R^n$ be a quasiregular mapping. Then*
(1) *f is either constant or sense-preserving, discrete, and open.*
(2) *f is differentiable a.e.*
(3) *f satisfies the condition (N).*

These important results are due to Rešetnjak [16, 18].

2.27. **Lemma.** *Suppose that $f: G \to R^n$ is quasiregular. Then $f'(x) = 0$ a.e. in B_f. Moreover, $m(fB_f) = 0$.*

Proof. We may assume that f is not constant. By 2.14, $J(x, f) = 0$ a.e. in B_f. Hence $|f'(x)| = 0$ a.e. in B_f. From [14, Lemma 3, p. 360] it follows that

$$\int_{R^n} N(y, f, B_f)\, dm(y) = \int_{B_f} J(x, f)\, dm(x) = 0\,.$$

Hence $m(fB_f) = 0$.

2.28. **Lemma.** *Suppose that $f: G \to R^n$ is a non-constant quasiregular mapping and that every point in $G \setminus B_f$ has a neighborhood U such that $K_O(f \mid U) \leqq a$, $K_I(f \mid U) \leqq b$. Then $K_O(f) \leqq a$, $K_I(f) \leqq b$.*

Proof. By 2.27, $f'(x) = 0$ a.e. in B_f. Hence, the inequalities (2.21) and (2.22) are automatically satisfied in B_f.

3. Path families and quasiregular mappings

3.1. Suppose that A is a subset of R^n. By a *path* in A we mean a continuous mapping $\gamma: \Delta \to A$ where Δ is a closed interval in R^1. If Γ is a family of paths in R^n, we let $F(\Gamma)$ be the family of all non-negative Borel functions $\varrho: R^n \to \dot{R}^1$ such that

$$\int_\gamma \varrho \, ds \geqq 1$$

for every rectifiable $\gamma \in \Gamma$. The *modulus* of Γ is the number

$$M(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{R^n} \varrho^n \, dm .$$

Suppose that Γ is a family of paths in a domain G and that $f : G \to R^n$ is a mapping. Then the family $f\Gamma$ of all paths $f \circ \gamma$, $\gamma \in \Gamma$, is called the image of Γ under f . If $f : G \to G'$ is a K-quasiconformal mapping, it is well known [23] that

$$M(\Gamma)/K \leqq M(f\Gamma) \leqq KM(\Gamma)$$

for every path family Γ in G . We conjecture that the right hand inequality is true also for K-quasiregular mappings. As yet, we have been able to prove it only in special cases, for example, if $n = 2$ or if $B_f = \emptyset$. The left hand inequality need not be true for non-homeomorphic quasiregular mappings, as is seen from the following counterexample: Let Γ be the family of all horizontal segments which join the vertical lines $\mathrm{Re}\, z = 0$ and $\mathrm{Re}\, z = 1$ in the complex plane R^2. If $f : R^2 \to R^2$ is the analytic function $f(z) = e^z$, we have $M(\Gamma) = \infty$, $M(f\Gamma) = 2\pi$. However, we can establish the following inequality:

3.2. **Theorem.** *Suppose that $f : G \to R^n$ is a quasiregular mapping and that A is a Borel set in G such that $N(f , A) < \infty$. If Γ is a family of paths in A ,*

$$M(\Gamma) \leqq N(f , A) K_O(f) M(f\Gamma) .$$

Proof. Set

$$L(x , f) = \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|}$$

for $x \in G$. Thus $L(x , f) = |f'(x)|$ whenever f is differentiable at x . It is easy to see that $x \mapsto L(x , f)$ is a Borel function.

Suppose that $\varrho' \in F(f\Gamma)$. Define $\varrho : R^n \to \dot{R}^1$ by setting

$$\varrho(x) = \varrho'(f(x)) L(x , f)$$

for $x \in A$ and $\varrho(x) = 0$ for $x \in \complement A$. Let Γ_0 be the family of all rectifiable paths $\gamma \in \Gamma$ such that f is absolutely continuous on γ . By this we mean that if γ^0 is the parametrization of γ by means of its path length, then $f \circ \gamma^0$ is absolutely continuous. By a result of Fuglede ([6] or [23, p. 16]), $M(\Gamma_0) = M(\Gamma)$. From the formula concerning change of variables in integrals it follows that

$$\int_{\gamma} \varrho \, ds \geqq \int_{f \circ \gamma} \varrho' \, ds \geqq 1$$

for all $\gamma \in \Gamma_0$. Thus $\varrho \in F(\Gamma_0)$. A more detailed proof is given in [26]. Hence we obtain

$$M(\Gamma) = M(\Gamma_0) \leqq \int_{R^n} \varrho^n \, dm = \int_{A} \varrho'(f(x))^n L(x\,,f)^n \, dm(x)$$
$$\leqq K_O(f) \int_{A} \varrho'(f(x))^n J(x\,,f) \, dm(x)\,.$$

Since f is ACL^n, $J(x\,,f)$ is integrable over every domain $D \in J(G)$. Thus, the transformation formula in [14, Theorem 3, p. 364] yields

$$\int_{A \cap D} \varrho'(f(x))^n J(x\,,f) \, dm(x) = \int_{R^n} \varrho'(y)^n N(y\,,f\,,A \cap D) \, dm(y)$$
$$\leqq N(f\,,A) \int_{R^n} \varrho'^n \, dm\,.$$

The theorem cited above is formulated in [14] for finite-valued functions, but we may apply it to $\min(k\,,\varrho'^n)$ and let then $k \to \infty$. Since $D \in J(G)$ is arbitrary, we obtain

$$M(\Gamma) \leqq N(f\,,A)\, K_O(f) \int_{R^n} \varrho'^n \, dm\,.$$

Since this holds for every $\varrho' \in F(f\Gamma)$, the theorem follows.

4. The metric definitions

4.1. *Notation.* Let $f : G \to R^n$ be discrete and open, and let $x \in G$. If $0 < r < d(x\,,\partial G)$, we denote

$$l(x\,,f\,,r) = \inf_{|x-y|=r} |f(y) - f(x)|\,,$$

$$L(x\,,f\,,r) = \sup_{|x-y|=r} |f(y) - f(x)| = \sup_{|x-y|\leqq r} |f(y) - f(x)|\,.$$

If $0 < r < d(f(x)\,,\,\partial fG)$, we denote

$$l^*(x\,,f\,,r) = \inf_{z\in\partial U(x,f,r)} |x-z|\,,$$

$$L^*(x\,,f\,,r) = \sup_{z\in\partial U(x,f,r)} |x-z| = \sup_{z\in\overline{U}(x,f,r)} |x-z|\,.$$

Recall that $U(x\,,f\,,r)$ is the x-component of $f^{-1}B^n(f(x), r)$.

4.2. *Definition.* Let $f: G \to R^n$ be discrete and open. If $x \in G$, we call

$$H(x\,,f) = \limsup_{r\to 0} \frac{L(x\,,f\,,r)}{l(x\,,f\,,r)}$$

the *linear dilatation* of f at x , and

$$H^*(x\,,f) = \limsup_{r\to 0} \frac{L^*(x\,,f\,,r)}{l^*(x\,,f\,,r)}$$

the *inverse linear dilatation* of f at x .

In this section we first establish upper bounds for $H(x\,,f)$ and $H^*(x\,,f)$ when f is a non-constant quasiregular mapping. The main results (Theorems 4.13 and 4.14) are that both these dilatations characterize non-constant quasiregular mappings. These characterizations are called the metric definitions for quasiregularity.

The inverse linear dilatation also plays an essential role in Section 7, where the important inner dilatation inequality for the capacities of condensers is proved.

We assume now that in all the lemmas which appear in this section $f: G \to R^n$ is a discrete open mapping. Given three sets $A\,,B\,,C$ in R^n , a path $\gamma: [a\,,b] \to R^n$ is said to *join* A and B in C if $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$ for $a < t < b$.

4.3. **Lemma.** *Let* $x \in G$ *and let* σ_x *be as in* 2.9. *Then* $l^*(x\,,f\,,L(x\,,f\,,r)) = L^*(x\,,f\,,l(x\,,f\,,r)) = r$ *for* $0 < r < l^*(x\,,f\,,\sigma_x)$.

Proof. Set $l = l(x\,,f\,,r)$ and $L = L(x\,,f\,,r)$. Obviously $l \leqq L < \sigma_x$. We prove $l^*(x\,,f\,,L) = r$. The proof for $L^*(x\,,f\,,l) = r$ is similar. Since $B^n(x\,,r) \subset U(x\,,f\,,L)$, $l^*(x\,,f\,,L) \geqq r$. Choose $a \in S^{n-1}(x\,,r)$ such that $|f(a)-f(x)| = L$. By 2.9, $\partial U(x\,,f\,,L) = U(x\,,f\,,\sigma_x) \cap f^{-1}S^{n-1}(f(x), L)$. Thus $a \in \partial U(x\,,f\,,r)$, which implies $l^*(x\,,f\,,L) \leqq |a-x| = r$.

