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SYMMETRIZATION AND EXTREMAL RINGS IN SPACE

BY

G. D. ANDERSON

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SUOMALAINEN TIEDEAKATEMIA

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KESKUSKIRJAPAINO
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1. Introduction

Suppose that f is a diffeomorphism of a 3-space domain Ω onto Ω'. Then f is locally affine; that is, if $P \varepsilon \Omega$ the differential mapping $df(P)$ carries the unit ball onto an ellipsoid with axes of lengths $a \geq b \geq c$. The dilatation functions

$$(1) \qquad H_I(P, f) = \left[\frac{ab}{c^2}\right]^{\frac{1}{2}}, \; H_0(P, f) = \left[\frac{a^2}{bc}\right]^{\frac{1}{2}}$$

measure how much infinitesimal balls are distorted, hence providing a natural measure of how much f differs from being conformal at P. These functions are bounded below by 1, and are 1 at a point P if and only if f is conformal there. We say that f is *quasiconformal* if either, and hence both, of these dilatations is bounded above in Ω.

One research goal in the study of quasiconformal mappings is to determine their distortion properties. This can be accomplished by assigning to each ring R a modulus $\operatorname{mod} R$ which is invariant under conformal (Möbius) transformations and which has the property that for each quasiconformal mapping f there is a number $K = K(f)$, $1 \leq K < \infty$, with

$$(2) \qquad \frac{1}{K} \operatorname{mod} R \leq \operatorname{mod} f(R) \leq K \operatorname{mod} R .$$

If one can show that among all rings with a certain geometric property a particular ring is extremal, that is, has the maximum modulus, then this fact can be used to determine distortion properties for quasiconformal mappings (Cf. [10], [11]).

It is comparatively easy to prove that certain plane rings are extremal, because one can employ conformal mappings [22]. Frequently it is also intuitively evident which rings in space ought to be extremal, but since the only conformal mappings in E^3 are the Möbius transformations (Cf. [11, § 29]), the proofs there become more difficult. The only method so far

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successful has been symmetrization. In § 2 of this paper we define the Steiner and Schwarz symmetrizations of space rings and state theorems, proved in [2], that these processes do not decrease the moduli of space rings. These symmetrization theorems enable us to show in § 3 that two given rings are extremal.

These extremal rings, designated by $R_{3,d}$ and $R_{3,s}$, may be described as follows. For fixed a, $0 < a < 1$, the ring $R_{3,d} = R_{3,d}(a)$ consists of the unit ball minus the disk $|x| \leq a, x_3 = 0$, while $R_{3,s} = R_{3,s}(a)$ is obtained from the unit ball by omission of the slit $|x_1| \leq a, x_2 = x_3 = 0$. In § 3 we show that these rings have the following extremal properties. Let R be a space ring consisting of the unit ball minus a continuum C. If the projection of C on some diametral plane is at least πa^2 in area, then $\operatorname{mod} R \leq \operatorname{mod} R_{3,d}(a)$. If the diameter of C is at least $2a$, then $\operatorname{mod} R \leq \operatorname{mod} R_{3,s}(a)$. In the same section we show that if the complement of a space ring R_3 lies in a plane Π and forms there a plane ring R_2, then point symmetrization in Π induces an operation on R_3 which does not decrease $\operatorname{mod} R_3$.

We continue the study of these extremal rings in § 4. We first show that $\operatorname{mod} R_{3,d} \leq \operatorname{mod} R_2$, where now R_2 denotes the plane ring consisting of the unit disk $|x_1 + ix_2| < 1$ minus the central slit $|x_1| \leq a$, $x_2 = 0$. This, together with a reference to [9], completes the double inequality $\operatorname{mod} R_{3,d} \leq \operatorname{mod} R_2 \leq \operatorname{mod} R_{3,s}$. In § 5 we obtain upper and lower bounds in terms of a for $\operatorname{mod} R_{3,d}(a)$ and $\operatorname{mod} R_{3,s}(a)$. Our main tool is the use of inequalities for elliptic functions [3]; these have also proved useful in [1]. In § 6 we use the bounds obtained in § 5 to study the asymptotic behavior of the moduli of these rings as a tends to 0. In § 7 we introduce a generalized notion of quasiconformality, together with some material on extremal lengths, and in the final two sections we employ this theory to investigate the behavior of $\operatorname{mod} R_{3,d}$ and $\operatorname{mod} R_{3,s}$ as a tends to 1. We discover that these moduli behave essentially like the modulus of the spherical annulus $a < |x| < 1$ as a tends to 0. But as a tends to 1 the asymptotic behavior of these three rings is markedly different one from another.

2. Symmetrization of space rings

2.1. *Space rings. A space ring* R is a domain in E^3 whose complement consists of a bounded component C_0 and an unbounded component C_1. The *conformal capacity* of R is defined [16] as

$$\operatorname{cap} R = \inf_u \int_R |\nabla u|^3 \, d\omega ,$$

where the infimum is taken over all real-valued functions $u = u(x)$ which are continuously differentiable in R and have boundary values 0 on ∂C_0 and 1 on ∂C_1.

Next, the *modulus* of R is defined [11] as

$$\text{mod}\, R = \left[\frac{4\pi}{\text{cap}\, R}\right]^{\frac{1}{2}}. \tag{3}$$

This is analogous to the modulus of a plane ring, usually defined by means of conformal mappings. The modulus is invariant under conformal (Möbius) transformations and satifies an inequality of the type (2) for each quasiconformal mapping f. If R is the spherical annulus $r_1 < |x| < r_2$, then the modulus of R is $\log r_2/r_1$ [13].

2.2. *Symmetrization methods.* Symmetrization is a geometric operation invented by Jacob Steiner and developed by Pólya and Szegö [19]. Two well-known kinds of symmetrization in the plane are the Steiner and point symmetrizations. In this section we consider analogues of these in 3-space — known as the Steiner and Schwarz symmetrizations [19], respectively — in which the corresponding plane symmetrization is performed in each plane normal to the x_3 axis.

If R is a bounded space ring and R' is obtained from it by one of these symmetrizations then $\text{mod}\, R \leq \text{mod}\, R'$. This inequality was proved by Gehring [13] for spherical and point symmetrization. The proofs for the Steiner and Schwarz symmetrizations, while embodying certain additional technical difficulties, follow the outline of Gehring's argument; proofs in detail are included in [2]. Similar results for radial symmetrization have been obtained by Pfaltzgraff [18].

2.3. *Steiner symmetrization of rings.* Let G be a bounded open set in E^3. We define a second set G^*, called the *Steiner symmetrization* of G with respect to the $x_1 x_2$ plane, as follows: Let $L = L(x_1, x_2)$ denote the line in E^3 through $(x_1, x_2, 0)$ that is parallel to the x_3 axis. Then $L \cap G^* = \emptyset$ if and only if $L \cap G = \emptyset$. If $L \cap G \neq \emptyset$, then $L \cap G^*$ is an open segment of length $m_1(L \cap G)$ which is bisected by the $x_1 x_2$ plane.

