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EXTENSION OVER QUASICONFORMALLY EQUIVALENT CURVES

BY

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1. Introduction

Two Jordan curves γ and γ' lying in the extended complex plane $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$ are called *quasiconformally equivalent* if there exists a quasiconformal mapping of $\mathbf{C}^*$ onto itself which maps γ onto γ'. In [5] there is given a metrical condition which characterizes this equivalence relation in terms of the existence of homeomorphisms $\varphi : \gamma \to \gamma'$ which can be extended to quasiconformal mappings of $\mathbf{C}^*$. Those Jordan curves which are equivalent with circles, called quasiconformal curves or quasicircles, form a special equivalence class denoted here by Γ_0. The class Γ_0 can also be described to consist exactly of those Jordan curves which permit a quasiconformal reflection ([2]).

Let G and G' be Jordan domains. We denote by $E(G, G')$ the family of quasiconformal mappings $f : G \to G'$ which can be quasiconformally extended to $\mathbf{C}^*$. By Satz II.8.1 in [3] it follows that if a quasiconformal mapping $f : G \to G'$ has a quasiconformal extension to a domain which contains the closure $\bar{G}$, then $f \in E(G, G')$. Quasiconformal curves have the following property with respect to quasiconformal extension: If the boundaries ∂G and $\partial G'$ belong to Γ_0, then every quasiconformal mapping $f : G \to G'$ belongs to $E(G, G')$. This result is an immediate consequence of the definition of Γ_0.

In this paper we study to which extent the extension property of Γ_0 is valid for the other equivalence classes. Especially we ask whether there is some equivalence class Γ different from Γ_0 such that every quasiconformal mapping $f : G \to G'$, where G and G' are Jordan domains with boundaries in Γ, belongs to $E(G, G')$. That no such class exists is established in Theorem 2. It is also natural to ask how small $E(G, G')$ can be. Theorem 3 gives an answer in this direction and shows that there is even an equivalence class Γ such that all mappings in $E(G, G')$, where the boundaries ∂G and $\partial G'$ are in Γ, coincide on the boundary, that is, $E(G, G')$ is smallest possible according to Corollary 1 of Theorem 1.

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2. Extension properties

We first restate Theorem 3 in [5] in a special case which is more appropriate for our present purposes:

Theorem 1. *Let* $f : \mathbf{C}^* \to \mathbf{C}^*$ *be a homeomorphism which is quasiconformal off a Jordan curve* γ. *If there exists a quasiconformal mapping* $g : \mathbf{C}^* \to \mathbf{C}^*$ *such that* $g|\gamma = f|\gamma$, *then* f *is quasiconformal.*

Corollary 1. *Let* f_1 *and* f_2 *be quasiconformal mappings of a Jordan domain* G *onto a Jordan domain* G' *such that they coincide on the boundary* ∂G. *Then* $f_1 \in E(G, G')$ *if and only if* $f_2 \in E(G, G')$.

Corollary 2. *Assume* $f \in E(G, G')$ *where* G *and* G' *are Jordan domains. If* $g : \mathbf{C}^* \to \mathbf{C}^*$ *is a homeomorphic extension of* f *which is quasiconformal in* $\mathbf{C}^* - \bar{G}$, *then* g *is a quasiconformal extension of* f.

We need some notation. The complement and the euclidean diameter of a set A are denoted by C_A and $d(A)$ respectively. We denote by $B_r(z)$ the open disc and by $S_r(z)$ the circle with radius r and center z. By $z_1 z_2$ we mean the open line segment with endpoints z_1 and z_2. If α is a Jordan arc and $w_1, w_2 \in \alpha$, then $\alpha(w_1, w_2)$ is the open subarc of α with endpoints w_1 and w_2. The absolute value of the cross-ratio of the sequence z_1, z_2, z_3, z_4 is denoted by $|z_1, z_2, z_3, z_4|$. For finite distinct points we have

$$|z_1, z_2, z_3, z_4| = \frac{|z_1 - z_3|\,|z_2 - z_4|}{|z_1 - z_4|\,|z_2 - z_3|}\,.$$

By $\mu(r)$, $0 < r < 1$, we denote the modulus of the unit disc slit along the real axis from 0 to r.

