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443

GRUNSKY TYPE OF INEQUALITIES, AND DETERMINATION OF THE TOTALITY OF THE EXTREMAL FUNCTIONS

BY

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1. Introduction

Starting from the idea of Grunsky there has recently been an important development in extremal problems of univalent functions. In these works the emphasis has been laid upon the functional side, while the question of the extremal function is often left without detailed discussion.

The present paper is concerned with the problem of determining *all* the extremal functions. It appears that, on each occasion when the functional in question can be maximized, the related conditions for the Grunsky parameters are able completely to characterize also the extremal function. This is a remarkable state of things, not encountered in extremal methods based on sequences [3].

In many cases, there pre-exists a conjecture of the extremal function. The state of things mentioned above accordingly provides an indication in attempts to effect further development of the Grunsky type of methods for more advances problems.

Let us concentrate on the class $S(b_1)$. This consists of functions f, for which we suppose that

$$(1) \qquad \begin{cases} f(z) = b_1 z + b_2 z^2 + \cdots ; \\ |z| < 1 \, , \; |f| < 1 \, ; \\ 0 < b_1 \leqq 1 \, . \end{cases}$$

The $S(b_1)$-functions are analytic, univalent and bounded in the above manner in the unit disc. In $S(b_1)$, the first positive coefficient b_1 is kept constant. This means the division of class S of all univalent functions in certain subclasses, which also approximate arbitrarily the unbounded univalent functions S . Clearly, the solution of an extremal problem for all b_1 implies determination of the extremal functional in S also. However, this leaves open the question of all the extremal functions in S. The complete solution in S requires that the corresponding Grunsky type of condition is first transformed to S.

2. The area inequality for $S(b_1)$

Let $g(w)$ be analytic in $\overline{D}$, where D is a simple domain of integration in the w-plane. The starting point of all the inequalities to be used is the integral inequality

$$0 \leqq \iint_D |g'(w)|^2 = \frac{1}{i} \int_{\partial D} \mathrm{Re}\{g(w)\}\, g'(w)\, dw\,. \tag{2}$$

This is a direct consequence of Green's basic formula and remains true also if $\mathrm{Im}\,\{g(w)\}$ is multivalued in D, while $\mathrm{Re}\,\{g(w)\}$ is single valued there.

In the case that $g(w)$ itself is single valued in D, (2) gives

$$0 \leqq \iint_D |g'(w)|^2 = \frac{1}{2i} \int_{\partial D} \overline{g(w)}\, g'(w)\, dw\,. \tag{3}$$

We first use this formula by choosing

$$\partial D = \partial K_1 \cup \gamma \cup -C \cup \gamma\,,$$

in accordance with Figure 1. Here

$$C = f(\partial\, K_r)\,, \quad r < 1\,.$$

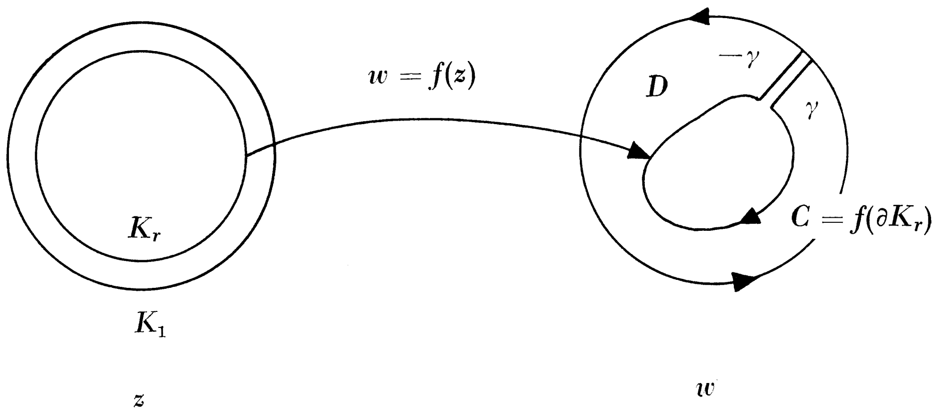

Figure 1.

The function $g(w)$ is choosen as

$$g(w) = b_1\left(\frac{1}{w} - w\right). \tag{4}$$

Since

$$\int_{\partial D} \overline{g(w)}\, g'(w)\, dw = \int_{\partial K_1} \overline{g(w)}\, g'(w)\, dw - \int_C \overline{g(w)}\, g'(w)\, dw$$

and $\int\limits_{\partial K_1}$ vanishes, there is obtained

$$(5)\qquad 0 \leqq \frac{1}{2i}\int\limits_{\partial D}\overline{g(w)}\,g'(w)\,dw = -\frac{1}{2i}\int\limits_{C}\overline{g(w)}\,g'(w)\,dw$$

$$= -\frac{1}{2}\int\limits_{0}^{2\pi}\overline{g(f(z))}\,\frac{d}{dz}\,g(f(z))\,z d\varphi\,,\quad z = re^{i\varphi}\,.$$