If F is a bounded closed set in E^3, we define F^* as above except in the second case, where we take $L \cap F^*$ to be a closed segment of length $m_1(L \cap F)$ which is bisected by the $x_1 x_2$ plane. If $m_1(L \cap F) = 0$, then $L \cap F^*$ is the single point $(x_1, x_2, 0)$.

If G and F are a bounded domain and a continuum, respectively, it is easily verified that G^* and F^* have the same properties. It is also easily shown that if E is a bounded open or closed set, then $C(E^*)$ is connected. Hence if R is a bounded ring in E^3 and C_0 and C_1 are the two components of $C(R)$ then the set

$$R^* = (R \cup C_0)^* - C_0{}^*$$

is a ring, and we define this to be the *Steiner symmetrization* of R. Then R^* has the following extremal property [2].

Theorem 1. *Let R be any bounded ring in E^3, and let R^* denote its Steiner symmetrization. Then* $\operatorname{mod} R \le \operatorname{mod} R^*$.

2.4. *Schwarz symmetrization of rings.* Let G be a bounded open set in E^3. Then G^{**}, the *Schwarz symmetrization* of G with respect to the x_3 axis, is defined as follows. Let $\Pi = \Pi(x_3)$ denote the plane through $(0, 0, x_3)$ that is normal to the x_3 axis. Then $\Pi \cap G^{**} = \emptyset$ if and only if $\Pi \cap G = \emptyset$. If $\Pi \cap G \neq \emptyset$, we take $\Pi \cap G^{**}$ to be an open disk of area $m_2(\Pi \cap G)$ with center on the x_3 axis.

If F is a bounded closed set in E^3, we define F^{**} as above except in the second case, where we take $\Pi \cap F^{**}$ to be a closed disk of area $m_2(\Pi \cap F)$ with center on the x_3 axis. If $m_2(\Pi \cap F) = 0$, then $\Pi \cap F^{**}$ is the single point $(0, 0, x_3)$.

If R is a bounded ring in E^3 and C_0 and C_1 are the two components of $C(R)$, then it is easily verified that the set

$$R^{**} = (R \cup C_0)^{**} - C_0^{**}$$

is a ring, and we define this to be the *Schwarz symmetrization* of R. It can be shown that R^{**} enjoys the following extremal property [2].

Theorem 2. *Let R be any bounded ring in E^3, and let R^{**} denote its Schwarz symmetrization. Then* $\operatorname{mod} R \le \operatorname{mod} R^{**}$.

3. Extremal space rings

3.1. *Space rings with complement in a plane.* An interesting type of ring R_3 in 3-space is one for which both components of $C(R_3)$ lie in a plane Π, say $x_3 = 0$, and for which the configuration $R_2 = \Pi \cap R_3$ is a plane ring.

If R_2 is a plane ring and C is the bounded component of $C(R_2)$, then point symmetrization in the plane replaces R_2 by a circular annulus $R_2' : r_1 < |x_1 + i x_2| < r_2$ with $m_2(C) = \pi r_1^2$ and $m_2(R \cup C) = \pi r_2^2$. It is known that $\operatorname{mod} R_2 \le \operatorname{mod} R_2'$ (Cf. [6]). But this process of plane symmetrization also replaces R_3 by a new space ring R_3' with $C(R_3') = C(R_2')$, and we shall show that the plane symmetrization increases the space modulus also. In the proof of this result we shall need the following.

Lemma 1. *Let* R_3 *be an unbounded space ring with nondegenerate boundary components* C_0 *and* C_1. *Given* $\varepsilon > 0$, *there exists a bounded ring* R *separating the components of* $C(R_3)$ *for which*

$$\text{mod}\, R_3 < (1 + \varepsilon)^{\frac{1}{2}} \,\text{mod}\, R \,. \tag{4}$$

Proof. This proof assumes a certain familiarity with the terminology of [13]. Since the boundary components of R_3 are non-degenerate, it follows by [16] that $\text{cap}\, R_3 > 0$. Therefore there exists a simple admissible function u [13, § 7] for R_3 such that

$$(1 + \varepsilon)\,\text{cap}\, R_3 > \int_{R_3} |\nabla\, u|^3 \, d\omega \,.$$

Now let E_i be the component of $\{x : u(x) = i\}$ which contains C_i, $i = 0, 1$. Since $u(x) = 1$ for sufficiently large $|x|$, $C(E_1)$ must be bounded. Next, we see that E_0 and E_1 are disjoint continua. Hence by Lemma 3.5 of [14] there exists a ring R, with C'_0 and C'_1 as the components of $C(R)$, such that $\partial C'_i \subset E_i \subset C'_i$ for $i = 0, 1$. Because $R \subset C(E_1)$, R is bounded. Since $C_i \subset E_i \subset C'_i$, R separates the components of $C(R_3)$. Finally, since $\partial C'_i \subset E_i$, $u = 0$ on $\partial C'_0$ and 1 on $\partial C'_1$. Thus u is an admissible function [13, § 3] for R and hence

$$\text{cap}\, R \le \int_{R} |\nabla\, u|^3 \, d\omega \le \int_{R_3} |\nabla u|^3 \, d\omega < (1 + \varepsilon)\,\text{cap}\, R_3 \,,$$

from which (4) follows.

Theorem 3. *Let* R_3 *be a space ring such that both components of* $C(R_3)$ *lie in a plane* Π *and determine a plane ring* R_2 *there. Let* R'_3 *be the space ring obtained by point symmetrizing the plane ring* R_2. *Then* $\text{mod}\, R_3 \le \text{mod}\, R'_3$.

Proof. For convenience let Π be the plane $x_3 = 0$. We may assume that $\text{mod}\, R_3 < \infty$, for otherwise [16] shows that $\text{mod}\, R'_3 = \infty$ and there is nothing to prove.

Given $\varepsilon > 0$, by Lemma 1 there exists a bounded ring R separating the components of $C(R_3)$ for which (4) holds. Then by Theorem 2,

$$\text{mod}\, R \le \text{mod}\, R^{**} \,, \tag{5}$$

where R^{**} is the Schwarz symmetrization of R with respect to the x_3 axis. But since R^{**} separates the boundary components of R'_3, Lemma 2 of [13] yields

(6) $$\operatorname{mod} R^{**} \leq \operatorname{mod} R_3' .$$

Combining (4), (5), and (6) we have $(1+\varepsilon)^{-\frac{1}{2}} \operatorname{mod} R_3 < \operatorname{mod} R_3'$, from which the theorem follows when we let ε approach zero.

