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ON BOUNDED UNIVALENT FUNCTIONS WHICH ARE CLOSE TO IDENTITY

BY

MENAHEM SCHIFFER and OLLI TAMMI

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SUOMALAINEN TIEDEAKATEMIA

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Introduction

The role of the Grunsky inequalities for the coefficient problem in the theory of univalent functions is well known. Using the inequalities, one could prove the Bieberbach conjecture for functions which are close enough to the Koebe function [2]. The purpose of the present paper is to give a similar treatment to the coefficient problem for bounded univalent functions which are close enough to the identity mapping $f(z) = z$. The Grunsky inequalities have been sharpened for the case of bounded univalent functions by Nehari [4]. Recently we have extended the result of Nehari [5] and this allows us to give a precise solution of the coefficient problem in this case.

In order to lay the groundwork and to exemplify our method, we deal in Section 1 with the special case of a_7. Here all the features of the general case are already present, while we can give a specific condition for proximity to the identity in order that our result be valid. In Section 2 we prepare some formal identities and asymptotic expressions needed in the general treatment. In Section 3 we consider the case of odd indexed coefficients a_{2n+1} under the assumption that $a_2, \ldots, a_{n-1}$ vanish. This special case is particularly easy to handle and prepares the more complicated approach to the general case. In Section 4, finally, the case of arbitrary a_{2n+1} is settled. Now we proceed to deal with even indexed coefficients a_{2n}. This is achieved in Sections 5—7 by introducing the odd function $\sqrt{f(z^2)}$ and applying the previous results.

1. The case of a_7

Let $S(b_1)$ denote the family of analytic functions

(1) $$f(z) = \sum_{\nu=1}^{\infty} b_\nu z^\nu = b_1 \sum_{\nu=1}^{\infty} a_\nu z^\nu , \quad a_\nu = \frac{b_\nu}{b_1} ,$$

which are univalent in $|z| < 1$ and bounded, that is,

(2) $$|f(z)| < 1 , \qquad 0 < b_1 \leqq 1 .$$

Our aim is to give an estimate for the coefficients a_ν when b_1 is close to 1.

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Our tools in the investigation are the following concepts and inequalities. Let

$$(3) \qquad \log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{i,k=0}^{\infty} A_{ik} z^i \zeta^k$$

and

$$(4) \qquad -\log (1 - f(z)\overline{f(\zeta)}) = \sum_{i,k=1}^{\infty} B_{ik} z^i \bar{\zeta}^k .$$

The matrices $((A_{ik}))$ and $((B_{ik}))$ are symmetric and hermitean, respectively. We recently showed the following necessary and sufficient condition for $f(z)$ to be univalent and bounded in $|z| < 1$: The inequality

$$(5) \qquad \mathrm{Re}\left\{ \sum_{m,n=0}^{N} A_{mn} x_m x_n \right\} + \sum_{m,n=1}^{N} B_{mn} x_m \bar{x}_n \leqq \sum_{m=1}^{N} \frac{|x_m|^2}{m} \quad (N = 1, 2, \ldots)$$

must hold for every complex vector $\{x_n\}$. The case $x_0 = 0$ is the Nehari condition for univalent bounded functions; it appears, however, that the introduction of the additional variable x_0 is very important in the discussion of the coefficient problem.

In the case $N = 3$, an easy calculation leads to the relations

$$A_{00} = \log b_1 ,$$

$$A_{11} = a_3 - a_2^2 ,$$

$$A_{22} = a_5 - \frac{3}{2} a_3^2 - 2 a_2 a_4 + 4 a_2^2 a_3 - \frac{3}{2} a_2^4 ,$$

$$A_{33} = a_7 - 2 a_4^2 - 3 a_3 a_5 - 2 a_2 a_6 + 4 a_2^2 a_5 + 12 a_2 a_3 a_4$$
$$+ \frac{7}{3} a_3^3 - 15 a_2^2 a_3^2 - 8 a_2^3 a_4 + 14 a_2^4 a_3 - \frac{10}{3} a_2^6 ,$$

$$A_{01} = a_2 ,$$

$$A_{02} = a_3 - \frac{1}{2} a_2^2 ,$$

$$A_{03} = a_4 - a_2 a_3 + \frac{1}{3} a_2^3 ,$$

$$A_{12} = a_4 - 2 a_2 a_3 + a_2^3 ,$$

$$A_{13} = a_5 - a_3^2 - 2 a_2 a_4 + 3 a_2^2 a_3 - a_2^4 ,$$

$$A_{23} = a_6 - 2 a_2 a_5 - 3 a_3 a_4 + 5 a_2 a_3^2 + 4 a_2^2 a_4 - 7 a_2^3 a_3 + 2 a_2^5 ;$$