The following result ([1]) is a consequence of a distortion theorem of Teichmüller:

Lemma 1. *If* $f : \mathbf{C}^* \to \mathbf{C}^*$ *is a* K-*quasiconformal mapping, then*

$$|f(z_1), \ldots, f(z_4)| \leq L(K, |z_1, \ldots, z_4|)$$

holds for distinct points $z_1, \ldots, z_4 \in \mathbf{C}^*$, *where we have used the notation*

$$L(K, t) = \frac{1}{(\mu^{-1}(K\mu((1+t)^{-1/2})))^2} - 1\,. \tag{2.1}$$

The fact that the class Γ_0 of quasiconformal curves is the only class which has the special extension property mentioned earlier is stated as follows.

Theorem 2. *Let* G *and* G' *be Jordan domains. Then every quasiconformal mapping* $f : G \to G'$ *has a quasiconformal extension to* $\mathbf{C}^*$ *if and only if* ∂G *and* $\partial G'$ *are quasiconformal curves. Moreover, if* ∂G *is not* *quasiconformal, then for every* $K > 1$ *there exists a* K*-quasiconformal mapping* $f : G \to G'$ *which cannot be extended quasiconformally over* ∂G.

Proof. We already pointed out that if $\gamma = \partial G$ and $\gamma' = \partial G'$ are quasiconformal curves, then every quasiconformal mapping $f : G \to G'$ belongs to $E(G, G')$. Since the inverse of a K-quasiconformal mapping is K-quasiconformal, it suffices to prove the last part of the theorem.

Let K be greater than 1. We can assume that $\infty \in \gamma \cap \gamma'$. Suppose that γ is not quasiconformal. If γ' is quasiconformal, the family $E(G, G')$ is empty. Suppose therefore that γ' is not quasiconformal at a point z' ([4]). If now γ is quasiconformal at a point z, the conformal mapping of G onto G' whose extension to the boundary maps z onto z' has no quasiconformal extension to $\mathbf{C}^*$. Consequently, we can assume that γ is not quasiconformal at any point.

Let now $Q = G(z_1, z_2, z_3, \infty)$ and $Q' = G'(z_1', z_2', z_3', \infty)$ be quadrilaterals with modulus 1, and let g and g' be the canonical mappings of Q and Q' respectively onto the square $A = A(0, 1, 1 + i, i)$. Using the same notation for the extension of g to the closure $\bar{Q}$ we denote $w_n = g^{-1}(2^{-2n+1})$, $n = 1, 2, \ldots$. Choose δ such that $0 < \delta < 1$ and $1/(1 - \delta) < \sqrt{K}$.

Let $\gamma_{n,1}, \gamma_{n,2}, \ldots$ be a sequence of open subarcs of $\gamma - \{\infty\}$ such that $\gamma_{n,m} \ni w_n$, $m = 1, 2, \ldots$, and $\lim_{m \to \infty} d(\gamma_{n,m}) = 0$. Since γ is not quasiconformal at w_n, there exist for every m successive points $a^1_{n,m}$, $a^2_{n,m}$, $a^3_{n,m}$ of $\gamma_{n,m}$ such that

$$\frac{|a^1_{n,m} - a^2_{n,m}|}{|a^1_{n,m} - a^3_{n,m}|} > m \, .$$

Denote by $b^j_{n,m}$ the first point of $\overline{\gamma_{n,m}(a^j_{n,m}, a^2_{n,m})}$ from $a^2_{n,m}$ which belongs to the line segment $\overline{a^1_{n,m}\, a^3_{n,m}}$, $j = 1, 3$. Let $H_{n,m}$ be the bounded Jordan domain whose boundary is $\overline{b^1_{n,m}\, b^3_{n,m}} \cup \gamma_{n,m}(b^1_{n,m}, b^3_{n,m})$ and let $U_{n,m}$ be a circular neighborhood of $a^2_{n,m}$ such that $U_{n,m} \cap ((\gamma - \gamma_{n,m}(b^1_{n,m}, b^3_{n,m})) \cup \overline{b^1_{n,m}\, b^3_{n,m}}) = \emptyset$. Then for every n at least one of the following cases occur:

(a) $\quad G \cap U_{n,m} \subset H_{n,m}$ for infinitely many m ,

(b) $\quad C_{\bar{G}} \cap U_{n,m} \subset H_{n,m}$ for infinitely many m .