We will utilize the expansion

$$(6)\qquad \frac{b_1}{f(z)} = z^{-1} + \sum_{0}^{\infty}\alpha_\nu z^\nu\,,\qquad 0 < |z| < 1\,,$$

which gives

$$\begin{cases} g(f(z)) = z^{-1} + \alpha_0 + \sum\limits_{1}^{\infty}\beta_\nu z^\nu\,, & 0 < |z| < 1\,, \\ \beta_\nu = \alpha_\nu - b_1\,b_\nu & (\nu = 1, 2, \ldots)\,; \end{cases}$$

$$(7)\qquad \left\{\overline{g(f(z))}\,z\,\frac{d}{dz}\,g(f(z))\right\}_{z=re^{i\varphi}} = -r^{-2} + \sum_{1}^{\infty}\nu|\beta_\nu|^2 r^{2\nu} + \sum_{\substack{-\infty \\ (\mu\neq 0)}}^{\infty} K_\mu e^{i\mu\varphi}\,.$$

From this and (5), there follows

$$\sum_{1}^{\infty}\nu|\beta_\nu|^2 r^{2\nu} \leqq \frac{1}{r^2}\,,\qquad 0 < r < 1\,;$$

$$\sum_{1}^{N}\nu|\beta_\nu|^2 r^{2\nu} \leqq \frac{1}{r^2}\,,\qquad 0 < r < 1\,.$$

Passing to the limit by letting, $r \to 1$, $N \to \infty$ is permitted in the present case and gives

$$(8)\qquad \sum_{1}^{\infty}|\alpha_\nu - b_1 b_\nu|^2 \leqq 1\,.$$

Apply now the rotation $\tau^{-1}f(\tau z)$, $|\tau| = 1$, to $f(z)$. This gives

$$(9)\qquad g(\tau^{-1}f(\tau z)) = z^{-1} + \tau\alpha_0 + \sum_{1}^{\infty}\tau^{\nu-1}(\tau^2\alpha_\nu - b_1 b_\nu)\,z^\nu$$

and hence we have got:

Theorem: *If* $f(z) \in S(b_1)$, *the following area-inequality holds*

$$(10)\qquad \sum_{1}^{\infty}|\gamma_\nu|^2 \leqq 1\,,\ \gamma_\nu = \tau^2\alpha_\nu - b_1 b_\nu\,,\ |\tau| = 1\,.$$

Here α_ν*:s are the coefficients of* (6).

The main meaning of (10) is the condition

$$|\gamma_1| \leqq 1 . \tag{11}$$

Equality here is possible exactly for

$$\gamma_2 = \gamma_3 = \cdots = 0 . \tag{12}$$

As will be seen, equality in (11) is actually achieved, and hence the extremal case is characterized by the conditions (12).

Choose τ so that $\tau^2\alpha_1$ is real and negative. The corresponding coefficients of $\tau^{-1}f(\tau z)$ are again denoted by b_ν. In this rotated extremal case, accordingly,

$$\alpha_1 = - |\alpha_1|$$

and (11) gives $|\alpha_1| + b_1^2 \leqq 1$;

$$|\alpha_1| = |a_3 - a_2^2| \leqq 1 - b_1^2 . \tag{13}$$

In the normalized extremal case, we have $\gamma_1 = -1$ and (12) is true. From (9) there follows for the extremal function:

$$g(f) = \frac{1}{z} - a_2 - z \qquad (\alpha_0 = - a_2) ,$$

$$b_1\left(\frac{1}{f} - f\right) = \frac{1}{z} - a_2 - z ,$$

$$\frac{f}{f^2 - 1} = b_1 \frac{z}{z^2 + a_2 z - 1} . \tag{14}$$

This is the necessary condition for the extremal f for which $\gamma_1 = -1$ is achieved. Because (14) actually yields functions of $S(b_1)$ we have checked that (14) is the necessary and sufficient condition for the extremal f. Consequently it gives *all* the extremal functions connected to (13) and normalized by rotation.

The right side of (14) is $\neq \infty$ for $|z| < 1$. This requires that the roots of $z^2 + a_2 z - 1 = 0$ are of the form $z_0 , -\frac{1}{z_0}$ with $|z_0| = 1$. Hence, (14) assumes the form

$$\begin{cases} \dfrac{f}{f^2 - 1} = b_1 \dfrac{z}{(z - z_0)\left(z + \dfrac{1}{z_0}\right)} , \\ a_2 = -\left(z_0 - \dfrac{1}{z_0}\right) = - 2 \operatorname{Im} \{z_0\} i . \end{cases} \tag{15}$$

The function

$$w = \frac{z}{(z - z_0)\left(z + \frac{1}{z_0}\right)}$$

maps the unit circle $K_1(0) : |z| < 1$ on to the w-plane slit along positive and negative imaginary axes, as illustrated in Figure 2.