3.2. *Extremal rings $R_{3,d}$ and $R_{3,s}$.* Now fix a, $0 < a < 1$. Throughout the rest of this paper R_2 will denote the plane ring consisting of the unit disk minus the symmetric slit $|x_1| \leq a$, $x_2 = 0$. By $R_{3,d}$ and $R_{3,s}$ we shall mean the space rings obtained from R_2 by rotation about the x_2 and x_1 axes, respectively. The ring $R_{3,d} = R_{3,d}(a)$ consists of the unit ball minus the closed central disk $|x| \leq a, x_3 = 0$, while the ring $R_{3,s} = R_{3,s}(a)$ consists of the unit ball minus the slit $|x_1| \leq a, x_2 = x_3 = 0$.

The ring $R_{3,d}$ is extremal in the following sense.

Theorem 4. *Let R be any space ring consisting of the unit ball minus a continuum C, and suppose that the projection of C on some diametral plane Π is at least πa^2 in area, $0 < a < 1$. Then* $\operatorname{mod} R \leq \operatorname{mod} R_{3,d}(a)$.

Proof. For convenience let Π be the plane $x_3 = 0$, and let R^* be the Steiner symmetrization of R with respect to Π. Then by Theorem 1, $\operatorname{mod} R \leq \operatorname{mod} R^*$. Now replace the continuum C by its projection on the $x_1 x_2$ plane. This yields a new ring R', and by the monotoneity of the space modulus [13, Lemma 2] $\operatorname{mod} R^* \leq \operatorname{mod} R'$. Finally, Schwarz symmetrization with respect to the x_3 axis replaces R' by a ring $R_{3,d}(b)$ for some b, $0 < a \leq b < 1$. But then by Theorem 1 and monotoneity we have $\operatorname{mod} R' \leq \operatorname{mod} R_{3,d}(b) \leq \operatorname{mod} R_{3,d}(a)$, and the proof is complete.

The ring $R_{3,s}$ enjoys the following extremal property.

Theorem 5. *Let R be any space ring consisting of the unit ball minus a continuum C whose diameter is at least $2a, 0 < a < 1$. Then* $\operatorname{mod} R \leq \operatorname{mod} R_{3,s}(a)$.

Proof. Let P_1 and P_2 be points of C such that $|P_1 - P_2| = 2b$, $b \geq a$. For convenience we may assume that $P_1 P_2$ is parallel to the x_1 axis. The Schwarz symmetrization of R with respect to the x_1 axis yields a new ring R' and, by Theorem 2, $\operatorname{mod} R \leq \operatorname{mod} R'$. This inequality follows also from Gehring's result on spherical symmetrization [13, Theorem 1] or the work of Šabat in [21].

Now the bounded component C' of $C(R')$ contains a segment $p \leq x_1 \leq p + 2b, x_2 = x_3 = 0$, and if C' is replaced by this segment a new ring R'' results such that $\operatorname{mod} R' \leq \operatorname{mod} R''$. Finally, Steiner symmetrization of R'' with respect to the $x_2 x_3$ plane yields $R_{3,s}(b)$ and, by Theorem 1 and monotoneity, $\operatorname{mod} R'' \leq \operatorname{mod} R_{3,s}(b) \leq \operatorname{mod} R_{3,s}(a)$.

4. A double inequality

In this section we begin our investigation of the properties of mod $R_{3,d}$ and mod $R_{3,s}$. Our first goal is the following result.

Theorem 6. *Let* R_2 *be the plane ring consisting of the unit disk minus the slit* $|x_1| \leq a, x_2 = 0$, *and let* $R_{3,d}$ *and* $R_{3,s}$ *be the space rings obtained by rotating* R_2 *about the* x_2 *and* x_1 *axes, respectively. Then*

$$\text{mod}\, R_{3,d} \leq \text{mod}\, R_2 \leq \text{mod}\, R_{3,s}\,. \tag{7}$$

Proof. For the first inequality in (7) we shall obtain a diffeomorphism f_d of a spherical annulus $A < |x| < 1$ onto $R_{3,d}$. This mapping can be shown to have positive Jacobian $J(x)$ and to map each radius of $A < |x| < 1$ onto a curve that is normal to the image of each surface $|x| = r$, $A < r < 1$. Under these conditions it follows from a result in [9] that

$$\int_A^1 D_1(f_d, r)\frac{dr}{r} \leq \text{mod}\, R_{3,d} \leq \int_A^1 D_2(f_d, r)\frac{dr}{r}, \tag{8}$$

where, for $A < r < 1$,

$$D_1(f_d, r) = \min_{|x|=r} \left[\frac{N(x)^3}{J(x)}\right]^{\frac{1}{2}}, \quad D_2(f_d, r) = \max_{|x|=r} \left[\frac{N(x)^3}{J(x)}\right]^{\frac{1}{2}}.$$

Here $N(x)$ is the stretching normal to $|x| = r$.

We shall then use some inequalities for elliptic functions to prove that $D_2(f_d, r) \leq 1$ and finally apply the second half of (8) to achieve the first half of (7). The functional $D_1(f_d, r)$ and the first half of (8) will be needed in the proof of a theorem in § 5. The second half of (7) follows directly from an inequality obtained by Gehring in [9].

The plane annulus $A < |x_1 + ix_2| < 1$ is mapped conformally (See [5, p. 28] or [17, pp. 280—295]) onto R_2 by the Jacobian elliptic sine function

$$y_1 + iy_2 = f(x_1 + ix_2) = k^{\frac{1}{2}}\, sn(i(u + iv), k), \tag{9}$$

where

$$u + iv = \frac{2K}{\pi} \log \frac{x_1 + ix_2}{A} - iK, \; A = \exp \frac{-\pi K'}{4K}, \; k = a^2, \tag{10}$$

$\log (x_1 + ix_2)$ denotes the principal branch of the logarithm, and

$$K = K(k) = \int_0^1 [(1-t^2)(1-k^2t^2)]^{-\frac{1}{2}}\, dt\,, \tag{11}$$
$$K' = K(k')\,,\ k' = (1-k^2)^{\frac{1}{2}}\,.$$

