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DEGREE AND POINT-INVERSES OF MAPPINGS ON SPHERES

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Degree and point-inverses of mappings on spheres

The purpose of this paper is to prove a theorem suggested by considerations in [1], p. 270. Although the theorem has some obvious generalizations (see theorem 2), we shall prove the following original form of it:

Theorem 1. *If* f *is a continuous mapping of the* n*-sphere* S^n *into* S^n *and if* $|\deg(f)| = k > 0$, *then the set of points* $y \in S^n$ *for which* $f^{-1}(y)$ *contains at least* k *points is dense in* S^n.

In what follows, f will always mean a continuous mapping of S^n into S^n. For a point $y \in S^n$, we call $f^{-1}(y)$ the *point-inverse* of y. By a *domain* we shall mean an open, non-empty, connected subset of S^n. The letter D stands as a symbol for a domain. A *neighbourhood* of a point $x \in S^n$, denoted by $U(x)$, is also always assumed to be a domain. The *boundary* of a set $A \subset S^n$ is denoted by ∂A.

The (*local*) *degree* (topological index) of a map f is an integer-valued function of triples (f, D, y) where $D \subset S^n$ is a domain and $y \in S^n - f(\partial D)$. For such a triple the value of the degree is denoted by $\deg(f, D, y)$. We shall need the following properties of the degree (for the definition of the degree and the proofs of the properties see e.g. [1], [2]):

(1) $\deg(f, D, y) = \deg(f, D, y')$, if y and y' belong to the same component of $S^n - f(\partial D)$.

(2) If $\deg(f, D, y) \neq 0$, then $y \in f(D)$.

(3) If $f|\bar{D}$ is injective and if $y \in f(D)$, then $|\deg(f, D, y)| = 1$.

(4) Let $D_1, \ldots, D_k \subset D$ be disjoint and let $y \in S^n - f(\partial D)$ such that $D \cap f^{-1}(y) \subset \bigcup_{i=1}^{k} D_i$. Then

$$\deg(f, D, y) = \sum_{i=1}^{k} \deg(f, D_i, y) .$$

From (1) it follows that $\deg(f, S^n, y)$ has the same value for all $y \in S^n$. This common value is the *global degree* of f, denoted by $\deg(f)$.

For the sake of convenience, we shall make the following additional definition:

Let f be a map, $x \in S^n$ and $y = f(x)$. We call x an *essential point* of f if x has a neighbourhood $U(x)$ such that $\overline{U(x)} \cap f^{-1}(y) = \{x\}$ and $\deg(f, U(x), y) \neq 0$.

Remark 1. Suppose that x is an essential point of f and that $U_1(x)$ and $U_2(x)$ are neighbourhoods of x such that $\overline{U_i(x)} \cap f^{-1}(f(x)) = \{x\}$. Choose a neighbourhood $U_3(x) \subset U_1(x) \cap U_2(x)$. Then (4) implies that

$$\deg(f, U_1(x), y) = \deg(f, U_3(x), y) = \deg(f, U_2(x), y),$$

where $y = f(x)$. Thus $\deg(f, U(x), y)$ has the same value for all $U(x)$ such that that $\overline{U(x)} \cap f^{-1}(y) = \{x\}$.

Remark 2. Suppose that $D \subset S^n$, $y \in S^n - f(\partial D)$ and $\deg(f, D, y) \neq 0$. Assume further that $f^{-1}(y) \cap D$ (which is non-empty by (2)) contains only a finite number of points, say $x_1, \ldots, x_k$. Choose disjoint neighbourhoods $U(x_i) \subset D$, $i = 1, \ldots, k$. (4) implies that

$$0 \neq \deg(f, D, y) = \sum_{i=1}^{k} \deg(f, U(x_i), y) .$$

Thus there must be essential points in $f^{-1}(y) \cap D$. Moreover, we have

$$\deg(f, D, y) = \sum_{j=1}^{h} \deg(f, U(x_{i_j}), y) ,$$

where $x_{i_1}, \ldots, x_{i_h}$ are the essential points of $f^{-1}(y) \cap D$.