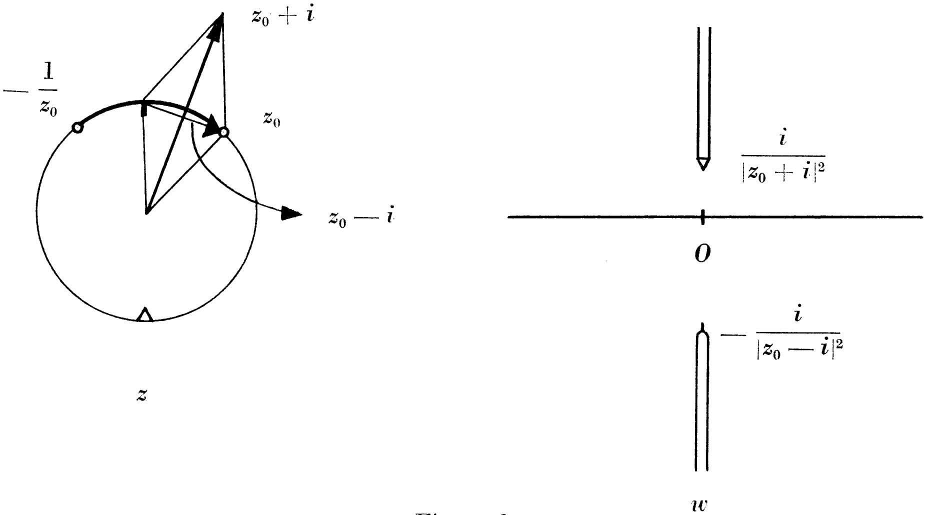

Figure 2.

With this basic mapping taken as the starting point, the maps of the unit circle given by the left and right side of (15) are drawn in Figure 3.

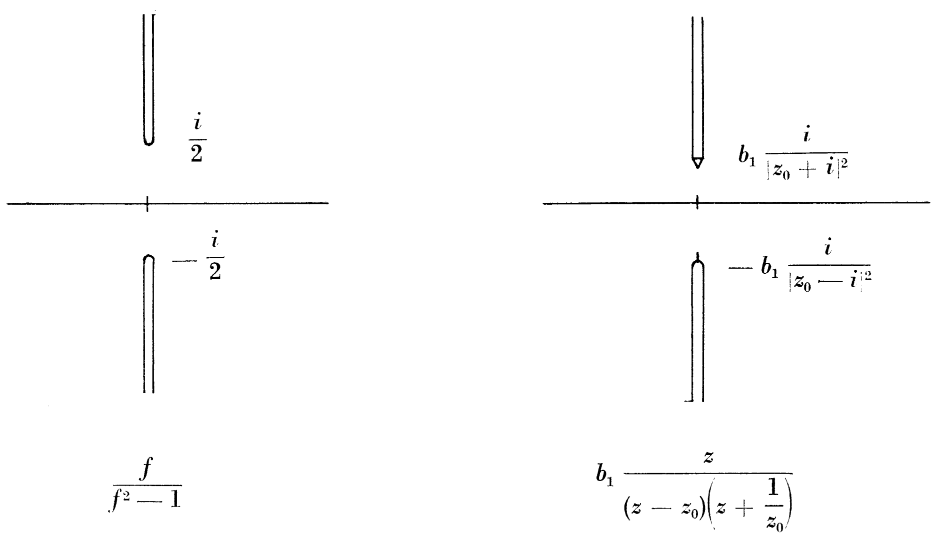

Figure 3.

The requirement needed for a $S(b_1)$-mapping is

$$\max\left(\frac{b_1}{|z_0+i|^2}, \frac{b_1}{|z_0-i|^2}\right) \leqq \frac{1}{2}. \tag{16}$$

If $0 \leqq \operatorname{Im}\{z_0\} \leqq 1$, then $\dfrac{1}{|z_0+i|^2} < \dfrac{1}{|z_0-i|^2}$. Thus, in this case

$$b_1 \frac{1}{|z_0-i|^2} \leqq \frac{1}{2};$$

$$b_1 \leqq \frac{1}{2}|z_0-i|^2 = 1 - \operatorname{Im}\{z_0\},$$

$$0 \leqq \operatorname{Im}\{z_0\} \leqq 1 - b_1. \tag{17}$$

Similarly, for $-1 \leqq \operatorname{Im}\{z_0\} \leqq 0$ we find

$$-(1-b_1) \leqq \operatorname{Im}\{z_0\} \leqq 0. \tag{18}$$

This leads to the extremal domains illustrated in Figure 4.

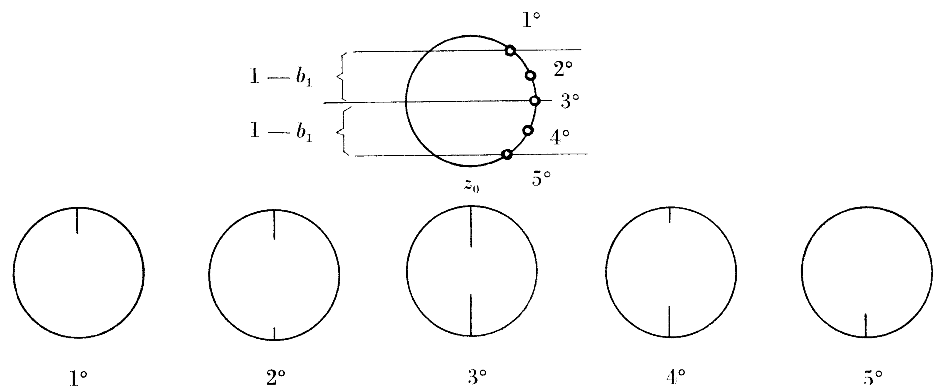

Figure 4.