For our purpose it will be convenient to employ Jacobi's imaginary transformation $sn(iz, k) = i\, tn(z, k')$ ([4, p. 37], [5, # 125.02]) to rewrite (9) as

$$y_1 + iy_2 = f(x_1 + ix_2) = ik^{\frac{1}{2}}\, tn(u + iv\,, k'). \tag{12}$$

We define the space mapping $y = f_d(x)$ from $A < |x| < 1$ onto $R_{3,d}$ by rotating $A < |x_1 + ix_2| < 1$ and R_2 about the x_2 and y_2 axes. That is, we let

$$t + iy_2 = f(s + ix_2)\,,\ \varphi = \Theta\,, \tag{13}$$

where (s, Θ) and (t, φ) are polar coordinates in the x_1x_3 and y_1y_3 planes, respectively. It is easily verified that this mapping satisfies the hypotheses for (8) and that

$$D_1(f_d\,, r) = \min_{|x_1+ix_2|=r} \left|\frac{x_1}{y_1} f'(x_1 + ix_2)\right|^{\frac{1}{2}}, \tag{14}$$
$$D_2(f_d\,, r) = \max_{|x_1+ix_2|=r} \left|\frac{x_1}{y_1} f'(x_1 + ix_2)\right|^{\frac{1}{2}}.$$

Now by (12) and the differentiation formula $\frac{d}{dz}(tn\, z) = (dn\, z)/(cn^2\, z)$ ([4, p. 9], [5, # 731.10]) we have

$$|f'| = \frac{2K}{\pi r} k^{\frac{1}{2}} \left|\frac{dn(u + iv\,, k')}{cn^2(u + iv\,, k')}\right|,$$

while the addition theorems ([4, p. 38], [5, # 125.01]) give

$$|f'| = \frac{2K}{\pi r} k^{\frac{1}{2}} \frac{(d^2C^2D^2 + k'^4s^2c^2S^2)^{\frac{1}{2}}(1 - S^2d^2)}{c^2C^2 + s^2d^2S^2D^2} \tag{15}$$

and

$$y_1 = -\,k^{\frac{1}{2}}\,\mathrm{Im}\,\frac{sD - icdSC}{cC + isdSD} = k^{\frac{1}{2}}\frac{Sd(c^2C^2 + s^2D^2)}{c^2C^2 + s^2d^2S^2D^2}\,, \tag{16}$$
$$y_2 = k^{\frac{1}{2}}\,\mathrm{Re}\,\frac{sD - isdSC}{cC + isdSD} = k^{\frac{1}{2}}\frac{scCD(1 - S^2d^2)}{c^2C^2 + s^2d^2S^2D^2}\,.$$

Here we have used the notation

(17)

$$s = sn\left(\frac{2K}{\pi}\log\frac{r}{A}, k'\right), \; c = dn\left(\frac{2K}{\pi}\log\frac{r}{A}, k'\right), \; d = dn\left(\frac{2K}{\pi}\log\frac{r}{A}, k'\right),$$

$$S = sn\left(\frac{2K}{\pi}\psi, k\right), \; C = cn\left(\frac{2K}{\pi}\psi, k\right), \; D = dn\left(\frac{2K}{\pi}\psi, k\right),$$

where $\left(r, \frac{\pi}{2} - \psi\right)$ are polar coordinates in the $x_1 x_2$ plane.

But using the identities $C^2 = 1 - S^2$, $D^2 = 1 - k^2 S^2$, $c^2 + s^2 = 1$, $c^2 + k^2 s^2 = d^2$ ([4, p. 9], [5, # 121.00]) we achieve

$$c^2 C^2 + s^2 D^2 = 1 - S^2 d^2 . \tag{18}$$

Hence from (15), (16), (18), and the fact that $x_1 = r \sin \psi$, there results

$$\left|\frac{x_1}{y_1} f'\right| = \frac{2K}{\pi} |\sin \psi| \left[\frac{C^2 D^2}{S^2} + k'^4 \frac{s^2 c^2}{d^2}\right]^{\frac{1}{2}} . \tag{19}$$

We wish to show that $D_2(f_d, r) \leq 1$, $A < r < 1$. For this it is sufficient to prove that $\sup_r D_2(f_d, r) \leq 1$. According to Landen's Transformation ([4, p. 72], [5, # 163.01]),

$$\frac{sc}{d} = (1 + k)^{-1} sn \left|(1 + k) \frac{2K}{\pi} \log \frac{r}{A}, \frac{1 - k}{1 + k}\right| . \tag{20}$$

Since $sn(u, k)$ is maximum when $u = K$ ([5, # 121.02], [24, p. 499]) and because $K((1 - k)/(1+k)) = (1 + k)K'/2$ ([4, p. 72], [5, # 164,02]) we see that sc/d has its maximum value $(1 + k)^{-1}$ when $\log r/A = \pi K'/4K = \log 1/A$, that is, when $r = 1$. Thus

$$\sup_r D_2(f_d, r)^2 = \sup_\psi \frac{2K}{\pi} (\sin \psi) \left[\frac{C^2 D^2}{S^2} + (1 - k)^2\right]^{\frac{1}{2}} . \tag{21}$$

Now by use of the identities $C^2 = 1 - S^2$, $D^2 = 1 - k^2 S^2$ ([5, # 121.00], [24, p. 493]) we see that

$$C^2 D^2 + (1 - k)^2 S^2 = (1 - kS^2)^2 , \tag{22}$$

so that (21) reduces to

$$\sup_r D_2(f_d, r)^2 = \sup_\psi \frac{2K}{\pi} \frac{(\sin \psi)}{S} (1 - kS^2) . \tag{23}$$

We have shown in [3] that the expression on the right of (23) is bounded above by 1. Hence $D_2(f_d, r) \leq 1$ for $A < r < 1$, and by (8) we have

$$\operatorname{mod} R_{3,d} \leq \int_A^1 D_2(f_d, r) \frac{dr}{r} \leq \log \frac{1}{A} = \operatorname{mod} R_2 ,$$

concluding the proof of the theorem.

5. Bounds for the moduli of $R_{3,d}$ and $R_{3,s}$

Next, using elliptic integrals, we obtain upper and lower bounds in terms of a for $\operatorname{mod} R_{3,d}(a)$.

Theorem 7. *For each* $a, 0 < a < 1$,

$$(24) \quad \frac{1}{2}\left(\frac{\pi}{2K}\right)^{\frac{1}{2}} \left[K\left(\left[\frac{1+k'}{2}\right]^{\frac{1}{2}}\right) - K\left(\left[\frac{1-k'}{2}\right]^{\frac{1}{2}}\right)\right] \leq \operatorname{mod} R_{3,d}(a) \leq \frac{\pi K'}{4K} ,$$

where $k = a^2, k' = (1 - a^4)^{\frac{1}{2}}$, *and* $K = K(k)$ *and* $K' = K(k')$ *denote the elliptic integrals in* (11).

Proof. Since $\operatorname{mod} R_2 = \log 1/A = \pi K'/4K$, the upper bound follows immediately from Theorem 6.