4.4 **Lemma.** *Let* $x \in G$ *and let* σ_x *be as in* 2.9. *For* $0 < s < t \leqq \sigma_x$ *let* $\Gamma(s\,,t)$ *be the family of all paths which join* $\partial U(x\,,f\,,s)$ *and* $\partial U(x\,,f\,,t)$

in $U(x, f, t) \setminus \bar{U}(x, f, s)$. *Suppose that there exist constants* b *and* σ, $1 \leqq b < \infty$, $0 < \sigma \leqq \sigma_x$, *such that* $M(\Gamma(s, t)) \leqq bM(f\Gamma(s, t))$ *for all* $0 < s < t < \sigma$. *Then* $H(x, f) \leqq C < \infty$ *where* C *depends only on* n *and* b.

Proof. Assume $0 < r < l^*(x, f, \sigma_x)$, and set $L = L(x, f, r)$, $l = l(x, f, r)$. Obviously $L < \sigma_x$. Suppose $l < L$. Then $f\Gamma(l, L)$ is a subfamily of the family of all paths joining $S^{n-1}(f(x), l)$ and $S^{n-1}(f(x), L)$ in $B^n(f(x), L) \setminus \bar{B}^n(f(x), l)$. Hence $M(f\Gamma(l, L)) \leqq \omega_{n-1}/(\log(L/l))^{n-1}$ [23, p. 7]. By 4.3, $\partial U(x, f, l)$ and $\partial U(x, f, L)$ meet $S^{n-1}(x, r)$. From this and the fact that $U(x, f, L) \setminus \bar{U}(x, f, l)$ is by 2.9 a ring it follows the estimate $M(\Gamma(l, L)) \geqq a_n > 0$ where a_n depends only on n [26, 11.7]. Since $M(\Gamma(l, L)) \leqq bM(f\Gamma(l, L))$, we obtain $L/l \leqq C$ where

$$C = \exp\left(\left(\frac{b\omega_{n-1}}{a_n}\right)^{1/(n-1)}\right).$$

This proves the lemma.

The upper bound for $H(x, f)$ when f is a non-constant quasiregular mapping follows now easily from the result of Section 3 and Lemma 4.4.

4.5. **Theorem.** *Let* $f : G \to R^n$ *be a non-constant quasiregular mapping. Then for every* $x \in G$

$$H(x, f) \leqq C < \infty,$$

where C *depends only on* n *and the product* $i(x, f)K_O(f)$.

Proof. Let $x \in G$. By 2.26, 2.12, and 3.2, the conditions in 4.4 are satisfied with $b = i(x, f)\, K_O(f) \geqq 1$ and some $\sigma > 0$. The result follows from 4.4.

A similar result holds for $H^*(x, f)$:

4.6. **Theorem.** *Let* $f : G \to R^n$ *be a non-constant quasiregular mapping. Then for every* $x \in G$

$$H^*(x, f) \leqq H(x, f)^{2i(x,f)K_O(f)} \leqq C^* < \infty,$$

where C^* *depends only on* n *and the product* $i(x, f)K_O(f)$.

Proof. Let $x \in G$. By 2.26, f is sense-preserving, discrete, and open. Choose σ_x as in 2.9, and set $D = U(x, f, \sigma_x)$. Let $t > 0$ be such that $L(x, f, t) < \sigma_x$ and let $r_0 > 0$ be such that $U(x, f, r) \subset B^n(x, t)$ if $0 < r \leqq r_0$. Assume $0 < r \leqq r_0$, and set $L^* = L^*(x, f, r)$, $l^* = l^*(x, f, r)$, $L = L(x, f, L^*)$, and $l = l(x, f, l^*)$. We choose a line J

through $f(x)$. Let $A' = \{y \mid l < |y-f(x)| < L\}$, let E' and F' be the components of $J \cap \bar{A}'$, and set $E = D \cap f^{-1}E'$, $F = D \cap f^{-1}F'$. If E_0 is any component of E, $fE_0 = E'$ by 2.6. By 2.9, $\partial U(x, f, l) = D \cap f^{-1}S^{n-1}(f(x), l)$ and $\partial U(x, f, L) = D \cap f^{-1}S^{n-1}(f(x), L)$. Hence E_0 meets both $\partial U(x, f, l)$ and $\partial U(x, f, L)$. Since $\partial U(x, f, l) \subset \bar{B}^n(x, l^*)$ and $\partial U(x, f, L) \subset \complement B^n(x, L^*)$, we get thus $\emptyset \neq S^{n-1}(x, u) \cap E_0 \subset S^{n-1}(x, u) \cap E$, and similarly $S^{n-1}(x, u) \cap F \neq \emptyset$ for every u, $l^* < u < L^*$.

Set $A = \{z \mid l^* < |z-x| < L^*\}$, and let Γ be the family of all paths which join E and F in A. Then $M(\Gamma) \geqq c_n \log (L^*/l^*)$ where $c_n > 0$ is the n-modulus of the family of all paths joining e_n and $-e_n$ in S^{n-1} [26, 10.12]. On the other hand, every path in $f\Gamma$ joins E' and F' in A', and hence $M(f\Gamma) \leqq c_n \log (L/l)$ by [26, 10.12]. By 3.2 and 2.12 we get thus

$$c_n \log \frac{L^*}{l^*} \leqq M(\Gamma) \leqq i(x, f) K_O(f) M(f\Gamma) \leqq i(x, f) K_O(f) c_n \log \left(\frac{L}{r} \frac{r}{l}\right).$$

But $r = l(x, f, L^*) = L(x, f, l^*)$, and letting $r \to 0$ we obtain

$$H^*(x, f) \leqq H(x, f)^{2i(x,f)K_O(f)}.$$

Theorem 4.5 completes the proof.

4.7. *Remark.* Define a K-quasiregular mapping $g : R^2 \to R^2$ by $g(z) = (x+iKy)^k$ where we have used the complex notation $z = x+iy$, where $K > 1$, and where k is a positive integer. Then $H(0, g) = K^k$ and $i(0, g) = k$, which shows that the linear dilatation depends in general on the local degree.

We turn now to the converse problem and establish characterizations of a non-constant quasiregular mapping f by $H(x, f)$ and $H^*(x, f)$. Recall that $f : G \to R^n$ is a discrete open mapping in the lemmas in this section.

4.8. **Lemma.** *Let $C \subset G$ be compact. Then there exists $t > 0$ such that the mapping $(x, s) \mapsto l^*(x, f, s)$ is continuous and the mapping $(x, s) \mapsto L^*(x, f, s)$ is lower semi-continuous in the set $C \times (0, t)$.*

Proof. For $x \in C$ let $s_x > 0$ be such that $\bar{U}(x, f, s_x) \subset B^n(x, a)$ where $a = d(C, \partial G)$. We cover C by sets $U(x_i, f, s_{x_i}/2)$, $i = 1, \ldots, k$. Assume $x \in C$ and let $x \in U(x_j, f, s_{x_j}/2)$. Then $U(x, f, s_{x_j}/2) \subset U(x_j, f, s_{x_j})$ and hence $\bar{U}(x, f, s) \subset C + aB^n$ if $0 < s < t = \min (s_{x_1}/2, \ldots, s_{x_k}/2)$. Assume $(x_0, s_0) \in C \times (0, t)$ and $0 < \varepsilon < \min (l^*(x_0, f, s_0), d(U(x_0, f, s_0), \partial G))$, and set $y_0 = f(x_0)$.

We show first that $(x, s) \mapsto l^*(x, f, s)$ is upper semicontinuous at (x_0, s_0). Let $z \in \partial U(x_0, f, s_0)$ be such that $|x_0 - z| = l^*(x_0, f, s_0)$. The set $fB^n(z, \varepsilon/2)$ is a neighborhood of $f(z)$ and there is therefore a point $v \in B^n(z, \varepsilon/2) \cap \complement \bar{U}(x_0, f, s_0)$ such that $v' = f(v) \notin \bar{B}^n(y_0, s_0)$. Set $\tau = (|v' - y_0| - s_0)/2$ and let $\delta, 0 < \delta < \varepsilon/2$, be such that $|x - x_0| < \delta$ implies $|f(x) - y_0| < \tau$. If now $(x, s) \in C \times (0, t]$ such that $|x - x_0| < \delta$ and $|s - s_0| < \tau$, then $v' \notin \bar{B}^n(f(x), s)$ and hence $v \notin \bar{U}(x, f, s)$. This implies $l^*(x, f, s) < |v - z| + |z - x_0| + |x_0 - x| < l^*(x_0, f, s_0) + \varepsilon$.

To show that $(x, s) \mapsto l^*(x, f, s)$ is lower semicontinuous at (x_0, s_0), set $r = l^*(x_0, f, s_0) - \varepsilon/2$. Then $\bar{B}^n(x_0, r) \subset U(x_0, f, s_0)$, and $2\tau = d(f\bar{B}^n(x_0, r), S^{n-1}(y_0, s_0))$ is positive. Let $\delta, 0 < \delta < \varepsilon/2$, be such that $|x - x_0| < \delta$ implies $|f(x) - y_0| < \tau$. Assume $(x, s) \in C \times (0, t]$, $|x - x_0| < \delta$, and $|s - s_0| < \tau$. Then $f\bar{B}^n(x_0, r) \subset B^n(f(x), s)$ and hence $\bar{B}^n(x_0, r) \subset U(x, f, s)$ because $x \in \bar{B}^n(x_0, r)$. From this it follows that $l^*(x, f, s) \geqq r - |x - x_0| \geqq l^*(x_0, f, s_0) - \varepsilon$.