Theorem. *In* $S(b_1)$, *there holds the inequality*

$$|a_3 - a_2^2| \leqq 1 - b_1^2. \tag{19}$$

Equality holds only for the two-radial slit functions which satisfy

$$\begin{cases} \dfrac{f}{f^2-1} = b_1 \dfrac{z}{(z-z_0)\left(z+\dfrac{1}{z_0}\right)}, \\ a_2 = -\left(z_0 - \dfrac{1}{z_0}\right) = -2\operatorname{Im}\{z_0\}\, i, \\ |z_0| = 1,\ |\operatorname{Im}\{z_0\}| \leqq 1 - b_1. \end{cases} \tag{20}$$

Thus, for each b_1, *there belongs a one parametric family of extremal mappings* (*Figure* 4), *where* a_2 *or* z_0 *is a parameter.*

It should be noticed that in the class S of unbounded functions $f(z) = z + a_2 z^2 + \ldots$, a similar result can be derived from the area inequality, obtained formally from (10) by taking $b_1 = 0$. There holds

$$|a_3 - a_2^2| \leqq 1 ,$$

and the normalized extremal function is

$$f(z) = \frac{z}{z^2 - a_2 z + 1} ,$$

where a_2 is a free real parameter.

In [7], [8], [9] a study was made of the functional $a_3 - \left(1 - \frac{1}{p}\right) a_2^2$ for $0 < p < \infty$. Only for one value of b_1 $(= e^{-p})$ the one parametric family of extremal functions was encountered in this case. The present result is peculiar, since a one parametric family of extremal functions is found to belong to each value of b_1.

3. The generalized Nehari inequality for $N = 1$.

In [5] formula (2) is applied by the choice of D as in Figure 1, and by taking

$$g(w) = x_0 \log w + \sum_{m=1}^{N} \left[\frac{\bar{x}_m}{m} \bar{F}_m(w) - \frac{x_m}{m} F_m\left(\frac{1}{w}\right) \right] . \tag{21}$$

Here x_0 is real and x_m are complex parameters. $F_m(w)$ is the m:th Faber polynomial of f.

For $g(f(z))$, the properties of Faber polynomials give the following development

$$\begin{cases} g(f(z)) = x_0 \log z - \sum\limits_{m=1}^{N} \dfrac{x_m}{m} z^{-m} + \sum\limits_{m=0}^{\infty} C_m z^m , \\ C_m = \sum\limits_{n=0}^{N} (x_n A_{mn} + \bar{x}_n B_{mn}) , \qquad m = 0, 1, 2, \ldots . \end{cases} \tag{22}$$

The coefficients A_{mn}, B_{mn} are certain combinations of the coefficients b_ν of $f(z)$, according to the definitions

$$(23)\qquad \begin{cases} \log \dfrac{f(z)-f(\zeta)}{z-\zeta} = \sum\limits_{m,n=0}^{\infty} A_{mn} z^m \zeta^n , \\ -\log\,(1 - f(z)\,\overline{f(\zeta)}) = \sum\limits_{m,n=1}^{\infty} B_{mn} z^m \bar{\zeta}^n . \end{cases}$$

Expression (21) of g implies that $g(w) \equiv 0$ for $w \in \partial K_1$. Hence (2) gives

$$0 \geqq \frac{1}{2\pi i} \int_C \operatorname{Re}\{g(w)\}\, g'(w)\, dw$$

$$= x_0^2 \log r + x_0 \operatorname{Re}\{C_0\} + \frac{1}{2}\left[-\sum_{m=1}^{N} \frac{|x_m|^2}{m} r^{-2m} + \sum_{m=0}^{\infty} m\, |C_m|^2 r^{2m} \right] .$$

As in the former case, it is deduced from this that

$$(24)\qquad \sum_{m=0}^{\infty} m |C_m|^2 \leqq \sum_{m=1}^{N} \frac{|x_m|^2}{m} - 2x_0 \operatorname{Re}\{C_0\} .$$

Clearly, if x_0 is so chosen that

$$(25)\qquad \operatorname{Re}\{C_0\} = \operatorname{Re}\{\sum_{m=0}^{N} x_m A_{m0}\} = 0 ,$$

then (24) gives

$$(26)\qquad \sum_{m=0}^{N} m |C_m|^2 \leqq \sum_{m=1}^{N} \frac{|x_m|^2}{m}$$

and equality here is possible only for

$$(27)\qquad C_{N+1} = C_{N+2} = \ldots = 0 .$$

It will appear, that those coefficient problems which can be solved by the use of (26) belong to cases (25), (27). Further, these conditions are able completely to characterize the extremal function.