To obtain the lower bound we apply the left side of (8) to $\operatorname{mod} R_{3,d}(a)$. Thus by (14) and (19) we must determine

$$(25) \qquad D_1(f_d, r)^2 = \min_{|x_1+ix_2|=r} \frac{2K}{\pi} |\sin \psi| \left[\frac{C^2 D^2}{S^2} + k'^4 \frac{s^2 c^2}{d^2}\right]^{\frac{1}{2}} ,$$

where $S, C, D, s, c,$ and d have the meanings assigned in (17). We assert that the minimum is achieved, for each r, when $\psi = \frac{\pi}{2}$. To see this, we use (20), (22), and the identity $sn^2 z + cn^2 z = 1$ ([5, # 121.00], [24, p. 493]) to rewrite (25) as

$$(26) \quad D_1(f_d, r)^2 = \frac{2K}{\pi} \min_{\psi} |\sin \psi| \left[\frac{(1-kS^2)^2}{S^2} - (1-k)^2 cn^2(z, k_1')\right]^{\frac{1}{2}} ,$$

where $z = (1 + k) \frac{2K}{\pi} \log \frac{r}{A}$ and $k_1' = (1 - k)/(1 + k)$. But since $|\sin \psi|$ is maximum when $\psi = \pi/2$, and since by (5) of [3] the expression $(\sin \psi)(1 - kS^2)/S$ achieves its minimum $1 - k$ when $\psi = \pi/2$, we conclude that the minimum in (26) occurs when $\psi = \pi/2$.

Hence by (25),

$$D_1(f_d, r)^2 = \frac{2K}{\pi} k'^2 \frac{sc}{d},$$

and combining this with (8) we obtain the inequality

$$\text{mod } R_{3,d} \geq k' \left(\frac{2K}{\pi}\right)^{\frac{1}{2}} \int_A^1 \left(\frac{sc}{d}\right)^{\frac{1}{2}} \frac{dr}{r}. \tag{27}$$

To evaluate the integral in (27) we first apply the half angle formulas ([5, # 124.02], [15, p. 120]) and the identity $dn^2 z + k^2 sn^2 z = 1$ ([5, # 121.00], [24, p. 493]) to write

$$\frac{sc}{d} = \frac{1 - dn\left(\frac{4K}{\pi} \log \frac{r}{A}, k'\right)}{k'^2 sn\left(\frac{4K}{\pi} \log \frac{r}{A}, k'\right)}. \tag{28}$$

Now making the change of variables

$$sn\left(\frac{4K}{\pi} \log \frac{r}{A}, k'\right) = \frac{t^2}{2 - t^2}$$

we have

$$\left[\frac{1 - dn\left(\frac{4K}{\pi} \log \frac{r}{A}, k'\right)}{k'^2 sn\left(\frac{4K}{\pi} \log \frac{r}{A}, k'\right)}\right]^{\frac{1}{2}} = \frac{\left[1 - \frac{1-k'}{2} t^2\right]^{\frac{1}{2}} - \left[1 - \frac{1+k'}{2} t^2\right]^{\frac{1}{2}}}{k't} \tag{29}$$

and

$$\frac{dr}{r} = \frac{\pi}{4K} \left[(1 - t^2)\left(1 - \frac{1-k'}{2} t^2\right)\left(1 - \frac{1+k'}{2} t^2\right)\right]^{-\frac{1}{2}} t\, dt. \tag{30}$$

When r is A, 1 then t is 0, 1, respectively, and by means of (28), (29), and (30) we may reduce (27) to

$$\text{mod } R_{3,d} \geq \frac{1}{2} \left(\frac{\pi}{2K}\right)^{\frac{1}{2}} \left[\int_0^1 \left[(1 - t^2)\left(1 - \frac{1+k'}{2} t^2\right)\right]^{-\frac{1}{2}} dt \right.$$

$$\left. - \int_0^1 \left[(1 - t^2)\left(1 - \frac{1-k'}{2} t^2\right)\right]^{-\frac{1}{2}} dt\right].$$

Finally, consulting the definition (11) of the elliptic integral $K = K(k)$ we arrive at the first half of (24), and the theorem is proved.

Our methods also yield the following bounds for $\operatorname{mod} R_{3,s}(a)$.

Theorem 8. *For each* $a, 0 < a < 1$,

$$(31)\quad \frac{\pi K'}{4K} \leq \operatorname{mod} R_{3,s}(a) \leq \frac{1}{2}\left(\frac{\pi}{2K}\right)^{\frac{1}{2}}\left[K\left(\left[\frac{1+k'}{2}\right]^{\frac{1}{2}}\right) + K\left(\left[\frac{1-k'}{2}\right]^{\frac{1}{2}}\right)\right],$$

where $k = a^2$, $k' = (1 - a^4)^{\frac{1}{2}}$, *and* $K = K(k)$ *and* $K' = K(k')$ *denote the elliptic integrals in* (11).

Proof. The lower bound follows directly from Theorem 6.

To obtain the upper bound we employ a technique introduced in the proof of Theorem 6. Let f be the conformal mapping (12) of the plane annulus $A < |x_1 + ix_2| < 1$ onto R_2, and let $y = f_s(x)$ be the space mapping of $A < |x| < 1$ onto $R_{3,s}$ obtained from f by rotating $a < |x_1 + ix_2| < 1$ and R_2 about the x_1 and y_1 axes. Then f_s is a diffeomorphism satisfying (8), where

$$D_1(f_s, r) = \min_{|x_1+ix_2|=r} \left|\frac{x_2}{y_2} f'(x_1 + ix_2)\right|^{\frac{1}{2}}$$

$$D_2(f_s, r) = \max_{|x_1+ix_2|=r} \left|\frac{x_2}{y_2} f'(x_1 + ix_2)\right|^{\frac{1}{2}}.$$

We now apply the right hand side of (8) to $\operatorname{mod} R_{3,s}$. Thus by (15) and (16) we must determine

$$(32)\qquad D_2(f_s, r)^2 = \max_{|x_1+ix_2|=r} \frac{2K}{\pi} |\cos\psi| \left[k'^4 \frac{S^2}{C^2 D^2} + \frac{d^2}{s^2 c^2}\right]^{\frac{1}{2}},$$

where $S, C, D, s, c,$ and d are the functions defined in (17).

We assert that this maximum is achieved, for each fixed r, when $\psi = 0$. Because of the special values $sn\ 0 = 0$, $cn\ 0 = dn\ 0 = 1$ ([4, p. 9], [5, # 122.01]), this is equivalent to the assertion that

$$(\cos^2\psi)\left[k'^4 \frac{S^2}{C^2 D^2} + \frac{d^2}{s^2 c^2}\right] \leq \frac{d^2}{s^2 c^2}$$

for all ψ.