$$B_{11} = b_1^2 ,$$

$$B_{22} = \frac{1}{2} b_1^4 + |b_2|^2 ,$$

$$B_{33} = \frac{1}{3} b_1^6 + 2 b_1^2 |b_2|^2 + |b_3|^2 ,$$

$$B_{12} = b_1 \bar{b}_2 ,$$

$$B_{23} = b_1^3 \bar{b}_2 + b_2 \bar{b}_3 ,$$

$$B_{13} = b_1 \bar{b}_3 .$$

In order to estimate a_7, take $x_3 = 1$. The coefficients a_6 and a_5 are eliminated by choosing

$$\tag{6} \begin{cases} x_2 = a_2 , \\ x_1 = \frac{1}{2} (3 a_3 - a_2^2) . \end{cases}$$

From [4] we know that the best possible estimate for a_7 is obtained for

$$\tag{7} x_0 = \frac{\mathrm{Re}\,\{\sum_1^3 A_{0\mu}\, x_\mu\}}{\log b_1^{-1}} = \frac{\mathrm{Re}\left\{a_4 + \frac{3}{2} a_2 a_3 - \frac{2}{3} a_2^3\right\}}{\log b_1^{-1}} .$$

Thus, for $N = 3$, (5) assumes the form

$$\tag{8} a_7 - \frac{1}{3}(1 - b_1^6)$$

$$\leqq \mathrm{Re}\left\{2 a_4^2 + a_2^3 a_4 - 3 a_2 a_3 a_4 - \frac{19}{12} a_3^3 + \frac{25}{4} a_2^2 a_3^2 \right.$$

$$\left. - \frac{19}{4} a_2^4 a_3 + \frac{13}{12} a_2^6 - \frac{1}{2}(3 + 5 b_1^2)\, a_3 \bar{a}_2^2 \right\}$$

$$+ \frac{1}{4}(9 - 25 b_1^2)\, |a_3|^2 + \frac{1}{4}(1 - b_1^2)\, |a_2|^4 + \frac{1}{2}(1 - 9 b_1^4)\, |a_2|^2$$

$$- \frac{\left[\mathrm{Re}\left\{a_4 + \frac{3}{2} a_2 a_3 - \frac{2}{3} a_2^3\right\}\right]^2}{\log b_1^{-1} .} .$$

We rewrite this condition by using the notation

$$t = a_4 + \frac{3}{2} a_2 a_3 - \frac{2}{3} a_2^3 ; \tag{9}$$

$$a_7 - \frac{1}{3}(1 - b_1^6) \leqq 2 \operatorname{Re}\{t^2\} - \frac{[\operatorname{Re}\{t\}]^2}{\log b_1^{-1}} \tag{10}$$

$$+ \operatorname{Re}\left\{- 9 a_2 a_3 t - \frac{19}{12} a_3^3 + \frac{11}{3} a_2^3 t + \frac{61}{4} a_2^2 a_3^2 - \frac{17}{2} a_2^4 a_3 + \frac{95}{36} a_2^6\right.$$

$$\left. - \frac{1}{2}(3 + 5 b_1^2) \bar{a}_2^2 a_3\right\} + \frac{1}{4}(1 - b_1^2) |a_2|^4$$

$$+ \frac{1}{4}(9 - 25 b_1^2) |a_3|^2 + \frac{1}{2}(1 - 9 b_1^4) |a_2|^2 .$$

In order to estimate the right side of (10), we will utilize the following bounds ([5], [6]):

$$|a_2| \leqq 2(1 - b_1) , \quad \text{for all} \quad b_1 ;$$

$$|a_3| \leqq 1 - b_1^2 \leqq 2(1 - b_1) , \quad \text{for} \quad e^{-1} \leqq b_1 \leqq 1 ;$$

$$|a_4| \leqq \frac{2}{3}(1 - b_1^3) \leqq 2(1 - b_1) , \quad \text{for} \quad 0{,}65 \leqq b_1 \leqq 1 ;$$

$$|a_\nu| \leqq 2(1 - b_1) , \quad \text{for} \quad e^{-1} \leqq b_1 \leqq 1 , \quad \nu = 2 , 3 , 4 .$$

This gives for t

$$|t| \leqq \left[2 + 6(1 - b_1) + \frac{16}{3}(1 - b_1)^2\right] (1 - b_1) .$$

When restricting ourselves to

$$0{,}9 \leqq b_1 \leqq 1 \tag{11}$$

we obtain $|t| < 2{,}7(1 - b_1)$. Thus, for the interval (11) the following inequalities are valid:

$$\begin{cases} |a_\nu| \leqq 2(1 - b_1) , & \nu = 2 , 3 , 4 ; \\ |t| < 2{,}7(1 - b_1) . \end{cases} \tag{12}$$