In the general case, one can proceed by estimating the linear combination

$$(28)\qquad S = \sum_{\nu=1}^{N} t_\nu C_\nu$$

with free complex parameters t_ν. This is effected with aid of Schwarz's inequality and (26):

$$|\operatorname{Re}\{S\}|^2 \leqq |\sum_1^N t_\nu C_\nu|^2 \leqq \sum_1^N \frac{|t_\nu|^2}{\nu} \cdot \sum_1^N \nu |C_\nu|^2$$

$$\leqq \sum_1^N \frac{|t_\nu|^2}{\nu} \cdot \sum_1^N \frac{|x_\nu|^2}{\nu} .$$

On specializing $t_\nu = x_\nu$, there is obtained the generalized Nehari inequality

$$(29)\qquad \begin{cases} \operatorname{Re}\{\sum_1^N x_\nu C_\nu\} \leqq \sum_1^N \dfrac{|x_\nu|^2}{\nu}, & \nu = 1, 2, \ldots; \\ \operatorname{Re}\{C_0\} = 0 . \end{cases}$$

Equality here is possible only if (27) is true.

Especial consideration is now given to the case $N = 1$. In [5], the coefficient a_3 was maximized for $e^{-1} \leqq b_1 \leqq 1$ by using the corresponding inequality (29). Since x_1 was chosen as 1, we observe that (21) reduces to the form

$$(30)\qquad g(w) = x_0 \log w + b_1\left(w - \frac{1}{w}\right).$$

This function is accordingly the most natural first generalization of (4) used in derivation of the area inequality. It should further be noticed that the use of Schwarz inequality may be omitted by the direct application of (26), which in the present case gives

$$|C_1| \leqq 1 ;$$

$$(31)\qquad \begin{cases} \operatorname{Re}\{A_{10}\, x_0 + A_{11} + B_{11}\} \leqq 1 , \\ \operatorname{Re}\{x_0\, A_{00} + A_{10}\} = 0 ; \end{cases}$$

$$\begin{cases} \operatorname{Re}\{\, a_2\, x_0 + a_3 - a_2^2 + b_1\} \leqq 1 , \\ x_0 = \dfrac{\operatorname{Re}\{a_2\}}{\log b_1^{-1}} . \end{cases}$$

By rotation normalize $a_3 > 0$ and find

$$a_3 - (1 - b_1^2) \leqq \operatorname{Re}\{a_2^2\} - \frac{[\operatorname{Re}\{a_2\}]^2}{\log b_1^{-1}} ;$$

$$(32)\qquad a_3 - (1 - b_1^2) \leqq \left[1 - \frac{1}{\log b_1^{-1}}\right] [\operatorname{Re}\{a_2\}]^2 - [\operatorname{Im}\{a_2\}]^2 .$$

From this the maximal a_3 for $e^{-1} \leqq b_1 \leqq 1$ is found to be $1 - b_1^2$, and the maximum occurs only for

$$(33)\qquad a_2 = 0 \text{ if } e^{-1} < b_1 \leqq 1 .$$

Since ecuality in (31) is in fact achieved in the maximum case, then in this case also necessarily

$$C_2 = C_3 = \cdots = 0 .$$

According to (33), $x_0 = 0$, and thus

$$C_1 = A_{11} + \bar{B}_{11} = a_3 + b_1^2 = 1 .$$

For the extremal f, presentations (22) and (30) give

$$b_1\left(f - \frac{1}{f}\right) = z - \frac{1}{z} .$$

Consideration is further given to the point $b_1 = e^{-1}$. Now, in the extremal case, a_2 is a free *real* parameter.

$$x_0 = \mathrm{Re}\,\{a_2\} = a_2 ,$$
$$C_1 = x_0 A_{10} + A_{11} + B_{11} = a_2^2 + a_3 - a_2^2 + b_1^2 = 1 .$$

For the extremal f, (22) and (30) now give

$$a_2 \log f + b_1\left(f - \frac{1}{f}\right) = a_2 \log z + z - \frac{1}{z} .$$

Theorem. *In* $S(b_1)$

$$0 < a_3 \leqq 1 - b_1^2 \text{ for } e^{-1} \leqq b_1 \leqq 1 .$$

For the totality of the extremal functions f *the following holds*

(34) $$e^{-1} < b_1 \leqq 1 : f - f^{-1} = b_1^{-1}(z - z^{-1}) ;$$

(35) $$b_1 = e^{-1} : b_1(f - f^{-1}) + a_2 \log f = z - z^{-1} + a_2 \log z .$$

Here, a_2 *is a free real parameter.*

It is known by the Löwner method that the above results (34) and (35) hold at least for some extremal f [8], [9]. The present completion is needed since the Löwner-method, as a sequence procedure, is unable to provide information of all the extremal functions.

In [7], the functional $a_3 - \left(1 - \frac{1}{p}\right) a_2^2$ for $0 < p < \infty$ was maximized. For this (32) gives

$$\mathrm{Re}\left\{a_3 - \left(1 - \frac{1}{p}\right) a_2^2\right\} - (1 - b_1^2)$$
$$\leqq \left[\frac{1}{p} - \frac{1}{\log b_1^{-1}}\right] [\mathrm{Re}\,\{a_2\}]^2 - \frac{1}{p}\,[\mathrm{Im}\,\{a_2\}]^2 .$$

It is checked from this that for $e^{-p} \leqq b_1 \leqq 1$ the totality of extremal functions agrees with those found in [8].