Let E_n be a disc such that $E_n \subset G'$ if (a) occurs for n and $E_n \subset C_{\bar{G}'}$ otherwise, and such that ∂E_n contains a point a_n' of $g'^{-1}(\omega_n)$ where ω_n is the interval

$$\left(\frac{1}{2^{2n-1}} - \frac{\delta}{2^{2n}} \, , \, \frac{1}{2^{2n-1}} + \frac{\delta}{2^{2n}} \right)$$

of the real axis. Now denote

$$\sigma_n = L\left(n, \frac{|z_1' - a_n'|}{|\zeta_n' - a_n'|}\right)$$

where L is defined by (2.1) and where ζ_n' is the center of E_n.

For every n we choose $m_n > n$ such that γ_{n,m_n} is contained in a disc D_n for which

$$\frac{|u - u^*|}{|u - z_1|} < \frac{1}{\sigma_n} \tag{2.2}$$

if $u, u^* \in D_n$, such that $g(\gamma_{n,m_n}) \subset \omega_n$, and that $G \cap U_{n,m_n} \subset H_{n,m_n}$ if (a) holds for n and $C_{\bar{G}} \cap U_{n,m_n} \subset H_{n,m_n}$ otherwise. For simplicity we write $a_n^j = a_{n,m_n}^j$, $b_n^j = b_{n,m_n}^j$, $j = 1, 2, 3$, $H_n = H_{n,m_n}$, and $U_n = U_{n,m_n}$.

We define now subsets of A for every $n, n = 1, 2, \ldots,$ as follows:

$$V_n = \left\{z \in A \mid \frac{1}{2^{2n}} < \operatorname{Re} z < \frac{1}{2^{2n-1}}\right\},$$

$$W_n = \left\{z \in A \mid \frac{1}{2^{2n-1}} < \operatorname{Re} z < \frac{1}{2^{2n-2}}\right\},$$

$$X_n = \left\{z \in A \mid \frac{1}{2^{2n}} < \operatorname{Re} z < v_n\right\},$$

$$Y_n = \left\{z \in A \mid v_n < \operatorname{Re} z < \frac{1}{2^{2n-2}}\right\},$$

where $v_n = g(a_n^2)$. Let $h : A \to A$ be a homeomorphism for which the restrictions $h|V_n$ and $h|W_n$ are the natural affine mappings onto X_n and Y_n respectively, $n = 1, 2, \ldots$. A homeomorphism $h' : A \to A$ is defined similarly by the use of the numbers $v_n' = g'(a_n')$.

According to our choice of δ, the mapping $f = g'^{-1} \circ h' \circ h^{-1} \circ g$ of G onto G' is K-quasiconformal. We claim that f has no quasiconformal extension over γ. Suppose such an extension exists, i.e. there is a K_0-quasiconformal mapping $F : \mathbf{C}^* \to \mathbf{C}^*$ for some $K_0 < \infty$ such that $F|G = f$. Assume $n > K_0$. If we apply Lemma 1 to the mapping F^{-1} and the sequence $z_1', \zeta_n', a_n', \infty$, we obtain from (2.2) that $F(D_n)$ and hence $F(H_n)$ do not contain the point ζ_n'. Let T_n be the circular annulus with boundary components $S_{r_n}(b_n^1)$ and $S_{t_n}(b_n^1)$ where $r_n = |b_n^1 - b_n^3|$ and $t_n = |b_n^1 - a_n^2|$. Suppose (a) holds for n. Then $E_n \subset G'$ and hence $l_n = F^{-1}(a_n' \zeta_n') \subset G$. Let c_n be a point of $U_n \cap l_n$. Since $c_n \in H_n$ and $F^{-1}(\zeta_n') \in C_{H_n}$, there is a point $d_n \in l_n \cap \partial H_n$. But this is possible only if $d_n \in b_n^1 b_n^3$ and there must therefore be a point $e_n \in l_n \cap S_{r_n}(b_n^1)$. The ring