Finally, to prove the lower semicontinuity of $(x, s) \mapsto L^*(x, f, s)$ at (x_0, s_0), let $z \in \partial U(x_0, f, s_0)$ be such that $|z - x_0| = L^*(x_0, f, s_0)$ and let $u \in U(x_0, f, s_0) \cap B^n(z, \varepsilon/2)$. There exists a continuum A in $U(x_0, f, s_0)$ such that $u \in A$ and $B^n(x_0, \varepsilon/2) \subset A$. Then fA is a compact set in $B^n(y_0, s_0)$ and $2\tau = d(fA, S^{n-1}(y_0, s_0))$ is positive. Again, let δ, $0 < \delta < \varepsilon/2$, be such that $|x - x_0| < \delta$ implies $|f(x) - y_0| < \tau$. Assuming $(x, s) \in C \times (0, t]$, $|x - x_0| < \delta$, and $|s - s_0| < \tau$, we have $fA \subset B^n(f(x), s)$ and therefore $A \subset U(x, f, s)$. Hence $L^*(x, f, s) \geqq |x - u| \geqq L^*(x_0, f, s_0) - \varepsilon$.

4.9. *Remark.* Let $g : R^3 \to R^3$ be the mapping defined in the cylindrical coordinates by $g(r, \varphi, z) = (r, 2\varphi, z)$. Then $x \mapsto L^*(x, g, s)$, $s > 0$, is discontinuous at points $x = (s, \varphi, z)$.

4.10. **Lemma.** *Let $C \subset G$ be compact. Then there exists $\varrho > 0$ such that for $0 < r < \varrho$*

(1) $x \mapsto l(x, f, r)$ *and* $x \mapsto L(x, f, r)$ *are continuous in* C,

(2) $x \mapsto l^*(x, f, L(x, f, r))$ *is continuous in* C,

(3) $x \mapsto L^*(x, f, L(x, f, r))$ *is lower semicontinuous in* C.

Proof. The condition (1) holds for $0 < r < d(C, \partial G)$. Let t be as in 4.8 and let $\varrho, 0 < \varrho < d(C, \partial G)$, be such that $|f(y) - f(z)| < t$ whenever $|y - z| < \varrho$, $y \in C$, and $z \in G$. Then (2) and (3) follow from 4.8 and (1).

4.11. **Lemma.** *If $H(x, f)$ or $H^*(x, f)$ is locally bounded, f is* ACL^n.

Proof. The proof is carried through in full detail for $H^*(x, f)$. By a simplified version one can prove the statement for $H(x, f)$. We shall show that the condition of Lemma 2.17 is satisfied. This is done by modifying the proof of Gehring [9, Lemma 9].

Let $D \in J(G)$, let $Q \subset D$ be an open n-interval, and let $P : R^n \to R^{n-1}$ be the orthogonal projection. For each Borel set $A \subset PQ$ we define $Z_A = Q \cap P^{-1}A$. Setting $\varphi(A, Q) = m(fZ_A)$ we obtain a q-quasiadditive set function $A \mapsto \varphi(A, Q)$ in PQ, where $q = N(f, D)$. Fix $z \in PQ$ such that the upper derivative $\overline{\varphi}'(z, Q)$ is finite, and set $J = Z_z$.

Let F be a compact subset of J. There exists $\varrho > 0$ such that (2) and (3) in 4.10 hold for $0 < r < \varrho$ and $C = \bar{Q}$. Since $H^*(x, f)$ is locally bounded, there is a constant $c < \infty$ such that $H^*(x, f) < c$ for $x \in D$. Given an integer k such that $0 < 1/k < \min(d(F, \partial Q), \varrho)$ let F_k be the set of all $x \in F$ such that $0 < r < 1/kc$ implies $L^*(x, f, L(x, f, r))/c \leqq l^*(x, f, L(x, f, r)) = r$. Then every F_k is compact by 4.10. Moreover, $F = \cup F_k$ by 4.3. Fix k, and choose $\varepsilon > 0$ and $t > 0$. By a well-known lemma [7, p. 6] there is a δ, $0 < \delta < 1/kc$, such that for every r, $0 < r < \delta$, there exists a covering of F_k by open segments $\Delta_1, \ldots, \Delta_p$ of J such that (1) $m_1(\Delta_i) = 2r$, (2) the center x_i of Δ_i belongs to F_k, (3) each point of F_k belongs to at most two different Δ_i, and (4) $pr < m_1(F_k) + \varepsilon$. Choose $r \in (0, \delta)$ such that $|f(u) - f(v)| < t/2$ whenever $|u-v| \leqq r$ and $u, v \in Q$. Set $s_i = L(x_i, f, r)$, $V_i = B^n(f(x_i), s_i)$. Then $fF_k \subset \cup V_i$ and $d(V_i) = 2s_i < t$. Hence $\Lambda_1^t(fF_k) \leqq \sum d(V_i) \leqq \sum 2s_i$. By Hölder's inequality this implies

$$\Lambda_1^t(fF_k)^n \leqq 2^n p^{n-1} \sum s_i^n = \frac{2^n p^{n-1}}{\Omega_n} \sum m(V_i) .$$

Since $x_i \in F_k$, we have $L^*(x_i, f, s_i) \leqq cl^*(x_i, f, s_i) = cr$, and therefore $U_i = U(x_i, f, s_i) \subset B^n(x_i, cr)$. Since $cr < 1/k$, this implies $V_i = fU_i \subset fZ_A$ where $A = B^{n-1}(z, cr)$. Observing that every point in Z_A belongs to at most $4c$ different U_i, we get thus $\sum m(V_i) \leqq 4cqm(fZ_A) = 4cq\varphi(A, Q)$. From this and from $pr < m_1(F) + \varepsilon$ it follows

$$\Lambda_1^t(fF_k)^n \leqq \frac{2^{n+2}c^n q(m_1(F) + \varepsilon)^{n-1}\, \Omega_{n-1}\, \varphi(A, Q)}{\Omega_n\, m_{n-1}(A)} .$$

Letting first $r \to 0$, then $\varepsilon \to 0$, and then $t \to 0$, we obtain $\Lambda_1(fF_k)^n \leqq qC\overline{\varphi}'(z, Q)m_1(F)^{n-1}$ where $C = 2^{n+2}c^n\Omega_{n-1}/\Omega_n$. Since fF is the limit of the expanding sequence of the compact sets fF_k, we have $\Lambda_1(fF) = \lim \Lambda_1(fF_k)$ and hence

$$\Lambda_1(fF)^n \leqq qC\overline{\varphi}'(z, Q)m_1(F)^{n-1} . \tag{4.12}$$

Let now I_j, $j = 1, \ldots, m$, be disjoint closed subintervals of J. We have $\sum d(fI_j) \leqq \sum \Lambda_1(fI_j) \leqq q\Lambda_1(f \cup I_j)$. Applying (4.12) to $F = \cup I_j$ we obtain

$$(\sum d(fI_j))^n \leqq q^{n+1} C\overline{\varphi}'(z, Q) (\sum m_1(I_j))^{n-1}.$$

Thus (2.18) is true for $i = n$. By symmetry, it holds also for $1 \leqq i \leqq n-1$. Hence f is ACL^n.

We are now in a position to prove the metric definitions for quasiregular mappings.

4.13. **Theorem.** *A non-constant mapping* $f: G \to R^n$ *is quasiregular if and only if it satisfies the following conditions:*

(1) f *is sense-preserving, discrete, and open.*

(2) $H(x, f)$ *is locally bounded in* G.

(3) *There exists* $a < \infty$ *such that* $H(x, f) \leqq a$ *for almost every* $x \in G \setminus B_f$.

Proof. Suppose first that f satisfies the conditions (1), (2), and (3). The mapping f is ACL^n by 4.11. An open ACL^n-mapping is differentiable a.e. [24, p. 9]. Let $D \in J(G)$. By (2) there exists $c < \infty$ such that $H(x, f) \leqq c$ for $x \in D$. If f is differentiable at $x \in D$, we have

$$|f'(x)|^n \leqq c^{n-1} J(x, f).$$

Since this holds a.e. in D, $f|D$ is quasiregular. If $x \in D \setminus B_f$ and if U is a connected neighborhood of x such that $U \subset D \setminus B_f$, it follows from (3) that $K_O(f|U)$, $K_I(f|U) \leqq a^{n-1}$. Hence $f|D$ is a^{n-1}-quasiregular by 2.28. Since this is true for every $D \in J(G)$, f is a^{n-1}-quasiregular.

Let now f be quasiregular. The condition (1) follows from 2.26, and (2) and (3) from 4.5 and 2.12.

4.14. **Theorem.** *A non-constant mapping* $f: G \to R^n$ *is quasiregular if and only if it satisfies the following conditions:*

(1) f *is sense-preserving, discrete, and open.*

(2) $H^*(x, f)$ *is locally bounded in* G.

(3) *There exists* $a^* < \infty$ *such that* $H^*(x, f) \leqq a^*$ *for almost every* $x \in G \setminus B_f$.