In view of (11), the first part of the right side of (10) may be estimated as follows:

$$2 \operatorname{Re}\{t^2\} - \frac{[\operatorname{Re}\{t\}]^2}{\log b_1^{-1}} \leqq [\operatorname{Re}\{t\}]^2 \left(2 - \frac{1}{\log b_1^{-1}}\right) \leqq 0 .$$

Hence, in the interval considered,

$$(13)\quad a_7 - \frac{1}{3}(1 - b_1^6) \leqq \frac{1}{4}(9 - 25b_1^2)|a_3|^2 + \frac{1}{2}(1 - 9b_1^4)|a_2|^2$$

$$+ \operatorname{Re}\left\{- 9a_2a_3t - \frac{19}{12}a_3^3 - \frac{1}{2}(3 + 5b_1^2)\bar{a}_2^2a_3 + \frac{11}{3}a_2^3t + \frac{61}{4}a_2^2a_3^2\right.$$

$$\left.- \frac{17}{2}a_2^4a_3 + \frac{1}{4}(1 - b_1^2)|a_2|^4 + \frac{95}{36}a_2^6\right\} = \mathrm{I} + \mathrm{II}\,.$$

Here we have divided the right side into the parts

$$\mathrm{I} = \frac{4}{5}\cdot\frac{1}{4}(9 - 25b_1^2)|a_3|^2 + \frac{2}{3}\cdot\frac{1}{2}(1 - 9b_1^4)|a_2|^2 + \operatorname{Re}\{- 9a_2a_3t\}\,,$$

$$\mathrm{II} = \frac{1}{5}\cdot\frac{1}{4}(9 - 25b_1^2)|a_3|^2 + \frac{1}{3}\cdot\frac{1}{2}(1 - 9b_1^4)|a_2|^2$$

$$+ \operatorname{Re}\left\{- \frac{19}{12}a_3^3 - \frac{1}{2}(3 + 5b_1^2)\bar{a}_2^2a_3 + \frac{11}{3}a_2^3t + \frac{61}{4}a_2^2a_3^2\right.$$

$$\left.- \frac{17}{2}a_2^4a_3 + \frac{1}{4}(1 - b_1^2)|a_2|^4 + \frac{95}{36}a_2^6\right\}.$$

For I we get in view of (12)

$$\mathrm{I} \leqq \frac{1}{5}(9 - 25b_1^2)|a_3|^2 + \frac{1}{3}(1 - 9b_1^4)|a_2|^2 + 25(1 - b_1)|a_2||a_3|\,.$$

The discriminant on the right side of the above expression, considered as a quadratic form in $|a_2|$ and $|a_3|$, is given by

$$\frac{1}{5}(9 - 25b_1^2)\cdot\frac{1}{3}(1 - 9b_1^4) - \left(\frac{25}{2}\right)^2(1 - b_1)^2\,.$$

This is positive in the interval defined by (11). Hence, in that interval $\mathrm{I} \leqq 0$.

Next, we decompose $\mathrm{II} = \mathrm{II}_1 + \mathrm{II}_2$:

$$\mathrm{II}_1 = \frac{1}{20}(9 - 25b_1^2)|a_3|^2 + \operatorname{Re}\left\{- \frac{19}{12}a_3^3\right\},$$

$$\mathrm{II}_2 = \frac{1}{6}(1 - 9b_1^4)|a_2|^2 + \operatorname{Re}\left\{- \frac{1}{2}(3 + 5b_1^2)\bar{a}_2^2a_3\right.$$

$$\left.+ \frac{11}{3}a_2^3t + \frac{61}{4}a_2^2a_3^2 - \frac{17}{12}a_2^4a_3 + \frac{1}{4}(1 - b_1^2)|a_2|^4 + \frac{95}{36}a_2^6\right\}.$$

$$\mathrm{II}_1 \leqq |a_3|^2 \left[\frac{1}{20}(9 - 25 b_1^2) + \frac{19}{6}(1 - b_1)\right].$$

Here $[\ ] < 0$ for the interval (11).

$$\mathrm{II}_2 \leqq |a_2|^2 \left[\frac{1}{6}(1 - 9 b_1^4) + (3 + 5 b_1^2)(1 - b_1) \right.$$

$$+ \frac{11}{3} \cdot 2 \cdot 2{,}7 \cdot (1 - b_1)^2 + 61 (1 - b_1)^2$$

$$\left. + \frac{17}{3} \cdot 2 (1 - b_1)^3 + 2 (1 - b_1)^3 + \frac{95}{9} \cdot 4 (1 - b_1)^4 \right].$$

The factor $[\ ]$ is seen to be < 0 for

$$0{,}93 \leqq b_1 \leqq 1 .$$

Hence we proved the

Theorem: $0 \leqq a_7 \leqq \frac{1}{3}(1 - b_1^6)$ *at least for* $0{,}93 \leqq b_1 \leqq 1$.