By a simple rearrangement and use of the identity $\cos^2\psi = 1 - \sin^2\psi$, this is reduced to the claim that

$$(33)\qquad k'^4 \frac{s^2 c^2}{d^2} \leq \frac{C^2 D^2}{S^2} \tan^2\psi .$$

Elsewhere [3] we have shown that

$$(34)\qquad \frac{2k'K}{\pi} \leq \frac{cn\left(\frac{2K}{\pi}\psi, k\right)}{sn\left(\frac{2K}{\pi}\psi, k\right)} \tan\psi$$

for all real ψ. Since $D \geq k'$ ([5, # 121.02], [24, pp. 493, 499]) it follows from (34) that

$$(35)\qquad \frac{CD}{S}\tan\psi \geq \frac{2k'^2 K}{\pi}.$$

Then, comparing (35) with (33), we see that to show that the maximum in (32) occurs when $\psi = 0$ it is sufficient to prove

$$(36)\qquad \frac{sc}{d} \leq \frac{2K}{\pi}.$$

We proved earlier, after (20), that $sc/d \leq (1+k)^{-1}$. Thus (36) is implied by

$$1 \leq (1+k)\frac{2K}{\pi}.$$

The latter inequality, however, is trivial, since $k \geq 0$ and $K \geq \pi/2$.

We conclude that (33) is valid, and that the maximum in (32) occurs for $\psi = 0$. Hence

$$D_2(f_s, r)^2 = \frac{2K}{\pi}\frac{d}{sc}.$$

By virtue of (8), this means that

$$\operatorname{mod} R_{3,s} \leq \left(\frac{2K}{\pi}\right)^{\frac{1}{2}} \int_a^1 \left(\frac{d}{sc}\right)^{\frac{1}{2}} \frac{dr}{r}.$$

Evaluation of this integral by the same change of variables used in the proof of Theorem 8 then yields the upper bound in (31).

6. Behavior of $\operatorname{mod} R_{3,d}$ and $\operatorname{mod} R_{3,s}$ for small a

Let us turn now to the study of the asymptotic behavior of these extremal rings. We first obtain, as applications of the theorems of the preceding section, a pair of asymptotic formulas for the moduli as a tends to 0.

Theorem 9. *As a function of* a, $(\operatorname{mod} R_{3,d}(a) - \log 1/a)$ *is monotone decreasing in the interval* $0 < a < 1$, *and*

$$\lim_{a\to 0} (\operatorname{mod} R_{3,d}(a) - \log 1/a) = c_1 ,$$

where $0.254\ldots = \frac{3}{2}\log 2 - \frac{\pi}{4} \leq c_1 \leq \log 2 = 0.693\ldots .$

Proof. For let $0 < a' < a < 1$. If R is the image of $R_{3,d}(a')$ under the conformal mapping $y = (a/a')\,x$, then $\operatorname{mod} R_{3,d}(a') = \operatorname{mod} R$. Since $R_{3,d}(a)$ and $1 < |x| < a/a'$ are two disjoint rings separating the boundary components of R, we see from Lemma 2 of [13] that

$$\operatorname{mod} R_{3,d}(a) + \log a/a' \leq \operatorname{mod} R = \operatorname{mod} R_{3,d}(a') ,$$

from which the monotoneity follows (Cf. Lemma 6 in [13]).

To obtain the upper bound we make use of Theorem 6, (10), and the limit $\lim_{k\to 0} k^2 \exp(\pi K'/K) = 16$ (Cf. [5, # 112.04, # 901.00], [7, p. 88]) to conclude that

$$\lim_{a\to 0} (\operatorname{mod} R_{3,d} - \log 1/a) \leq \lim_{a\to 0} (\operatorname{mod} R_2 - \log 1/a)$$

(37)
$$= \lim_{k\to 0} \left[\frac{\pi K'}{4K} + \log k^{\frac{1}{2}}\right] = \log 2 .$$

To obtain the lower bound we use Theorem 7 and the fact that $K(0) = \pi/2$ to obtain

(38)
$$\lim_{a\to 0} (\operatorname{mod} R_{3,d} - \log 1/a) \geq \frac{1}{2} \lim_{k\to 0} \left[[\log k + K\left(\left[\frac{1+k'}{2}\right]^{\frac{1}{2}}\right)\right] - \frac{\pi}{4} .$$

Then by means of the limit $\lim_{k\to 0} (K' - \log 4/k) = 0$ ([5, # 112.01], [24, p. 521]) we may reduce the right side of (38) to $\frac{1}{2}\log 8 - \pi/4$, which is the lower bound in the theorem.

Theorem 10. *As a function of* a, $(\operatorname{mod} R_{3,s}(a) - \log 1/a)$ *is monotone decreasing in the interval* $0 < a < 1$, *and*

$$\lim_{a\to 0} (\operatorname{mod} R_{3,s}(a) - \log 1/a) = c_2 ,$$

where $0.693\ldots = \log 2 \leq c_2 \leq \frac{3}{2}\log 2 + \frac{\pi}{4} = 1.82\ldots .$

Proof. The argument for monotoneity is the same as that given in Theorem 9. To obtain the lower bound we make use of Theorem 6 as in (37) to conclude that

$$\lim_{a\to 0}\,(\operatorname{mod} R_{3,s} - \log 1/a) \geq \lim_{a\to 0}\,(\operatorname{mod} R_2 - \log 1/a) = \log 2\,.$$

Finally, use of Theorem 8, together with a proof similar to that given for the lower bound in Theorem 9, gives

$$\lim_{a\to 0}\,(\operatorname{mod} R_{3,s} - \log 1/a) \leq \frac{3}{2}\log 2 + \frac{\pi}{4}\,,$$

which is the upper bound in the theorem.

We remark that the bounds in Theorem 10 were obtained by Gehring ([9, § 9], [13, § 21]), the upper bound by a quite different method.

7. Quasiconformality and extremal lengths

Before we can study the asymptotic behavior of the moduli of $R_{3,d}$ and $R_{3,s}$ as a tends to 1, we need some additional tools. We begin by returning to (1), the dilatation functions $H_I(P, f)$ and $H_0(P,f)$ defined in § 1. If f is a diffeomorphism of a 3-space domain Ω onto Ω', then the functionals

$$(39)\qquad K_I(f) = \sup_{P\in\Omega} H_I(P\,,f)\,,\; K_0(f) = \sup_{P\in\Omega} H_0(P\,,f)$$

are known as the *inner* and *outer dilatations* of f, respectively. These dilatations are simultaneously infinite or finite. If both are finite, then f is called a *quasiconformal mapping.*