Proof. Suppose that f satisfies (1), (2), and (3). As in the proof of 4.13 we conclude that f is ACL^n and differentiable a.e. Each point in $G \setminus B_f$ has a connected neighborhood U such that $f|U$ is injective. Hence $H(f(z), (f|U)^{-1}) = H^*(z, f)$ for z in such a U. This together with (2) and (3) imply that $(f|U)^{-1}$ and hence $f|U$ are a^{*n-1}-quasiconformal. If, in addition to this, we show that $|f'(x)| = J(x, f) = 0$ for almost every $x \in B_f$, we can conclude that f is a^{*n-1}-quasiregular. Let $x_0 \in B_f$ be a point where f is differentiable. By 2.14, $J(x_0, f) = 0$. Since f is differentiable a.e., it is sufficient to show that $f'(x_0) = 0$. We have

$$f(x_0+h) - f(x_0) = f'(x_0)h + |h|\varepsilon(h) \tag{4.15}$$

where $\varepsilon(h) \to 0$ as $h \to 0$. Suppose that $|f'(x_0)| = \lambda > 0$. Choose unit vectors h_1, h_2 such that $|f'(x_0)h_1| = \lambda$ and $f'(x_0)h_2 = 0$. Consider $r > 0$ such that $U(x_0, f, r)$ is a normal neighborhood of x_0. Choose $\alpha_1, \alpha_2 > 0$ such that $x_0 + \alpha_i h_i \in \partial U(x_0, f, r)$, $i = 1, 2$. Then $|f(x_0 + \alpha_i h_i) - f(x_0)| = r$, and we obtain from (4.15)

$$|\alpha_1 f'(x_0)h_1 + \alpha_1 \varepsilon(\alpha_1 h_1)| = r = \alpha_2 |\varepsilon(\alpha_2 h_2)| .$$

Thus

$$\frac{L^*(x_0, f, r)}{l^*(x_0, f, r)} \geqq \frac{\alpha_2}{\alpha_1} \geqq \frac{\lambda - |\varepsilon(\alpha_1 h_1)|}{|\varepsilon(\alpha_2 h_2)|},$$

which implies $H^*(x_0, f) = \infty$. By (2), this is a contradiction.

If f is quasiregular, the conditions (1), (2), and (3) follow from 2.26, 4.6, and 2.12.

5. Condensers

5.1. In this section we generalize the concept of a ring domain and its capacity. This generalization is called a condenser, and we state some properties of the capacity of a condenser.

5.2. *Definition.* A *condenser* is a pair $E = (A, C)$ where $A \subset R^n$ is open and C is a non-empty compact set contained in A. E is a *ringlike condenser* if $A \setminus C$ is a ring (see 2.1). E is a *bounded condenser* if A is bounded. A condenser $E = (A, C)$ is said to be in a domain G if $A \subset G$.

The following lemma is immediate.

5.3. **Lemma.** *If* $f: G \to R^n$ *is open and* $E = (A, C)$ *is a condenser in* G, *then* (fA, fC) *is a condenser in* fG.

In the above situation we denote $fE = (fA, fC)$.

5.4. *The capacity of a condenser.* Let $E = (A, C)$ be a condenser. We set

$$\operatorname{cap} E = \operatorname{cap}(A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^n \, dm$$

and call it the *capacity* of the condenser E. The set $W_0(E) = W_0(A, C)$ is the family of all non-negative functions $u : A \to R^1$ such that (1) $u \in C_0(A)$, (2) $u(x) \geqq 1$ for $x \in C$, and (3) u is ACL. In the above formula

$$|\nabla u| = (\sum_{i=1}^{n} (\partial_i u)^2)^{1/2}.$$

We mention some properties of the capacity of a condenser.

5.5. **Lemma.** *If* $E = (A, C)$ *is a condenser, then*

$$\operatorname{cap} E = \inf_{u \in W_0^\infty(E)} \int_A |\nabla u|^n \, dm$$

where $W_0^\infty(E) = W_0^\infty(A, C) = W_0(E) \cap C_0^\infty(A)$.

Proof. Obviously

$$\operatorname{cap} E \leqq \inf_{u \in W_0^\infty(E)} \int_A |\nabla u|^n \, dm .$$

The converse inequality is proved by a standard approximating argument. The construction involves first multiplying $u \in W_0(E)$ by $1 + \varepsilon$, $\varepsilon > 0$, so that the resulting function is $\geqq (1+\varepsilon)$ on C, and then forming a smooth integral average, cf. e.g. [26, Section 27]. The details may be omitted.

5.6. **Lemma.** *If* $E = (A, C)$ *is a ringlike condenser, then* $\operatorname{cap} E = \operatorname{cap}(A \setminus C)$ *in the sense of Gehring* [8, p. 500].

This is a direct consequence of n-dimensional versions of [8, Lemma 1, p. 501] and [8, Remark, p. 502].

5.7. **Lemma.** *If* $E = (A, C)$ *is a condenser, then*

$$\operatorname{cap} E = \inf \operatorname{cap}(U, C),$$

where the infimum is taken over all open sets U *such that* $\bar{U}$ *is compact in* A *and* $U \supset C$.

Proof. Obviously $\operatorname{cap} E \leqq \operatorname{cap}(U, C)$ for all sets U of the above type, hence

$$\operatorname{cap} E \leqq \inf \operatorname{cap}(U, C).$$

Let $\varepsilon > 0$. Then there exists a function $u \in W_0(E)$ such that

$$\operatorname{cap} E > \int_A |\nabla u|^n \, dm - \varepsilon .$$

Since $\operatorname{spt} u$ is compact in A, there exists an open set U such that $\operatorname{spt} u \subset U$ and $\bar{U}$ is compact in A. Then $u \in W_0(U, C)$ and we obtain

$$\operatorname{cap}(U, C) \leqq \int_A |\nabla u|^n \, dm < \operatorname{cap}(A, C) + \varepsilon .$$

The lemma follows.

5.8. **Lemma.** *The inequality*

$$\operatorname{cap} E \leqq \frac{m(A)}{d(C, \partial A)^n}$$

holds for the capacity of a bounded condenser $E = (A, C)$.

Proof. Let $0 < \varepsilon < d(C, \partial A)^n$. There exists an open set U such that $C \subset U \subset \bar{U} \subset A$ and $d(C, \partial A)^n \leqq d(C, \partial U)^n + \varepsilon$. If we define $u(x) = d(x, \complement U)/d(C, \partial U)$, then $|u(x)-u(y)| \leqq |x-y|/d(C, \partial U)$ for all x, $y \in R^n$. Thus $u \in W_0(E)$ and $|\nabla u| \leqq 1/d(C, \partial U)$ a.e., which implies

$$\operatorname{cap} E \leqq \int_A d(C, \partial U)^{-n} \, dm = \frac{m(A)}{d(C, \partial U)^n} \leqq \frac{m(A)}{d(C, \partial A)^n - \varepsilon} .$$

Letting $\varepsilon \to 0$ gives the desired result.

5.9. **Lemma.** *Suppose that* $E = (A, C)$ *is a condenser such that* C *is connected. Then*

$$(\operatorname{cap} E)^{n-1} \geqq b_n \frac{d(C)^n}{m(A)}$$

where b_n *is a positive constant which depends only on* n.

Proof. By 5.7 we may suppose that A is bounded. We may also assume that $d(C) = r > 0$ and that C contains the origin and the point re_n. Let $u \in W_0^\infty(E)$. For $0 < t < r$ we let $T(t)$ denote the hyperplane $x_n = t$. Using the method of [23, p. 9], we estimate the integral

$$\int_{T(t)} |\nabla u|^n \, dm_{n-1} .$$

Fix $z \in C \cap T(t)$. For $y \in S^{n-2}$ let $R(y)$ be the supremum of all $t_0 > 0$ such that $z + ty \in A$ for $0 \leqq t < t_0$. Then

$$\int_0^{R(y)} |\nabla u(z+ty)|\, dt \geqq u(z) - u(z+R(y)y) \geqq 1$$

for all $y \in S^{n-2}$. By Hölder's inequality this implies

$$1 \leqq (n-1)^{n-1} R(y) \int_0^{R(y)} |\nabla u(z+ty)|^n t^{n-2}\, dt .$$

Integrating over $y \in S^{n-2}$ yields

$$\begin{aligned}(n-1)^{1-n} \int_{S^{n-2}} R^{-1}\, dm_{n-2} &\leqq \int_{S^{n-2}} dm_{n-2}(y) \int_0^{R(y)} |\nabla u(z+ty)|^n t^{n-2}\, dt \\ &\leqq \int_{T(t)} |\nabla u|^n\, dm_{n-1} .\end{aligned} \tag{5.10}$$

On the other hand, we obtain by Hölder's inequality

$$\begin{aligned}\omega_{n-2}^n = \left(\int_{S^{n-2}} dm_{n-2}\right)^n &\leqq \int_{S^{n-2}} R^{n-1}\, dm_{n-2} \left(\int_{S^{n-2}} R^{-1}\, dm_{n-2}\right)^{n-1} \\ &\leqq (n-1) m_{n-1}(A \cap T(t)) \left(\int_{S^{n-2}} R^{-1}\, dm_{n-2}\right)^{n-1} .\end{aligned} \tag{5.11}$$