$T'_n = F(T_n)$ separates the points $F(b_n^1)$ and $F(e_n)$ from the points a'_n and ∞. Since $|F(b_n^1) - F(e_n)| \geq |a'_n - F(e_n)|$, we get by the modulus theorem of Teichmüller ([3], p. 58) the following estimate

$$M(T'_n) \leq 2\mu\left(\frac{1}{\sqrt{2}}\right) = \pi\ .$$

The case that (a) does not hold for n is treated similarly. But $\lim_{n\to\infty} M(T_n) = \infty$ because

$$\frac{|b_n^1 - a_n^2|}{|b_n^1 - b_n^3|} > m_n - 1$$

and $m_n > n, n = 1, 2, \ldots$, and we have a contradiction with the quasiconformality of F. The theorem is proved.

Remark. By a similar but more complicated argument one can actually show that the statement in Theorem 2 holds also for $K = 1$.

3. An example

Let γ and γ' be quasiconformally equivalent and let $g : \mathbf{C}^* \to \mathbf{C}^*$ be a quasiconformal mapping such that $g(\gamma) = \gamma'$. Let G be one of the complementary domains of γ and denote $G' = g(G)$. Then $f_0 = g|G \in E(G, G')$, and Corollary 1 of Theorem 1 says that every quasiconformal mapping $f : G \to G'$ which coincides with f_0 on the boundary belongs to $E(G, G')$. Note that if Jordan domains D and D' with $\partial D = \gamma$ and $\partial D' = \gamma'$ are chosen so that $g(D) \neq D'$, then $E(D, D')$ may be empty. We shall now establish an example which gives the following result.

Theorem 3. *There is an equivalence class* Γ *such that if* G *and* G'

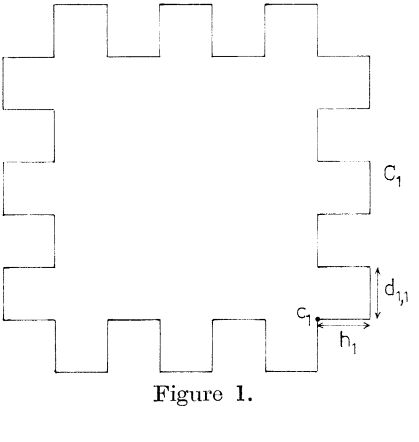

Figure 1.

are Jordan domains such that $\partial G, \partial G' \in \Gamma$, *then all mappings in* $E(G, G')$ *coincide on the boundary.*

This follows immediately from the following theorem.

Theorem 4. *There is a Jordan curve* γ *such that if* $f: \mathbf{C}^* \to \mathbf{C}^*$ *is a quasiconformal mapping which maps* γ *onto itself, then* $f|\gamma$ *is the identity mapping.*

Proof. To construct a Jordan curve with the required property we start with the segmental Jordan curve C_1 and a point c_1 on C_1 as shown in Figure 1 where $d_{1,1} = h_1 = 1/5$. We shall form a sequence $C_1, C_2, \ldots$ of segmental Jordan curves inductively as follows. Suppose C_{l-1} is constructed and the point c_{l-1} on C_{l-1} chosen. To construct C_l we deform, starting from the point c_{l-1}, the line segment parts of C_{l-1} into successive wave-like closed arcs $C_{l,1}, \ldots, C_{l,n_l}$ as shown in Figure 2 (which is an illustration of the case where l is even) such that the height and the width of the waves satisfy certain conditions stated below. We set

$$C_l = \bigcup_{n=1}^{n_l} C_{l,n}$$

and define c_l to be the common point of $C_{l,1}$ and C_{l,n_l}. With this construction every C_k, $k \geq 2$, is the union of successive wavelike arcs $C_{k,1}, \ldots, C_{k,n_k}$ which come from successive line segment parts of C_{k-1}. We denote by h_k the height and by $d_{k,n}$ the width of the waves of the arc $C_{k,n}$