4. The Nehari inequality for $\sqrt{f(z^2)}$ and $N = 3$

In [4] the problem of a_4 in $S(b_1)$ was solved for b_1 close to 1, and close to 0. The result was arrived at by replacing $f(z)$ by the related odd function

$$\sqrt{f(z^2)} = B_1(z + A_3 z^3 + \cdots) .$$

Here

$$\begin{cases} B_1 = b_1^{1/2} , \\ A_{2\nu} = 0 \qquad (\nu = 1, 2, \ldots) , \\ A_3 = \dfrac{a_2}{2} , \\ A_5 = \dfrac{a_3}{2} - \dfrac{a_2^2}{8} , \\ A_7 = \dfrac{a_4}{2} - \dfrac{1}{4} a_2 a_3 + \dfrac{1}{16} a_2^3 . \end{cases}$$

We have to take $N = 3$, $x_0 = x_2 = 0$, $x_3 = 1$. This leaves one free parameter x_1, and $g(w)$ assumes the form

$$g(w) = \bar{x}_1 \bar{F}_1(w) - x_1 F_1\left(\frac{1}{w}\right) + \frac{1}{3}\left[\bar{F}_3(w) - F_3\left(\frac{1}{w}\right)\right] \tag{36}$$

$$= B_1(\bar{x}_1 + A_3)\, w - B_1(x_1 + A_3)\, \frac{1}{w} + \frac{B_1^3}{3}\left(w^3 - \frac{1}{w^3}\right) .$$

According to [4], p. 77, for $\dfrac{19}{34} \leqq b_1 \leqq 1$

$$a_4 \leqq \frac{2}{3}\,(1 - b_1^3) , \tag{37}$$

with equality only for

$$x_1 = a_2 = 0, \quad a_3 = 0 . \tag{38}$$

According to (22), in this case

$$\begin{cases} C_0 = A_{03} , \\ C_1 = A_{13} + B_{13} , \\ C_2 = A_{23} + B_{23} , \\ C_3 = A_{33} + B_{33} . \end{cases}$$

Since for odd $\mu + \nu$

$$A_{\mu\nu} = B_{\mu\nu} = 0 ,$$

we obtain

$$C_0 = C_1 = C_2 = 0 \, ,$$

$$C_3 = A_7 - 3A_3A_5 + \frac{7}{3} A_3^3 + \frac{1}{3} B_1^6 + B_1^2 |A_3|^2 = \frac{a_4}{2} + \frac{1}{3} b_1^3 \, .$$

On a combination of (22) to (36), there is found for the extremal f

$$\frac{b_1^{3/2}}{3} [f(z^2)^{3/2} - f(z^2)^{-3/2}]$$

$$= - \frac{1}{3} z^{-3} + \left[\frac{1}{3} (1 - b_1^3) + \frac{1}{3} b_1^3 \right] z^2 \, ,$$

which implies for $f = f(z)$:

$$\frac{f}{(1 - f^3)^{2/3}} = b_1 \frac{z}{(1 - z^3)^{2/3}} \, . \tag{39}$$

Theorem. *In* $S(b_1)$ $0 < a_4 \leqq \frac{2}{3} (1 - b_1^3)$ *at least for* $\frac{19}{34} \leqq b_1 \leqq 1$. *Equality holds only for the two-radial slit function* f *which satisfies* (39).

Next we want to establish, that for b_1 close enouch to 0 the condition for g with a proper x_1 implies the radial slit mapping f defined by

$$\frac{f}{(1 - f)^2} = b_1 \frac{z}{(1 - z)^2} \, . \tag{40}$$

In [4], [10], there was derived an estimate for a_4 when b_1 is close to 0. This estimate is true in all the other cases but the radial slit case (40). x_1 was chosen to be

$$x_1 = \frac{\operatorname{Re} \left\{ a_3 - \frac{3}{4} a_2^2 \right\} + b_1 \operatorname{Re} \{a_2\}}{2(1 - b_1) - \operatorname{Re} \{a_2\}} \, . \tag{41}$$

This estimate allowed to exclude the corresponding a_4. Thus it was found that the radial slit case (40) was the only possible maximum case. Consequently, the extremal function question is completely solved in this problem. The expression (41) is undetermined for (40). We are interested in the correct value of x_1 needed to determine $g(w)$ belonging to the radial slit case.

In case (40), we have the following coefficients

$$\begin{cases} a_2 = 2 - 2b_1 , \\ a_3 = 3 - 8b_1 + 5b_1^2 , \\ a_4 = 4 - 20b_1 + 30b_1^2 - 14b_1^3 ; \end{cases}$$

$$\begin{cases} A_3 = 1 - b_1 , \\ A_5 = (1 - b_1)(1 - 2b_1) , \\ A_7 = (1 - b_1)(1 - 5b_1 + 5b_1^2) . \end{cases}$$

Take $x_1 = \bar{x}_1$ and write (36) in the form

$$g(w) = B_1 (x_1 + 1 - b_1) \left(w - \frac{1}{w}\right) + \frac{1}{3} B_1^3 \left(w^3 - \frac{1}{w^3}\right) .$$