Setting $f(t) = m_{n-1}(A \cap T(t))$, we obtain from (5.10) and (5.11)

$$\int_{T(t)} |\nabla u|^n\, dm_{n-1} \geqq (n-1)^{1-n-1/(n-1)} \omega_{n-2}^{n/(n-1)} f(t)^{1/(1-n)} .$$

Integrating over $0 < t < r$ we obtain

$$\int_A |\nabla u|^n\, dm \geqq (n-1)^{1-n-1/(n-1)} \omega_{n-2}^{n/(n-1)} \int_0^r f(t)^{1/(1-n)}\, dt . \tag{5.12}$$

Hölder's inequality gives

$$r^n = \left(\int_0^r dt\right)^n \leqq \left(\int_0^r f(t) dt\right) \left(\int_0^r f(t)^{1/(1-n)}\, dt\right)^{n-1} \leqq m(A) \left(\int_0^r f(t)^{1/(1-n)}\, dt\right)^{n-1} .$$

By (5.12), this implies

$$\left(\int_A |\nabla u|^n \, dm\right)^{n-1} \geqq (n-1)^{-2+2n-n^2} \omega_{n-2}^n \frac{r^n}{m(A)} .$$

Since this holds for every $u \in W_0^\infty(E)$, the lemma follows.

6. The condenser definition for $K_O(f)$

6.1. Suppose that $f : G \to R^n$ is a mapping. A condenser $E = (A , C)$ is said to be a *normal condenser* of f if A is a normal domain of f. If E is a normal condenser, we set $N(f , E) = N(f , A) = \mu(f , A)$ (cf. 2.4).

This section is devoted to the proof of the following result:

6.2. **Theorem.** *Suppose that* $f : G \to R^n$ *is sense-preserving, discrete, and open, and that* $1 \leqq K < \infty$. *Then the following conditions are equivalent:*

(1) $K_O(f) \leqq K$.

(2) $\operatorname{cap} E \leqq KN(f , E) \operatorname{cap} fE$ *for all normal condensers* E *in* G .

(3) $\operatorname{cap} E \leqq KN(f , E) \operatorname{cap} fE$ *for all ringlike normal condensers* E *in* G .

Since (2) implies (3) trivially, it suffices to prove that (3) $\Rightarrow$ (1) $\Rightarrow$ (2).

6.3. *Proof for* (3) $\Rightarrow$ (1). We show first, using the metric definition 4.13, that f is quasiregular. Let $x \in G$, and choose $\sigma_x > 0$ as in 2.9. Choose $0 < r_1 < r_2 < \sigma_x$, and set $U_i = U(x , f , r_i)$, $i = 1 , 2$. Then $E = (U_2 , \bar{U}_1)$ is a ringlike normal condenser. Let Γ be the family of all paths joining ∂U_1 and ∂U_2 in $U_2 \setminus \bar{U}_1$, and let Γ_1 be the family of all paths joining the boundary components of the spherical ring $A = B^n(f(x) , r_2) \setminus \bar{B}^n(f(x) , r_1))$ in A . By 2.9, $\partial U_i = \bar{U}_2 \cap f^{-1}S^{n-1}(f(x), r_i)$, $i = 1 , 2$. From the path lifting lemma 2.7 it follows that $\Gamma_1 = f\Gamma$.

By 5.6 and by a generalized version of [10, Theorem 1], $\operatorname{cap} E = M(\Gamma)$ and $\operatorname{cap} fE = M(\Gamma_1)$. Hence (3) implies $M(\Gamma) \leqq Ki(x , f)M(f\Gamma)$. From 4.4 it follows that $H(x , f) \leqq C < \infty$ where C depends only on n , K, and $i(x , f)$. Since $i(x , f) = 1$ for $x \in G \setminus B_f$, it follows from 4.13 that f is quasiregular.

If $x \in G \setminus B_f$, then there is a neighborhood $V = B^n(x , r)$ such that $f|V$ is injective. By (3) and by 5.6, $\operatorname{cap} R \leqq K \operatorname{cap} fR$ for all rings $R \subset V$. Hence $K_O(f|V) \leqq K$ by the corresponding result for quasiconformal mappings [26, 36.1]. By 2.28, $K_O(f) \leqq K$.

6.4. *Proof for* (1) $\Rightarrow$ (2). Suppose that $E = (A , C)$ is a normal condenser in G . Let $\varepsilon > 0$. By 5.5, there is $v \in W_0^\infty(fE)$ such that

$$\int_{fA} |\nabla v|^n \, dm < \operatorname{cap} fE + \varepsilon .$$

Define $u : A \to R^1$ by $u(x) = v(f(x))$. Then $u(x) \geqq 1$ for $x \in C$. Since f is ACL and a.e. differentiable, u has also these properties. Since A is a normal domain of f, it follows from 2.6 that $\operatorname{spt} u \subset A \cap f^{-1}(\operatorname{spt} v)$ is compact. Hence $u \in W_0(E)$, which implies

$$\operatorname{cap} E \leqq \int_A |\nabla u|^n \, dm .$$

Here $|\nabla u(x)| \leqq |\nabla v(f(x))| \, |f'(x)|$ a.e. Using [14, Theorem 3, p. 364] we obtain

$$\operatorname{cap} E \leqq K_O(f) \int_A |\nabla v(f(x))|^n J(x, f) \, dm(x)$$

$$= K_O(f) \int_{R^n} |\nabla v(y)|^n N(y, f, A) \, dm(y)$$

$$\leqq N(f, E) \, K_O(f) \int_{fA} |\nabla v|^n \, dm \leqq N(f, E) \, K_O(f) \, (\operatorname{cap} fE + \varepsilon) .$$

Since ε is arbitrary, (2) follows.

7. The condenser definition for $K_I(f)$

This section is devoted to the proof of the following result:

7.1. **Theorem.** *Suppose that $f : G \to R^n$ is sense-preserving, discrete, and open, and that $1 \leqq K < \infty$. Then the following conditions are equivalent:*

(1) $K_I(f) \leqq K$.

(2) $\operatorname{cap} fE \leqq K \operatorname{cap} E$ *for all condensers* E *in* G.

(3) $\operatorname{cap} fE \leqq K \operatorname{cap} E$ *for all ringlike condensers* E *in* G.

7.2. *Remarks.* This result differs from 6.2 in two respects. First, in 7.1 the factor $N(f, E)$ does not appear. Second, the inequality (2) holds for all condensers while the corresponding inequality $\operatorname{cap} E \leqq N(f, E) K_O(f) \operatorname{cap} fE$ of 6.2 is given only for normal condensers. In particular, (2) holds for condensers $E = (A, C)$ such that $\bar{A}$ is not compact in G. This makes 7.1 to a useful tool when the boundary behavior of quasiregular mappings is studied. For example, suppose that $f : R^n \to R^n$ is a bounded non-constant quasiregular mapping. If E is the condenser $(R^n, \bar{B}^n)$, we have $\operatorname{cap} E = 0$, while 5.9 implies $\operatorname{cap} fE > 0$. This contradicts (2), and we have proved Liouville's theorem in n dimensions (cf. Rešetnjak [19, p. 661]). We intend to return to related questions in a later paper. The proof of 7.1 is considerably more difficult than that of 6.2. Since (2) implies (3) trivially, it suffices to prove that (3) $\Rightarrow$ (1) $\Rightarrow$ (2).

7.3. *Proof for* (3) $\Rightarrow$ (1). We show first with the help of Lemma 2.17 that f is ACL^n. Let $D \in J(G)$, and let Q be an open n-interval in D. Write $Q = Q_0 \times J$, where Q_0 is an $(n-1)$-interval in R^{n-1} and J is an open segment of the x_n-axis. Using the notation of 2.16, we have $\varphi_n(E, Q) = m(f(E \times J))$ for Borel sets $E \subset Q_0$. Fix $z \in Q_0$ such that $\bar{\varphi}_n'(z, Q) < \infty$, and let $\Delta_1, \ldots, \Delta_k$ be disjoint closed subintervals of the segment $J_z = \{z\} \times J$. Set $A_i = \Delta_i + rB^n$ where r is a positive number such that (i) the domains A_i are disjoint, (ii) $A_i \subset Q$, and (iii) $\Omega_n r \leqq \omega_{n-1} m_1(\Delta_i)$ for $1 \leqq i \leqq k$. Then (A_i, Δ_i) is a ringlike condenser, and we obtain from 5.8 and (iii) the estimate

$$\operatorname{cap}(A_i, \Delta_i) \leqq m(A_i)/r^n \leqq 2\omega_{n-1}\, m_1(\Delta_i)/r .$$

On the other hand, 5.9 implies

$$(\operatorname{cap}(fA_i, f\Delta_i))^{n-1} \geqq b_n \frac{d(f\Delta_i)^n}{m(fA_i)} .$$

Together with (3) these yield

$$d(f\Delta_i) \leqq C r^{(1-n)/n} m(fA_i)^{1/n} m_1(\Delta_i)^{(n-1)/n} ,$$

where C depends only on n and K.