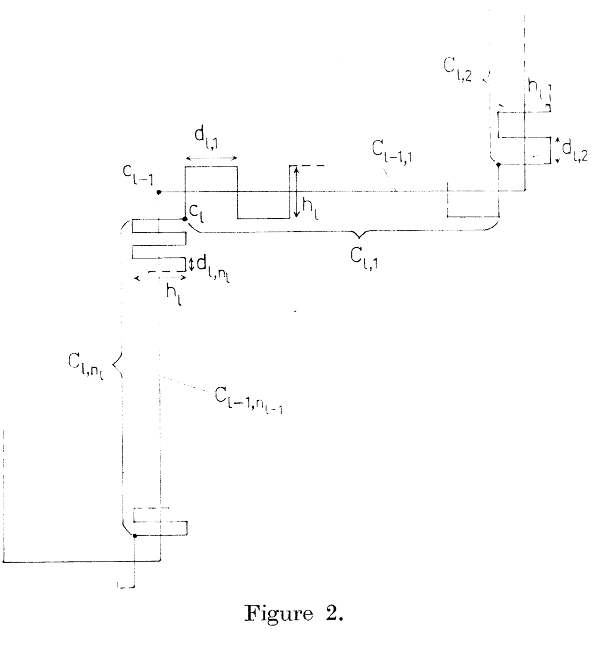

Figure 2.

(see Fig. 2). In the construction we require now the following additional conditions for $k \geq 2$:

$$h_k < \min \left\{ \frac{d_{k-1,n_{k-1}}}{k\, n_k}, \frac{1}{18} \left(\frac{h_{k-1}}{2k} \right)^k \right\}, \tag{3.1}$$

$$d_{k,1} = h_k\,; \quad \frac{d_{k,n+p}}{d_{k,n}} < \frac{1}{24} \left(\frac{1}{2p} \right)^p h_{k-1}^{p-1}, \; p \geq 1\,, \tag{3.2}$$

$$L\left(p, \frac{4h_k}{d_{k,n}}\right) < \frac{h_k}{8\, d_{k,n+p}}, \; p \geq 1\,. \tag{3.3}$$

Here L is the function defined by (2.1). It should be noted in (3.1) that n_k is determined by C_{k-1}.

As a limit of the sequence $C_1, C_2, \ldots$ we obtain a Jordan curve γ. We claim that γ has the property in the theorem. To prove this suppose there is a K-quasiconformal mapping $f: \mathbf{C}^* \to \mathbf{C}^*$ such that $f(\gamma) = \gamma$ and such that $f|\gamma$ is not the identity. Let $c \in \gamma$ be the limit of the sequence $c_1, c_2, \ldots$. Because of the assumption, there is a point $z \in \gamma$ such that $z' = f(z) \neq z$ and $z, z' \neq c$. Suppose that the sequence c, z', z is positively oriented with respect to the bounded complementary domain of γ. By the Hölder continuity of f there is a constant $M < \infty$ such that $|f(w_1) - f(w_2)| \leq M |w_1 - w_2|^{1/K}$ holds for points $w_1, w_2 \in \gamma$.

We denote by $\gamma_{k,n}$ the subarc of γ which comes as a limit from successive deformations of $C_{k,n}$ in the obvious sense. Let now $p > M, K$ be an integer. We choose $\varrho > 0$ such that $\overline{B_\varrho(z)} \cap \overline{B_\varrho(z')} = \emptyset$ and such that $4\varrho < \min\{|z - c|, |z' - c|, |z' - f(c)|\}$. Let $\gamma_z \ni z$ be a subarc of γ such that $\gamma_z \subset B_\varrho(z)$ and $\gamma_{z'} = f(\gamma_z) \subset B_\varrho(z')$. We can then choose an integer $k > 4\, L(p, 16)$ such that $d(\gamma_{z'}) > 2\, h_{k-1}$ and such that there is an integer n such that, if $w \in \gamma_z$, then $w \in \gamma_{k,n+p+m}$ for some $m \geq 0$, and if $w' \in \gamma_{z'}$, then $w' \in \gamma_{k,n-m'}$ for some $m' \geq 0$.