In the present case (22) gives

$$\begin{cases} C_0 = C_2 = 0 , \\ C_1 = x_1 , \\ C_3 = \dfrac{1}{3} . \end{cases}$$

Hence, for w (22) and (36) imply:

$$3B_1 (x_1 + 1 - b_1)(w - w^{-1}) + B_1^3 (w^3 - w^{-3}) \\ = 3x_1 (z - z^{-1}) + z^3 - z^{-3} .$$

By squaring we obtain from this

$$b_1^3 (f^3 + f^{-3}) + 6b_1^2 (x_1 + \varepsilon)(f^2 + f^{-2}) \\ + [9b_1 (x_1 + \varepsilon)^2 - 6b_1^2 (x_1 + \varepsilon)] (f + f^{-1}) \\ - 2[9b_1 (x_1 + \varepsilon)^2 + b_1^3] \\ = \zeta^3 + \zeta^{-3} + 6x_1 (\zeta^2 + \zeta^{-2}) + (9x_1^2 - 6x_1)(\zeta + \zeta^{-1}) - 2(9x_1^2 + 1) .$$

Here, we have denoted

$$w^2 = f(\zeta) , \; z^2 = \zeta , \; \varepsilon = 1 - b_1 .$$

Finally, compare the result with the condition

$$b_1^3 (f + f^{-1} - 2)^3 = (z + z^{-1} - 2)^3$$

obtained from (40). This shows that complete identity is achieved by taking

$$x_1 = -1 .$$

Result. *In the inequality method for* $\sqrt{f(z^2)}$ *with* $N = 3$ *conditions* (21) *and* (22) *determine the radial slit mapping* f *by the choice of*

$$x_0 = x_2 = 0 \, , \, x_3 = 1 \, ; \, x_1 = -1 \, . \tag{42}$$

5. The generalized Nehari inequality for $N = n$ and $a_2 = \ldots = a_n = 0$.

In [6], there was solved the problem of maximizing a_n when b_1 is close to one. In particular, for a_{2n+1} with the side conditions $a_2 = \ldots = a_n = 0$ the extremal conditions $a_{n+1} = \ldots = a_{2n} = 0$ were determined. Let us check the uniqueness of the extremal domain in this case.

From the recursion formula

$$\sum_1^\infty \frac{1}{\nu} F_\nu(t) \, z^\nu = -\log\,(1 - t f(z)) = \sum_1^\infty \frac{1}{\nu} t^\nu f(z)^\nu \tag{43}$$

for the Faber polynomials, there follows for the function

$$f(z) = b_1 \, (z + a_{2n+1} \, z^{2n+1} + \ldots) \tag{44}$$

in question

$$F_\nu(t) = b_1^\nu \, t^\nu \quad (\nu = 1, \ldots, n) \, . \tag{45}$$

Because

$$a_2 = \ldots = a_{2n} = 0 \, , \, a_{2n+1} = \frac{1}{n} \, (1 - b_1)^{2n} \, ,$$

we get for the coefficients A_{mn} and B_{mn} of (23), according to the formulae of [6]

$$\begin{cases} A_{nn} = a_{2n+1} = \dfrac{1}{n} \, (1 - b_1^{2n}) \, , \\ A_{ik} = 0 \, , \, i = 0, \ldots, n-1 \, ; \, i \leqq k = 0 \, , \ldots n \, , \\ B_{ik} = 0 \, , \, 0 \leqq i < k \leqq n \, , \\ B_{kk} = \dfrac{1}{k} \, b_1^{2k} \, , \, k = 1, \ldots, n \, . \end{cases} \tag{46}$$

According to [6], in the maximum case

$$x_0 = x_1 = \ldots = x_{n-1} = 0, \, x_n = 1 \, .$$

Thus, from (21)

$$g(w) = \frac{1}{n} \bar{F}_n(w) - \frac{1}{n} F_n\left(\frac{1}{w}\right) = \frac{b_1^n}{n} \, (w_n - w^{-n}) \, . \tag{47}$$

From (22) we get

$$C_1 = \ldots = C_{n-1} = 0\,,\ C_n = A_{nn} + B_{nn} = \frac{1}{n}\,;$$

$$g(f(z)) = -\frac{1}{n} z^{-n} + \frac{1}{n} z^{n}\,. \tag{48}$$

Thus, comparison of (47) and (48) yields

$$b_1^n (f^n - f^{-n}) = z^n - z^{-n}\,;$$

$$\frac{f}{(1-f^{2n})^{\frac{1}{n}}} = b_1 \frac{z}{(1-z^{2n})^{\frac{1}{n}}}\,.$$

The case a_{2n} with $a_2 = \ldots = a_n = 0$ is further solved in [6]. Determination of the extremal function succeeds in the above manner. Thus, we arrive at the conclusion.