Summing over $1 \leqq i \leqq k$ and using Hölder's inequality we obtain

$$\Big(\sum_{i=1}^{k} d(f\Delta_i)\Big)^n \leqq C_1 m_{n-1}(B)^{-1} \Big(\sum_{i=1}^{k} m(fA_i)\Big) \Big(\sum_{i=1}^{k} m_1(\Delta_i)\Big)^{n-1}$$

where $B = B^{n-1}(z, r)$ and C_1 depends only on n and K. Setting $q = N(f, D)$ we have $\sum m(fA_i) \leqq q m(\bigcup fA_i) \leqq q\varphi_n(B, Q)$. Letting $r \to 0$ we thus obtain

$$\Big(\sum_{i=1}^{k} d(f\Delta_i)\Big)^n \leqq q C_1 \bar{\varphi}_n'(z, Q) \Big(\sum_{i=1}^{k} m_1(\Delta_i)\Big)^{n-1} .$$

By symmetry, Lemma 2.17 implies that f is ACL^n.

As an open ACL^n-mapping, f is differentiable a.e. by [24, p. 9]. We prove that f is quasiregular by showing that if f is differentiable at x_0, then

$$(7.4) \qquad |f'(x_0)|^n \leqq K_1 J(x_0, f)$$

where K_1 depends only on n and K. For brevity, we set $L = |f'(x_0)|$ and $J = J(x_0, f)$. We may assume that $L > 0$. Let $0 < \varepsilon < L$, and let e be a unit vector such that $|f'(x_0)e| = L$. For $0 < r < d(x_0, \partial G)/2$ let C_r be the closed line segment with end points x_0 and $x_0 + re$. Setting $A_r = C_r + rB^n$ we obtain a ringlike condenser $E_r = (A_r, C_r)$ in G. Since the condensers E_r are similar for all r, $\operatorname{cap} E_r = c$ is a positive constant, independent of r. We choose r so small that $|f(x_0 + re) - f(x_0)| > (L - \varepsilon)r$ and $m(fA_r) \leqq (J + \varepsilon)m(A_r)$. Since $m(A_r) \leqq 3\omega_{n-1}r^n$, 5.9 implies

$$(\operatorname{cap} fE_r)^{n-1} \geqq b_n \frac{d(fC_r)^n}{m(fA_r)} \geqq C \frac{(L-\varepsilon)^n}{J+\varepsilon}$$

where C depends only on n. Since $\operatorname{cap} fE_r \leqq K \operatorname{cap} E_r = Kc$, we obtain

$$(L - \varepsilon)^n \leqq C^{-1}K^{n-1}c^{n-1}(J + \varepsilon) .$$

Letting $\varepsilon \to 0$ yields (7.4). Hence f is quasiregular.

Let $x \in G \setminus B_f$, and choose a connected neighborhood D of x such that $f \mid D$ is injective. Applying Theorem 6.2 to the mapping $(f \mid D)^{-1}$ we obtain

$$K_I(f \mid D) = K_O((f \mid D)^{-1}) \leqq K .$$

By 2.28, $K_I(f) \leqq K$.

7.5. We now turn to the proof of (1) $\Rightarrow$ (2) in Theorem 7.1. Assume that $K_I(f) \leqq K$, and let $E = (A, C)$ be a condenser in G. Let u be an arbitrary function in $W_0^\infty(E)$. We define $v : fA \to R^1$ by

$$v(y) = \sup_{x \in f^{-1}(y)} u(x) .$$

Since f is discrete and since $\operatorname{spt} u$ is compact, $f^{-1}(y) \cap \operatorname{spt} u$ is finite. Hence for each $y \in fA$ there is $x \in f^{-1}(y)$ such that $v(y) = u(x)$. We are going to show that $v \in W_0(fE)$. For this purpose we prove some lemmas.

7.6. **Lemma.** *The function v has the following properties:*
(1) $v(y) \geqq 1$ *for* $y \in fC$.
(2) $\operatorname{spt} v = f(\operatorname{spt} u)$.
(3) $\operatorname{spt} v$ *is compact in* fA.
(4) v *is continuous.*

Proof. The property (1) is trivial. For (2), set $U = \{x \in A \mid u(x) \neq 0\}$ and $V = \{y \in fA \mid v(y) \neq 0\}$. Then $fU = V$. Since f is continuous and $\operatorname{spt} u = \bar{U}$ is compact, $f(\operatorname{spt} u) = \bar{V} = \operatorname{spt} v$. Since (3) is a consequence of (2), it remains to verify (4).

Let $y \in fA$. If $y \notin \operatorname{spt} v$, then $v = 0$ in a neighborhood of y. Next assume $y \in \operatorname{spt} v = f(\operatorname{spt} u)$. Let $\varepsilon > 0$. Choose a neighborhood U of $\operatorname{spt} u$ such that $\bar{U}$ is a compact subset of A and such that $\partial U \cap f^{-1}(y) = \emptyset$. Let $U \cap f^{-1}(y) = \{x_1, \ldots, x_j\}$, where $u(x_1) = v(y)$. By the continuity of u we can find neighborhoods $U(x_i)$ of x_i, $1 \leqq i \leqq j$, such that $|u(x) - u(x_i)| < \varepsilon$ for $x \in U(x_i)$. Then $F = \bar{U} \setminus \bigcup_{i=1} U(x_i)$ is compact, $y \notin fF$, and

$$V = \left(\bigcap_{i=1}^{j} fU(x_i)\right) \setminus fF$$

is a neighborhood of y. We show that $|v(z) - v(y)| < \varepsilon$ if $z \in V$.

Since $z \in fU(x_1)$, $z = f(x_0)$ for some $x_0 \in U(x_1)$. Hence

$$v(z) \geqq u(x_0) \geqq u(x_1) - \varepsilon = v(y) - \varepsilon .$$

On the other hand, $v(z) = u(x)$ for some x in $U \cap f^{-1}(z) \subset U \setminus F$. Hence $x \in U(x_i)$ for some i, which implies

$$v(z) = u(x) < u(x_i) + \varepsilon \leqq v(y) + \varepsilon .$$

The lemma is proved.

7.7. To show that $v \in W_0(fE)$, we still must prove that v is ACL. Since this property is local, it suffices to show that v is ACL in a neighborhood of each point of $\operatorname{spt} v$. Fix $y_0 \in \operatorname{spt} v$, and let $f^{-1}(y_0) \cap \operatorname{spt} u = \{x_1, \ldots, x_q\}$. Choose r_0 such that $0 < r_0 < d(y_0, \partial fA)$ and such that the domains $U(x_i, f, r_0)$ are disjoint normal neighborhoods of x_i for $1 \leqq i \leqq q$ (see 2.9). Next choose a positive number $r_1 \leqq r_0$ such that $B^n(y_0, r_1) \cap f(\operatorname{spt} u \setminus \bigcup_{i=1}^{q} U(x_i, f, r_0)) = \emptyset$. Then the components of $f^{-1}B^n(y_0, r_1)$ which meet $\operatorname{spt} u$ are the sets $U(x_i, f, r_1)$, $1 \leqq i \leqq q$. Set $U_i = U(x_i, f, r_1)$ and $U = \bigcup_{i=1}^{q} U_i$. Choose an open n-interval Q such that $\bar{Q} \subset B^n(y_0, r_1)$. Write $Q = Q_0 \times J$, where Q_0 is an $(n-1)$-interval in R^{n-1}, and $J = (a, b)$ is an open segment of the x_n-axis. For each Borel set $A \subset Q_0$ put $\varphi(A) = m(U \cap f^{-1}(A \times J))$. Then φ is a 1-quasiadditive (in fact, completely additive) set function in Q_0. By 2.3, $\overline{\varphi}'(z) < \infty$ for almost every $z \in Q_0$. Fix such z, and set $J_z = \{z\} \times J$. To show that v is ACL, it is sufficient to prove the following result:

7.8. **Lemma.** *The function v is absolutely continuous on $\bar{J}_z$.*

7.9. For the proof of 7.8 we need another lemma, which states, roughly speaking, that the one-to-many correspondence f^{-1} is absolutely continuous on $\bar{J}_z$. We let Φ denote the set of all continuous mappings $g : \bar{J}_z \to U$ such that $f \circ g$ is the identity mapping of $\bar{J}_z$. Observe that for any $g \in \Phi$, $g\bar{J}_z$ is contained in some U_i.

7.10. **Lemma.** *For every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$\sum_{i=1}^{p} |g_i(\bar{y}_i) - g_i(y_i)| < \varepsilon$$

whenever $[y_1, \bar{y}_1], \ldots, [y_p, \bar{y}_p]$ *are disjoint closed intervals of* J_z, $\sum_{i=1}^{p} |\bar{y}_i - y_i| < \delta$, *and* $g_i \in \Phi$, $1 \leqq i \leqq p$.