Since $d(\gamma_{z'}) > 2\, h_{k-1}, \gamma_{z'}$ contains a subarc $\gamma_{k,q}$ with $d(\gamma_{k,q}) > h_{k-1}/2$. We denote the common points of $C_{k,q}$ and C_{k-1} by $a_1, \ldots, a_r$ so that the indices correspond to the successive order on $C_{k,q}$. Let $\tilde{a}_i$ be a point on $\gamma_{k,q}$ closest to a_i and let b_i be a point on C_k closest to $\tilde{b}_i = f^{-1}(\tilde{a}_i)$, $i = 1, \ldots, r$. Then $|a_i - a_{i+1}| = d_{k,q}$, $i = 1, \ldots, r-1$, and $|\tilde{a}_i - a_i|$, $|\tilde{b}_i - b_i| \leq 2\, h_{k+1}$, $i = 1, \ldots, r$.

Suppose that the arc $\alpha = (C_k - \{c_k\})\,(b_i, b_{i+1})$ contains at least 12 successive line segment parts of C_k (Fig. 3). Let $e_0 = b_i, e_1, \ldots, e_{s-1}$, $e_s = b_{i+1}$ be a sequence of successive points on α such that every $\alpha(e_\mu, e_{\mu+1})$, $\mu = 1, \ldots, s-2$, consists of four successive line segment parts of C_k and such that $\alpha(e_0, e_1)$ and $\alpha(e_{s-1}, e_s)$ do not contain four successive line segment parts of C_k. Let $\tilde{e}_\mu \in \gamma_z$ be a point such that $|\tilde{e}_\mu - e_\mu| \leq 2\, h_{k+1}$ if

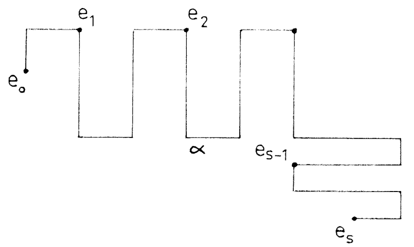

Figure 3.

$\mu = 1, \ldots, s-1,$ and denote $\tilde{e}_0 = \tilde{b}_i$ and $\tilde{e}_s = \tilde{b}_{i+1}$. Then for every $\mu, \mu = 0, \ldots, s-1$, there exists an integer $\nu, \nu = 0, \ldots, s-1, |\nu - \mu| \leq 2$, such that the following condition holds:

($*$) The arc $\beta_\nu = \gamma_z(\tilde{e}_\nu, \tilde{e}_{\nu+1})$ has a point w_ν such that

$$\frac{|\tilde{e}_\nu - w_\nu|}{|\tilde{e}_\nu - \tilde{e}_{\nu+1}|} > \frac{h_k}{4\, d_{k,n+p}} \, .$$

By applying Lemma 1 to the mapping f^{-1} and the sequence $f(w_\nu), f(\tilde{e}_{\nu+1}), f(\tilde{e}_\nu), f(c)$ we get by the choice of ϱ the inequality

$$\frac{|\tilde{e}_\nu - w_\nu|}{2\,|\tilde{e}_\nu - \tilde{e}_{\nu+1}|} \leq L\left(K, \frac{2\,|f(\tilde{e}_\nu) - f(w_\nu)|}{|f(\tilde{e}_\nu) - f(\tilde{e}_{\nu+1})|}\right) .$$

If now ($*$) is satisfied for ν, we have by the choice of p that

$$\frac{h_k}{8\, d_{k,n+p}} < L\left(p, \frac{2\,|f(\tilde{e}_\nu) - f(w_\nu)|}{|f(\tilde{e}_\nu) - f(\tilde{e}_{\nu+1})|}\right) .$$

The condition (3.3) implies then that

$$\frac{2\,|f(\tilde{e}_\nu) - f(w_\nu)|}{|f(\tilde{e}_\nu) - f(\tilde{e}_{\nu+1})|} > \frac{4\, h_k}{d_{k,n}} \, .$$

But, by the construction of γ, this is possible only if

$$d(f(\beta_\nu)) \leq 2\, h_{k+1} \, . \tag{3.4}$$

If the condition ($*$) is not satisfied for $\nu = \mu$, we estimate as follows. Let $\nu, |\nu - \mu| \leq 2$, be an integer such that ($*$) is satisfied for ν. Then there is a point $\zeta_\nu \in \beta_\nu$ such that

$$\frac{\sup\limits_{w \in \beta_\nu} |\tilde{e}_\nu - w|}{|\tilde{e}_\nu - \zeta_\nu|} \leq 8 \, .$$