For the proof of theorem 1 we still need a lemma:

Lemma. *Let f be a map, $D \subset S^n$, $y_0 \in f(D) - f(\partial D)$, and let $U(y_0) \subset f(D) - f(\partial D)$ be such that for every $y \in U(y_0)$, $f^{-1}(y) \cap D$ is finite and contains exactly one essential point. Then*

$$|\deg(f, D, y_0)| = 1 .$$

Proof. Define a map $g\colon U(y_0) \to D$ by sending every $y \in U(y_0)$ to the essential point of $f^{-1}(y) \cap D$. We shall show that g is continuous. For this purpose, let $y \in U(y_0)$, $x = g(y)$, and let $U(x) \subset D$ be a neighbourhood of x, of which we may assume that $\overline{U(x)} \cap f^{-1}(y) = \{x\}$. Choose $U(y) \subset U(y_0) - f(\partial U(x))$. Then (1) implies that for any $y' \in U(y)$,

$$\deg(f, U(x), y') = \deg(f, U(x), y) \neq 0 .$$

But this means that the essential point of $f^{-1}(y') \cap D$ lies in $U(x)$. Thus $g(U(y)) \subset U(x)$, which proves the continuity of g.

Since g is continuous and certainly also injective, $g(U(y_0))$ is open. Consequently, $x_0 = g(y_0)$ has a neighbourhood $U(x_0)$ such that $\overline{U(x_0)} \subset g(U(y_0))$. Then $f \mid \overline{U(x_0)}$ is injective, and by (3) and remark 2 we have

$$|\deg(f, D, y_0)| = |\deg(f, U(x_0), y_0)| = 1 .$$

The proof of theorem 1. Suppose that there exists an open, non-empty set $B \subset S^n$, such that the point-inverse of any point of B has less than

k points. By remark 2, $f^{-1}(y)$ contains essential points when $y \in B$. Take a point $y_0 \in B$, for which $f^{-1}(y_0)$ contains a maximal number $m(<k)$ of essential points, say $x_1, \ldots, x_m$. Choose disjoint neighbourhoods $U(x_i)$, $i = 1, \ldots, m$ such that $f^{-1}(y_0) \cap \overline{U(x_i)} = \{x_i\}$, $i = 1, \ldots, m$. Then y_0 has a neighbourhood $U(y_0) \subset B$ such that for any $y \in U(y_0)$ we have

$$\deg(f, U(x_i), y) = \deg(f, U(x_i), y_0) \neq 0 ,$$

$i = 1, \ldots, m$. This means that when $y \in U(y_0)$, $f^{-1}(y) \cap U(x_i)$ contains essential points for all $i = 1, \ldots, m$. But since the number of them cannot exceed m there is exactly one of them in each $U(x_i)$. The preceding lemma and remark 2 then yield

$$|\deg(f)| = |\sum_{i=1}^{m} \deg(f, U(x_i), y_0)| \leq \sum_{i=1}^{m} |\deg(f, U(x_i), y_0)| = m < k .$$

This contradiction proves the theorem.

Theorem 1 can be generalized at least to n-dimensional orientable manifolds, for which degree theory can also be defined (see e.g. [3]):

Theorem 2. *Suppose that X and Y are oriented n-manifolds and that $f: \bar{D} \to Y$ is a continuous mapping, where D is a relatively compact domain of X. Let $y_0 \in Y - f(\partial D)$, and let C be the component of $Y - f(\partial D)$ containing y_0. If $|\deg(f, D, y_0)| = k > 0$, then the set of points $y \in C$, for which $f^{-1}(y)$ contains at least k points, is dense in C.*

The proof of the theorem is completely analogous to the one given above.

References

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[3] VÄISÄLÄ, J.: Discrete open mappings on manifolds. - Ann. Acad. Sci. Fenn. A I 392 (1966), 1—10.

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