Proof. The proof is closely related to the proof of 4.11. By 4.6, there is a constant c such that $H^*(x, f) < c$ for $x \in U$. Suppose that $[y_i, \bar{y}_i]$ are disjoint closed intervals of J_z and that $g_i \in \Phi$, $1 \leqq i \leqq p$. More precisely, $y_i = (z, t_i)$, $\bar{y}_i = (z, \bar{t}_i)$, and $a < t_1 < \bar{t}_1 < \ldots < t_p < \bar{t}_p < b$. Set $F^i = [y_i, \bar{y}_i]$ and $F = F^1 \cup \ldots \cup F^p$. Choose an integer k_0 such that $0 < 1/k_0 < d(F, \partial Q)$. For $k \geqq k_0$ and for $1 \leqq i \leqq p$ let F^i_k denote the set of all points $y \in F^i$ such that $0 < r < 1/k$ implies $L^*(g_i(y), f, r) \leqq cl^*(g_i(y), f, r)$. Then $F^i_k \subset F^i_{k+1}$ and $F^i = \bigcup_{k=k_0}^{\infty} F^i_k$. By 4.8, the function $y \mapsto L^*(g_i(y), f, r)$ is lower semicontinuous, and the function $y \mapsto l^*(g_i(y), f, r)$ is continuous. Hence each F^i_k is compact.

Let η and t be arbitrary positive numbers, and fix $k \geqq k_0$. Using the same lemma as in the proof of 4.11 ([7, p. 6] or [26, 31.1]), we can find positive numbers $\delta_1, \ldots, \delta_p$ such that for all $r \in (0, \delta_i)$ there exists a covering of F^i_k by open intervals $\Delta^i_1, \ldots, \Delta^i_{l(i)}$ such that (1) $m_1(\Delta^i_m) = 2r$, (2) the center y^i_m of Δ^i_m belongs to F^i_k, (3) every point of F^i_k belongs to at most two different Δ^i_m, and (4) $l(i)r < m_1(F^i_k) + \eta/p$. Set $\alpha = \min_{1 \leqq i \leqq p-1} |y_{i+1} - \bar{y}_i|$. Choose $r > 0$ such that $r \leqq \min(\delta_1, \ldots, \delta_p, 1/k, \alpha/2)$ and such that $|g_i(y) - g_i(y')| < t/2c$ whenever $|y - y'| \leqq 2r$, $y, y' \in \bar{J}_z$, and $1 \leqq i \leqq p$. Then $\{\Delta^i_m \mid 1 \leqq i \leqq p, 1 \leqq m \leqq l(i)\}$ is a covering of $\bigcup_{i=1}^{p} F^i_k$ such that every point is covered at most twice.

Set $x^i_m = g_i(y^i_m)$ and $U^i_m = U(x^i_m, f, r)$, $1 \leqq i \leqq p$, $1 \leqq m \leqq l(i)$. Since $y^i_m \in F^i_k$, we have $L^*_{im} \leqq cl^*_{im}$ where $L^*_{im} = L^*(x^i_m, f, r)$, $l^*_{im} = l^*(x^i_m, f, r)$. On the other hand, $l^*_{im} \leqq d(g_i\Delta^i_m) < t/2c$, which implies $d(U^i_m) \leqq 2L^*_{im} < t$. Since $g_i\Delta^i_m \subset U^i_m$, this yields

$$\Lambda^t_1 \left(\bigcup_{i=1}^{p} g_i F^i_k\right)^n \leqq \sum_{i=1}^{p} \sum_{m=1}^{l(i)} d(U^i_m) \leqq 2c \sum_{i=1}^{p} \sum_{m=1}^{l(i)} l^*_{im}.$$

By Hölder's inequality we obtain

$$\Lambda_1^t\,(\bigcup_{i=1}^{p} g_i F_k^i)^n \leqq 2^n\, c^n\, (\sum_{i=1}^{p} \sum_{m=1}^{l(i)} l_{im}^{*n})\, (\sum_{i=1}^{p} l(i))^{n-1}\,.$$

Since $l(i)r \leqq m_1(F_k^i) + \eta/p \leqq m_1(F^i) + \eta/p$ and since $\Omega_n l_{im}^{*n} \leqq m(U_m^i)$, this implies

$$\Lambda_1^t\,(\bigcup_{i=1}^{p} g_i F_k^i)^n \leqq \frac{2^n\, c^n\, (m_1(F) + \eta)^{n-1} \sum\sum m(U_m^i)}{\Omega_n\, r^{n-1}}\,.$$

Set $B = B^{n-1}(z\,,r)$. Then each U_m^i is contained in $U \cap f^{-1}(B \times J)$. Since every point belongs to not more than two different U_m^i, we have $\sum\sum m(U_m^i) \leqq 2m(U \cap f^{-1}(B \times J)) = 2\varphi(B)$. Thus

$$\Lambda_1^t\,(\bigcup_{i=1}^{p} g_i F_k^i)^n \leqq C(m_1(F) + \eta)^{n-1}\, \varphi(B)/m_{n-1}(B)$$

where $C = 2^{n+1}c^n\Omega_{n-1}/\Omega_n$. Letting first $r \to 0$, then $\eta \to 0$, then $t \to 0$, and then $k \to \infty$ yields

$$(\sum_{i=1}^{p} \Lambda_1(g_i F^i))^n = \Lambda_1\,(\bigcup_{i=1}^{p} g_i F^i)^n \leqq C\overline{\varphi}'(z)\, m_1(F)\,.$$

Since $|g_i(\bar{y}_i) - g_i(y_i)| \leqq \Lambda_1(g_i F^i)$, this proves 7.10.

7.11. *Proof for Lemma* 7.8. We show first that for every pair $y, \bar{y} \in J_z$ there are $g\,, \bar{g} \in \Phi$ such that

$$(7.12)\qquad |v(\bar{y}) - v(y)| \leqq |u(g(\bar{y})) - u(g(y))| + |u(\bar{g}(\bar{y})) - u(\bar{g}(y))|\,.$$

By 2.7, there are $g\,, \bar{g} \in \Phi$ such that $v(y) = u(g(y))$ and $v(\bar{y}) = u(\bar{g}(\bar{y}))$. If $v(y) \leqq v(\bar{y})$, then

$$|v(\bar{y}) - v(y)| = u(\bar{g}(\bar{y})) - u(g(y)) \leqq u(\bar{g}(\bar{y})) - u(\bar{g}(y))\,,$$

and if $v(\bar{y}) \leqq v(y)$, then

$$|v(\bar{y}) - v(y)| = u(g(y)) - u(\bar{g}(\bar{y})) \leqq u(g(y)) - u(g(\bar{y}))\,.$$

These inequalities prove (7.12).

Let $\varepsilon > 0$, and let $\delta > 0$ be the number given by 7.10. Since $u \in C_0^\infty(A)$, u satisfies a Lipschitz condition

$$(7.13)\qquad |u(x) - u(x')| \leqq M|x - x'|$$

for all $x\,, x' \in A$. Let $[y_1\,, \bar{y}_1]\,, \ldots\,, [y_p\,, \bar{y}_p]$ be disjoint closed intervals of J_z such that $\sum |\bar{y}_i - y_i| < \delta$. By (7.12) there are $g_i\,, \bar{g}_i \in \Phi$ such that

$$|v(\bar{y}_i) - v(y_i)| \leqq |u(g_i(\bar{y}_i)) - u(g_i(y_i))| + |u(\bar{g}_i(\bar{y}_i)) - u(\bar{g}_i(y_i))|$$

for $1 \leqq i \leqq p$. By 7.10 and (7.13) we obtain

$$\sum_{i=1}^{p} |v(\bar{y}_i) - v(y_i)| \leqq 2M\varepsilon .$$

Hence v is absolutely continuous on $\bar{J}_z$, and 7.8 is proved. This implies:

7.14. **Lemma.** $v \in W_0(fE)$.

7.15. **Lemma.** *Suppose that* $y \in \operatorname{spt} v \setminus f(\operatorname{spt} u \cap B_f)$. *Then there is a neighborhood* V_0 *of* y *such that for every connected neighborhood* $V \subset V_0$ *of* y, *the following conditions are satisfied:*

(1) $V \cap f(B_f \cap \operatorname{spt} u) = \emptyset$.

(2) *The components of* $f^{-1}V$ *which meet* $\operatorname{spt} u$ *form a finite collection* $D_1, \ldots, D_k$.

(3) f *defines homeomorphisms* $f_i : D_i \to V$.

(4) v *is differentiable a.e. in* V.

(5) $|\nabla v(z)| \leqq \max_{1 \leqq i \leqq k} |\nabla u(g_i(z))|\,|g_i'(z)|$ *for almost every* $z \in V$ *where* $g_i = f_i^{-1}$.

Proof. Choose disjoint neighborhoods $U_1, \ldots, U_k$ of the points of $\operatorname{spt} u \cap f^{-1}(y)$ such that $U_i \in J(G)$ and such that $f \mid \bar{U}_i$ is injective, $1 \leqq i \leqq k$. We claim that

$$V_0 = \left(\bigcap_{i=1}^{k} fU_i\right) \setminus f\left(\operatorname{spt} u \setminus \bigcup_{i=1}^{k} U_i\right)$$

is the required neighborhood of y.