These definitions may be generalized, by means of the theory of rings, to include an arbitrary homeomorphism f of Ω onto Ω'. The *inner* and *outer dilatations* of a homeomorphism f are defined as

$$(40)\qquad K_I(f) = \sup_R \frac{\operatorname{mod} R}{\operatorname{mod} f(R)}\,,\qquad K_0(f) = \sup_R \frac{\operatorname{mod} f(R)}{\operatorname{mod} R}\,,$$

where the suprema are taken over all bounded rings R with $\bar{R}\subset\Omega$ for which $\operatorname{mod} R$ and $\operatorname{mod} f(R)$ are not both infinite. If one of these dilatations is finite the other is also (Cf. (1.10) in [14]) and f is said to be a *quasiconformal mapping.* In case f is a diffeomorphism, this definition reduces to the one previously given [14, Lemma 1.1]. If f is a homeomorphism of a ring R onto a ring R', then it follows from Lemma 1 and (40) that

$$(41)\qquad K_I(f) \geq \frac{\operatorname{mod} R}{\operatorname{mod} R'}\,,\qquad K_0(f) \geq \frac{\operatorname{mod} R'}{\operatorname{mod} R}\,.$$

Next, we shall need a result from the theory of extremal lengths. Let Γ be a family of arcs in E^3, and let $F(\Gamma)$ denote the family of density functions ϱ which are nonnegative and Borel measurable in E^3 and for which

(42) $$\int_\gamma \varrho \, ds \geq 1$$

for each arc $\gamma \varepsilon \Gamma$. Here the integral is taken with respect to linear measure [20] if γ is not locally rectifiable. Then following Väisälä [23] we define the modulus $M(\Gamma)$ of the family Γ as

(43) $$M(\Gamma) = \inf_{\varrho \varepsilon F(\Gamma)} \int_{E^3} \varrho^3 \, d\omega .$$

(See also [8] and [14]). If R is a space ring and Γ is the family of arcs which join the components of ∂R in R, then by Theorem 1 of [9],

(44) $$\operatorname{cap} R = M(\Gamma) .$$

8. Behavior of mod $\mathbf{R}_{3,d}$ as a tends to 1

Theorem 11. *The modulus of* $R_{3,d}(a)$ *has the following order as* a *tends to* 1:

(45) $$\limsup_{a \to 1} \, (1 - a)^{-\frac{1}{2}} \operatorname{mod} R_{3,d}(a) \leq \frac{\pi}{2} = 1.57 \ldots$$

and

(46) $$\liminf_{a \to 1} \, (1 - a)^{-\frac{1}{2}} \operatorname{mod} R_{3,d}(a) \geq 0.65 \ldots .$$

Proof. To obtain the upper bound (45) let Γ_1 be the family of arcs in $R_{3,d}$ joining the components of $\partial R_{3,d}$, and let Γ be the subfamily of circular arcs γ which are normal to the boundary components. Since $\Gamma_1 \supset \Gamma$ we have $F(\Gamma_1) \subset F(\Gamma)$ and, by (43), $M(\Gamma_1) \geq M(\Gamma)$. We will prove that

(47) $$M(\Gamma) \geq \frac{32}{\pi} \frac{a^2}{1 - a^2} .$$

Then, using (43), (44), and (47), we will have

$$\operatorname{cap} R_{3,d} = M(\Gamma_1) \geq M(\Gamma) \geq \frac{32}{\pi} \frac{a^2}{1 - a^2} .$$

By definition (3) of the modulus, this is equivalent to

$$(\operatorname{mod} R_{3,d}(a))^2 \leq \frac{\pi^2}{8} \frac{1 - a^2}{a^2} ,$$

from which (45) follows directly.

To prove (47) let Ω be the intersection of the plane quadrant $x_1 > 0$, $x_2 > 0, x_3 = 0$ with the union of the curves in Γ considered as point sets, and let Ω' be the image of Ω under

$$y_1 + iy_2 = f(x_1 + ix_2) = \frac{1 + (x_1 + ix_2)}{1 - (x_1 + ix_2)}.$$

Then Ω' is the quarter annulus $1 < |y_1 + iy_2| < b = (1 + a)/(1 - a)$, $y_1 > 0, y_2 > 0, y_3 = 0$. If g is the inverse of f, then it is easy to see that

$$(48) \qquad |g'(y_1 + iy_2)| = \frac{2}{|y_1 + iy_2 + 1|^2}, \quad x_1 = \frac{|y_1 + iy_2|^2 - 1}{|y_1 + iy_2 + 1|^2}.$$

Let $\varrho \,\varepsilon\, F(\Gamma)$. Then for each $\gamma \,\varepsilon\, \Gamma$ in Ω we have, by virtue of (42),

$$(49) \qquad 1 \leq \left[\int_\gamma \varrho \, ds\right]^3 = \left[\int_{\gamma'} \varrho' \, |g'| \, ds'\right]^3,$$

where $\varrho = \varrho(x_1 + ix_2)$, $\varrho' = \varrho'(y_1 + iy_2) = \varrho(g(y_1 + iy_2))$, and $\gamma' = f(\gamma)$ is a quarter circle $r = r_0, 0 < \varphi < \pi/2$, for some fixed r_0, $1 < r_0 < b$. Then an application of Hölder's inequality to (49) yields

$$(50) \qquad 1 \leq \left[\int_{\gamma'} \varrho'^3 \, x_1 \, |g'|^2 \, ds'\right] \left[\int_{\gamma'} |g'|^{\frac{1}{2}} x_1^{-\frac{1}{2}} \, ds'\right]^2.$$

Using (48) we may reduce (50) to

$$(51) \qquad \int_{\gamma'} \varrho'^3 \, x_1 \, |g'|^2 \, ds' \geq \frac{2}{\pi^2} \, \frac{r^2 - 1}{r^2},$$

where (r, φ) are polar coordinates in the $y_1 + iy_2$ plane.

Next, we see that

$$(52) \qquad \int_{E^3} \varrho^3 \, d\omega \geq 2 \int_0^{2\pi} \left(\int_\Omega \varrho^3 \, x_1 \, d\sigma\right) d\Theta = 4\pi \int_{\Omega'} \varrho'^3 \, x_1 \, |g'|^2 \, d\sigma'.$$

But using (51) and the fact that $b = (1 + a)/(1 - a)$ we determine that

$$\int_{\Omega'} \varrho'^3 \, x_1 \, |g'|^2 \, d\sigma' \geq \frac{8}{\pi^2} \, \frac{a^2}{1 - a^2}.$$

Hence (52) gives the estimate

$$\int_{E^3} \varrho^3 \, d\omega \geq \frac{32}{\pi} \, \frac{a^2}{1 - a^2},$$

from which (47) follows when we take the infimum over all $\varrho \,\varepsilon\, F(\Gamma)$.