Theorem. *In* $S(b_1)$ *the problem of maximizing* a_{2n+1} *with the side conditions* $a_2 = \ldots = a_n = 0$ *leads to the only extremal function which satisfies*

$$\frac{f}{(1-f^{k-1})^{\frac{2}{k-1}}} = b_1 \frac{z}{(1-z^{k-1})^{\frac{2}{k-1}}} \tag{49}$$

for $k = 2n+1$ $(n = 1, 2, \ldots)$ *and*

$$e^{-\frac{2}{n+1}} \leqq b_1 \leqq 1\,.$$

Similarly, the problem of maximizing a_{2n} *with* $a_2 = \ldots = a_n = 0$ *has the extremal function determined by* (49) *for* $k = 2n$ $(n = 1, 2, \ldots)$ *and* $0 < b_1 \leqq 1$.

6. Discussion on the choice of $g(w)$.

Finally, let us discuss about modifications of the function $g(w)$. We omit the question of irrational functions, which evidently is needed for a_3 with b_1 close to 0. We ask here the meaning of the most natural generalization of the above use of *Faber polynoms* (3°). By this is meant the procedure, in which F_m is replaced by a *general polynom of m:th degree* (1°). This choice is compared with the *power method* (2°), which is obtained by replacing F_m simply by w^m. All these choices appear to be mutually equivalent.

1°. The polynom method.

We omit the effect of the term $x_0 \log w$ and consider the combination

$$g(w) = \sum_1^N \left[\bar{y}_m \bar{P}_m(w) - y_m P_m\left(\frac{1}{w}\right)\right], \tag{50}$$

where

$$P_m(w) = \sum_{\nu=1}^{m} C_{m\nu} w^\nu \tag{51}$$

is a polynom of m:th degree and has free complex coefficients. The numbers y_m are supposed to be free complex parameters. This freedoom of y_m and C_m leaves for the coefficients of

$$y_m P_m(w) = \sum_{\nu=1}^{m} (y_m C_{m\nu}) w^\nu$$

the role of new free parameters. This shows us, that the most general polynom method is arrived at by taking

$$g(w) = \sum_{m=1}^N \left[\bar{P}_m(w) - P_m\left(\frac{1}{w}\right)\right] \tag{52}$$

with free complex coefficients $C_{m\nu}$.

2°. The power method.

Rearrange the sum of (52) as follows:

$$\sum_{m=1}^N P_m(w) = \sum_{m=1}^N \sum_{\nu=1}^N C_{m\nu} w^\nu = \sum_{m=1}^N \left(\sum_{\nu=m}^N C_{\nu m}\right) w^m . \tag{53}$$

The freedoom of the numbers $C_{m\nu}$ further shows, that the only effective free parameters in (52) are

$$t_m = \sum_{\nu=m}^N C_{\nu m} \qquad (m = 1, \ldots, N) . \tag{54}$$

Accordingly, instead of (52) we are led to the equivalent choice

$$g(w) = \sum_{m=1}^N (\bar{t}_m w^m - t_m w^{-m}) . \tag{55}$$

3°. Connection with Faber polynom method.

In particular, start now from the Faber polynom form for g (equation (21)):

$$g(w) = \sum_{m=1}^N \left[\frac{\bar{x}_m}{m} \bar{F}_m(w) - \frac{x_m}{m} F_m\left(\frac{1}{w}\right)\right]. \tag{56}$$

This means that in (52) we take

$$P_m(w) = \frac{x_m}{m} F_m(w) \,. \tag{57}$$

The m:th Faber polynom belonging to f is written

$$F_m(w) = \sum_{\nu=1}^{m} k_\nu^{(m)} w^\nu \qquad (m = 1, 2, \ldots) \,. \tag{58}$$

Thus we obtain

$$\sum_{m=1}^{N} \bar{P}_m(w) = \sum_{m=1}^{N} \frac{\bar{x}_m}{m} \bar{F}_m(w) = \sum_{m=1}^{N} \frac{\bar{x}_m}{m} \sum_{\nu=1}^{m} \overline{k_\nu^{(m)}} w^\nu$$

$$= \sum_{m=1}^{N} \left(\sum_{\nu=m}^{N} \frac{\bar{x}_\nu}{\nu} \overline{k_m^{(\nu)}} \right) w^m \,.$$

This shows that we are led to the form (55) by taking as new complex parameters

$$t_m = \sum_{\nu=m}^{N} x_\nu \frac{1}{\nu} k_m^{(\nu)} \qquad (m = 1, \ldots, N) \,. \tag{59}$$

Clearly, the connection (59) between the complex parameter spaces

$$C^{(N)} = \{t = (t_1, \ldots, t_N) \mid t_\nu \in C\} \,,$$

$$C^{(N)} = \{x = (x_1, \ldots, x_N) \mid x \in C\}$$

is surjective.

Result. *Consider the methods* 1°, 2°, 3° *defined by the choices* (52), (55), (56) *of function* $g(w)$. *These methods are so connected with each other that*

$$1° \Rightarrow 2° \Leftrightarrow 3° \,.$$

As a conclusion, it may be noted that in Grunsky type of inequalities the use of Faber polynoms may be avoided by the simple power choice (55). Furthermore, to construct more effective choices of g than 1°, 2°, 3°, $g(w)$ must be extended outside the range of polynoms. The coice (21) with the additional term $x_0 \log w$ provides an example of this. Additional examples of extensions of this kind are given by [1] and [2].

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