Again by applying Lemma 1 to the mapping f and the sequence $w, \zeta_\nu, \tilde{e}_\nu, c$, where $w \in \beta_\nu$, we get by the choice of ϱ and p the inequality

$$\frac{\sup\limits_{w' \in f(\beta_\nu)} |f(\tilde{e}_\nu) - w'|}{|f(\tilde{e}_\nu) - f(\zeta_\nu)|} \leq 2L(p, 16) .$$

This together with (3.4) implies

$$d(f(\beta_\nu)) \leq 4\, L(p, 16)\, d(f(\beta_\nu)) \leq 8\, L(p, 16)\, h_{k+1} ,$$

and since $s < n_{k+1}/4$, we get thus

$$|\tilde{a}_i - \tilde{a}_{i+1}| = |f(\tilde{b}_i) - f(\tilde{b}_{i+1})| \leq 2\, n_{k+1}\, L(p, 16)\, h_{k+1} .$$

From $n_{k+1}\, h_{k+1} < d_{k,n_k}/k$ and from the choice of k it follows that $|\tilde{a}_i - \tilde{a}_{i+1}| < d_{k,n_k}/2$. But then

$$\begin{aligned} |a_i - a_{i+1}| &\leq |a_i - \tilde{a}_i| + |\tilde{a}_i - \tilde{a}_{i+1}| + |\tilde{a}_{i+1} - a_{i+1}| < 4\, h_{k+1} + d_{k,n_k}/2 \\ &< d_{k,n_k}/2 + d_{k,n_k}/2 < d_{k,q} , \end{aligned}$$

which is a contradiction. Hence the arc α contains less than 12 line segment parts of C_k.

We have now proved that the arc $A = (C_k - \{c_k\})\,(b_1, b_r)$ contains at most $12(r - 1)$ line segment parts of C_k. Let now $z_0 = b_1, z_1, \ldots, z_{\lambda-1}$, $z_\lambda = b_r$ be a sequence of successive points on C_k which divide A into subarcs similarly as the arc α was divided by the sequence $e_0, \ldots, e_s$ above. Then $\lambda \leq 3\, r$. It is now observed that if $z_\mu, z_{\mu+1} \in C_{k,t}$ for some t and if $\mu = 1, \ldots, \lambda - 2$, then $|z_\mu - z_{\mu+1}| = 2\, d_{k,t}$. Otherwise we have $|z_\mu - z_{\mu+1}| \leq 2\, h_k$. Since in any case $d_{k,n+p}/d_{k,n} < 1/48$, one can conclude that

$$A \subset \bigcup_{i=0}^{2} C_{k,t-i}$$

for some t. It then follows the estimate

$$|\tilde{b}_1 - \tilde{b}_r| \leq 4\, h_{k+1} + 8\, h_k + 6\, r\, d_{k,n+p} < 9\, h_k + 6\, r\, d_{k,n+p} .$$

By the use of (3.1) and (3.2) we get

$$|\tilde{b}_1 - \tilde{b}_r| < \frac{1}{2} \left(\frac{h_{k-1}}{2k}\right)^k + \frac{r}{4} \left(\frac{1}{2p}\right)^p h_{k-1}^{p-1}\, d_{k,n} ,$$

and from $k > 4\, L(p, 16) > p$ and $r < 2\, h_{k-1}/d_{k,n}$ the inequality

$$|\tilde{b}_1 - \tilde{b}_r| < \left(\frac{h_{k-1}}{2p}\right)^p .$$

Combining this with $|\tilde{a}_1 - \tilde{a}_r| > h_{k-1}/2$ we are led to the contradiction

$$|\tilde{a}_1 - \tilde{a}_r| = |f(\tilde{b}_1) - f(\tilde{b}_r)| > p|\tilde{b}_1 - \tilde{b}_r|^{1/p} > M|\tilde{b}_1 - \tilde{b}_r|^{1/K} .$$

In the case that the sequence c, z, z' is positively oriented with respect to the bounded complementary domain of γ, the Hölder continuity of f^{-1} is used. The theorem is proved.

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