Next we obtain the lower bound (46). For this, let (r, Θ, x_2) be cylindrical coordinates in E_3 and define

$$u(x) = \begin{cases} |x_2|/(1-r) & \text{if } |x_2| \leq 1 - r\,,\ 0 \leq r \leq a\,, \\ [(r-a)^2 + x_2^2]^{\frac{1}{2}}/(1-a) & \text{if } (r-a)^2 + x_2^2 \leq (1-a)^2,\ a \leq r \leq 1\,, \\ 1 \text{ elsewhere.} \end{cases}$$

Then u is admissible for $R_{3,d}(a)$ [13, § 3], and an elementary integration gives

$$\int\limits_{R_{3,d}(a)} |\nabla u|^3 \, d\omega = \pi(4p + \pi) \frac{a}{1-a} - 4\pi p \log \frac{1}{1-a} + \frac{4\pi}{3}\,,$$

where $p = [7\sqrt{2} + \log(7 + 5\sqrt{2})]/8$. Then, appealing to Lemma 1 in [13] we have

$$\limsup_{a \to 1}\ (1-a) \operatorname{cap} R_{3,s}(a) \leq \pi(4p + \pi)\,,$$

from which, because of (3), (46) follows immediately.

9. Behavior of mod $R_{3,s}$ as a tends to 1

We conclude this paper by proving the following asymptotic formula for mod $R_{3,s}$.

Theorem 12. *As a function of* a, $\left[\log \frac{1+a}{1-a}\right]^{\frac{1}{2}} \operatorname{mod} R_{3,s}(a)$ *is monotone increasing in the interval* $0 < a < 1$, *and*

$$\lim_{a \to 1} \left[\log \frac{1+a}{1-a}\right]^{\frac{1}{2}} \operatorname{mod} R_{3,s}(a) = c_3\,,$$

where $1.03\ldots \leq c_3 \leq q$,

$$q = \int\limits_0^{\pi/2} (\sin t)^{-\frac{1}{2}}\, dt = 2^{\frac{1}{2}} K((\tfrac{1}{2})^{\frac{1}{2}}) = 2.62\ldots\,. \tag{53}$$

Proof. We first prove the asserted monotoneity. There exists a Möbius transformation carrying $R_{3,s}(a)$ conformally onto the space ring $R(b)$ consisting of the half space $x_2 > 0$ minus the slit $1 \leq x_2 \leq b$, $x_1 = x_3 = 0$, where

$$b = \left[\frac{1+a}{1-a}\right]^2\,. \tag{54}$$

The mapping

$$g(x) = x\,|x|^{p-1}, \quad p = (\log b')/(\log b)$$

is a diffeomorphism of $R(b)$ onto $R(b')$. Assuming for definiteness that $1 < b' < b$, we have $0 < p < 1$. Applying (39) and (41) it is easily seen that

$$K_0(g) = p^{-\frac{1}{2}} = \left[\frac{\log b}{\log b'}\right]^{\frac{1}{2}} \geq \frac{\operatorname{mod} R(b')}{\operatorname{mod} R(b)},$$

from which it follows that $(\log b)^{\frac{1}{2}} \operatorname{mod} R(b)$ is a monotone increasing function of b. Since by (54) a and b increase together, this shows that the expression in the theorem increases monotonically with a as asserted.

To obtain the upper bound in the theorem let E be the slit $1 \leq x_2 \leq b$, $x_1 = x_3 = 0$ and Γ' the family of arcs γ' in $1 < |x| < b$, $x_2 > 0$ which join E to the plane annulus $1 < |x_1 + i x_3| < b$, $x_2 = 0$. Then Lemma 3.8 of [14] gives

$$M(\Gamma') = \frac{2\pi}{q^2} \log b, \tag{55}$$

where q is the elliptic integral in (53).

Now let Γ be the family of arcs γ joining the boundary components of $R(b)$ in $R(b)$. It follows from (44) and (55) that

$$\operatorname{cap} R(b) = M(\Gamma) \geq M(\Gamma') = \frac{2\pi}{q^2} \log b,$$

and combining this with the fact that $\operatorname{cap} R_{3,s} = \operatorname{cap} R(b)$, one arrives at

$$(\operatorname{mod} R_{3,s})^2 = \frac{4\pi}{\operatorname{cap} R_{3,s}} \leq \frac{2q^2}{\log b} = \frac{q^2}{\log \dfrac{1+a}{1-a}}. \tag{56}$$

The upper bound in the theorem follows from (56) and the monotoneity already proved.

To obtain the lower bound we use the same method as for (46) in § 8. Let (r, Θ, x_1) be cylindrical coordinates in E_3, and define

$$u(x) = \begin{cases} r/(1 - |x_1|) & \text{if } 0 \leq r \leq 1 - |x_1|,\ |x_1| \leq a \\ [(|x_1| - a)^2 + r^2]^{\frac{1}{2}}/(1 - a) & \text{if } (|x_1| - a)^2 + r^2 \leq (1 - a)^2,\ a \leq |x_1| \leq 1, \\ 1 & \text{elsewhere.} \end{cases}$$

Then u is admissible for $R_{3,s}$ [13, § 3], and an elementary integration gives

$$\int_{R_{3,s}(a)} |\nabla u|^3 \, d\omega = \frac{4\pi(4\sqrt{2}-1)}{5} \log \frac{1}{1-a} + \frac{2\pi}{3} .$$

Then, appealing again to Lemma 1 in [13] we have

$$\limsup_{a\to 1} \frac{\operatorname{cap} R_{3,s}(a)}{\log \dfrac{1+a}{1-a}} \leq \frac{4\pi(4\sqrt{2}-1)}{5} ,$$

from which, in view of (3), the lower bound in the theorem follows.

Remark. By point symmetrization in space [13, Theorem 2] one can verify that the spherical annular ring $R_{3,b}(a)$, $0 < a < 1$, consisting of the unit ball minus the closed ball $|x| \leq a$ is extremal in the following sense. Let R be any space ring consisting of the unit ball minus a continuum C, and suppose $m_3(C) \geq 4\pi a^3/3$. Then $\operatorname{mod} R \leq \operatorname{mod} R_{3,b}(a)$.

Next, since $\operatorname{mod} R_{3,b}(a) = \log 1/a$, it is easily checked that as a function of a,

$$\frac{1+a}{1-a} \operatorname{mod} R_{3,b}(a) \tag{57}$$

is monotone decreasing in the interval $0 < a < 1$ and that (57) has limit 2 as a tends to 1.

The pattern exhibited by (57) and Theorems 11 and 12 indicates how strongly influenced is the asymptotic behavior of these extremal rings, as a tends to 1, by the dimension of the central set omitted from the unit ball in the formation of the ring. Theorems 9 and 10 in this paper show, however, that all three of these rings have the same order as a tends to 0, so that for small a the modulus is much less sensitive to the dimension of the bounded component of the complement. One naturally wonders if a pattern similar to this prevails for the analogous extremal rings in higher dimensions.

Michigan State University
East Lansing, Michigan, USA

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