Let $V \subset V_0$ be a connected neighborhood of y. Then (1) holds since $U_i \cap B_f = \emptyset$ for $1 \leqq i \leqq k$. If D is a component of $f^{-1}V$ such that $D \cap \operatorname{spt} u \neq \emptyset$, then D meets some U_i. Since $f \mid \bar{U}_i$ is injective, we have $V_0 \cap f\partial U_i = \emptyset$, and hence $D \cap \partial U_i = \emptyset$. This implies $D \subset U_i$, which proves (2) and (3).

Since the mappings g_i are quasiconformal, they are differentiable a.e. in V. Let $z \in V$ be a point at which every g_i, $1 \leqq i \leqq k$, is differentiable. Let I be the set of all indexes i such that $v(z) = u(g_i(z))$. If $h \in R^n$ is small enough, then by the continuity of v, $v(z+h) = \max_{j \in I} u(g_j(z+h))$. Thus

$$|v(z+h) - v(z)| = |(u(g_j(z+h)) - u(g_j(z))| \tag{7.16}$$

for some $j \in I$. By (7.13), this yields

$$|v(z+h) - v(z)| \leqq M \max_{1 \leqq i \leqq k} |g_i(z+h) - g_i(z)| .$$

Since the mappings g_i are differentiable at z, this implies

$$\limsup_{h\to 0} \frac{|v(z+h)-v(z)|}{|h|} < \infty ,$$

and (4) follows from the theorem of Rademacher and Stepanov [26, 29.1]. Finally, (5) follows easily from (7.16).

7.17. We shall now complete the proof of Theorem 7.1. Let $u \in W_0^\infty(E)$ and v be as in 7.5. By Lemma 7.14 we have

$$\operatorname{cap} fE \leqq \int_{fA} |\nabla v|^n \, dm .$$

There exists a countable net of open disjoint cubes $Q_1, Q_2, \ldots$ such that $fA \setminus f(\operatorname{spt} u \cap B_f) = \bigcup_{j=1}^{\infty} \bar{Q}_j$ and such that if Q_j meets $\operatorname{spt} v$, then the conditions (1)—(5) of 7.15 are satisfied for $V = Q_j$. Since $m(fB_f) = 0$ by 2.27, $m(fA \setminus \bigcup_{j=1}^{\infty} Q_j) = 0$. Hence

$$\operatorname{cap} fE \leqq \sum_{j=1}^{\infty} \int_{Q_j} |\nabla v|^n \, dm . \tag{7.18}$$

Consider a fixed cube Q_j. If Q_j does not meet $\operatorname{spt} v$, then

$$\int_{Q_j} |\nabla v|^n \, dm = 0 .$$

If Q_j meets $\operatorname{spt} v$, consider the inverse mappings $g_i : Q_j \to D_i$, $1 \leqq i \leqq k$, given by 7.15. For almost every $z \in Q_j$ we have

$$\begin{aligned} |\nabla v(z)|^n &\leqq \max_{1\leqq i\leqq k} |\nabla u(g_i(z))|^n \, |g_i'(z)|^n \leqq \sum_{i=1}^{k} |\nabla u(g_i(z))|^n \, K_O(g_i) J(z, g_i) \\ &\leqq K_I(f) \sum_{i=1}^{k} |\nabla u(g_i(z))|^n \, J(z, g_i) . \end{aligned}$$

Since $K_I(f) \leqq K$, this implies

$$\begin{aligned} \int_{Q_j} |\nabla v|^n dm &\leqq K \sum_{i=1}^{k} \int_{Q_j} |\nabla u(g_i(z))|^n \, J(z, g_i) \, dm(z) \\ &= K \sum_{i=1}^{k} \int_{g_i Q_j} |\nabla u|^n \, dm = K \int_{f^{-1} Q_j} |\nabla u|^n \, dm . \end{aligned}$$

Hence we obtain from (7.18)

$$\operatorname{cap} fE \leqq K \sum_{j=1}^{\infty} \int_{f^{-1}Q_j} |\nabla u|^n \, dm \leqq K \int_A |\nabla u|^n \, dm .$$

Since $u \in W_0^\infty(E)$ was arbitrary, this proves $\operatorname{cap} fE \leqq K \operatorname{cap} E$. Theorem 7.1 is thus completely proved.

8. Applications

In this section we show that the composite mapping of two quasiregular mappings is quasiregular and that $m(B_f) = 0$ for a non-constant quasiregular mapping.

8.1. **Theorem.** *Suppose that* $f : G \to R^n$ *and* $g : G' \to R^n$ *are quasiregular and that* $fG \subset G'$. *Then* $g \circ f : G \to R^n$ *is quasiregular, and*

$$K_O(g \circ f) \leqq K_O(g)\, K_O(f) \,,\; K_I(g \circ f) \leqq K_I(g)\, K_I(f) \,.$$

Proof. If either f or g is constant, the result is trivial. If not, then $g \circ f$ is sense-preserving, discrete, and open. Let E be a condenser in G. Then 7.1 implies

$$\operatorname{cap} gfE \leqq K_I(g) \operatorname{cap} fE \leqq K_I(g) K_I(f) \operatorname{cap} E .$$

Hence, by 7.1, $g \circ f$ is quasiregular and $K_I(g \circ f) \leqq K_I(g) K_I(f)$. Since the corresponding result is well known to be valid for quasiconformal mappings, the inequality for K_O follows from 2.28.

8.2. **Theorem.** *If* $f : G \to R^n$ *is a non-constant quasiregular mapping, then* $J(x, f) > 0$ *a.e.*

Proof. We shall use an argument similar to the proof of Theorem 6 in Gehring [9]. It suffices to show that $A = \{x \in G \mid J(x, f) = 0\}$ has no points of density.

Let $x_0 \in G$, and choose a positive number r_0 such that if $0 < r \leqq r_0$, then $L^*(x_0, f, r) \leqq 2H^*(x_0, f) l^*(x_0, f, r)$ and $U(x_0, f, r)$ is a normal neighborhood of x_0. Fix $r \in (0, r_0]$, and set $U_0 = U(x_0, f, r_0)$, $U = U(x_0, f, r)$, $L^* = L^*(x_0, f, r)$, and $l^* = l^*(x_0, f, r)$. Consider the condenser $E = (U, C)$ where $C = \bar{B}^n(x_0, l^*/2)$. Let $v \in W_0^\infty(fE)$, and define $u : U \to R^1$ by $u(x) = v(f(x))$. Since U is a normal domain, $\operatorname{spt} u \subset U$. Moreover, u is ACL, and $u(x) \geqq 1$ for $x \in C$. Let $P : R^n \to$

R^{n-1} be the orthogonal projection, and set $D = B^{n-1}(P(x_0), l^*/2)$. Then for almost every $z \in D$ we have

$$2 \leqq \int_{U \cap P^{-1}(z)} |\nabla u| \, dm_1 .$$

Integrating over $z \in D$ yields

$$\alpha = 2^{2-n} \Omega_{n-1} l^{*n-1} \leqq \int_U |\nabla u| \, dm .$$

Since $|\nabla u(x)| \leqq |\nabla v(f(x))| \, |f'(x)|$ a.e. and since $f'(x) = 0$ a.e. in A, we obtain

$$\alpha \leqq \int_{U \setminus A} |\nabla v(f(x))| \, |f'(x)| \, dm(x) .$$

By Hölder's inequality this implies

$$\alpha^n \leqq m(U \setminus A)^{n-1} \int_U |\nabla v(f(x))|^n \, |f'(x)|^n \, dm(x)$$

$$\leqq K_O(f) m(U \setminus A)^{n-1} \int_U |\nabla v(f(x))|^n J(x, f) \, dm(x) .$$

By [14, Theorem 3, p. 364], this yields

$$\alpha^n \leqq K_O(f) N(f, U_0) \, m(U \setminus A)^{n-1} \int_{fU} |\nabla v|^n \, dm .$$

Since this holds for all $v \in W_0^\infty(fE)$, we obtain

$$\alpha^n \leqq K_O(f) \, N(f, U_0) \, m(U \setminus A)^{n-1} \operatorname{cap} fE .$$

Here $\operatorname{cap} fE \leqq K_I(f) \operatorname{cap} E$ by 7.1, and $\operatorname{cap} E \leqq \omega_{n-1}/(\log 2)^{n-1}$. Set $B = B^n(x_0, L^*)$. Then $m(B) = \Omega_n L^{*n} \leqq \Omega_n 2^n H^*(x_0, f)^n l^{*n}$, and we obtain

$$m(B) \leqq \beta m(U \setminus A) \leqq \beta m(B \setminus A)$$

where the constant β is independent of r. Hence x_0 cannot be a point of density of A.

8.3. **Theorem.** *If $f : G \to R^n$ is a non-constant quasiregular mapping, then $m(B_f) = 0$.*

Proof. By 2.14 and 2.26, $J(x, f) = 0$ a.e. in B_f. The theorem follows from 8.2.

8.4. **Theorem.** *Suppose that* $f : G \to R^n$ *is a non-constant quasiregular mapping. If* A *is a measurable set in* G, *then* fA *is measurable. Moreover,* $m(fA) = 0$ *if and only if* $m(A) = 0$.

Proof. We express $G \setminus B_f$ as a countable union of domains in which f is injective. Since $m(B_f) = m(fB_f) = 0$, the theorem follows from the corresponding result for quasiconformal mappings.

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Printed October 1969