2. The structure of the coefficients A_{ik} and B_{ik}

We shall discuss the asymptotic character of the coefficients A_{ik} and B_{ik} in the series developments (3) and (4) in the case that the first coefficient b_1 is close to 1. This will allow us to utilize the inequalities (5) to estimate the coeffcients a_j of the function $f(z) \in S(b_1)$ considered. To clarify the situation, we set up

$$f(z) = b_1 \left(z + \sum_{\nu=2}^{n} a_\nu z^\nu + \sum_{\nu=n+1}^{\infty} a_\nu z^\nu\right). \tag{14}$$

It will appear that in the asymptotics for A_{ik} and B_{ik} with $i \leq n$, $k \leq n$ the first set of coefficients plays a different role than the second set. Indeed, we obtain in view of (3) the identity

$$\sum_{i,k=0}^{\infty} A_{ik} z^i \zeta^k = \log b_1 + \log \left[1 + \sum_{\nu=2}^{n} \sum_{\alpha=0}^{\nu-1} a_\nu z^\alpha \zeta^{\nu-\alpha-1} \right. \tag{15}$$
$$\left. + \sum_{\nu=n+1}^{\infty} \sum_{\alpha=0}^{\nu-1} a_\nu z^\alpha \zeta^{\nu-\alpha-1}\right].$$

If we develop the right-hand side into a formal power series in z and ζ and wish to express in this way all A_{ik} with $i, k \leq n$, we may disregard all terms which are of degree of homogeneity $> 2n$. We thus find

(16) $$\sum_{i,k=0}^{\infty} A_{ik} z^i \zeta^k = \log b_1 + \sum_{\nu=2}^{n} \sum_{\alpha=0}^{\nu-1} a_\nu z^\alpha \zeta^{\nu-\alpha-1}$$

$$+ \sum_{\nu=n+1}^{\infty} \sum_{\alpha=0}^{\nu-1} a_\nu z^\alpha \zeta^{\nu-\alpha-1} - \frac{1}{2} [(\sum_{\nu=2}^{n} \sum_{\alpha=0}^{\nu-1} a_\nu z^\alpha \zeta^{\nu-\alpha-1})^2$$

$$+ 2 \sum_{\nu=2}^{n} \sum_{\alpha=0}^{\nu-1} a_\nu z^\alpha \zeta^{\nu-\alpha-1} \cdot \sum_{\nu=n+1}^{\infty} \sum_{\alpha=0}^{\nu-1} a_\nu z^\alpha \zeta^{\nu-\alpha-1}$$

$$+ a_{n+1}^2 (\sum_{\alpha=1}^{n} z^\alpha \zeta^{n-\alpha})^2] + D ,$$

where D is small of the third order in all a_k and is of second order at least in $a_2, \ldots, a_n$ and does not contain terms of degree $\leqq 2n$. Comparing now the coefficients of $z^i \zeta^k$ on both sides of (14), we find

(17) $$A_{ik} = a_{i+k+1} - \sum_{\nu=2}^{i} \nu a_\nu a_{i+k+2-\nu}$$

$$- \frac{1}{2} \sum_{\nu=i+1}^{k+1} (i+1) a_\nu a_{i+k+2-\nu} + \ldots ,$$

for $i \leqq k = 0, \ldots, n$. If a_ν with $\nu < 2$ occurs in these formulas, it is understood to be zero.

In particular,

(17′) $$A_{nn} = a_{2n+1} - \sum_{\nu=2}^{n} \nu a_\nu a_{2n+2-\nu} - \frac{n+1}{2} a_{n+1}^2 + \ldots .$$

The omitted terms are of third order in all a_k and quadratic in $a_2, \ldots, a_n$.

Similarly, we consider the generating function for the matrix $((B_{ik}))$. We have

(18) $$\sum_{i,k=1}^{\infty} B_{ik} z^i \bar{\zeta}^k = - \log \left(1 - f(z) \overline{f(\zeta)}\right)$$

and in view of the representation (14) for $f(z)$ we find

(19) $$\sum_{i,k=1}^{\infty} B_{ik} z^i \bar{\zeta}^k = - \log (1 - b_1^2 z \bar{\zeta})$$

$$- \log \left(1 - b_1^2 \frac{z \sum_2^\infty \bar{a}_\nu \bar{\zeta}^\nu + \bar{\zeta} \sum_2^\infty a_\nu z^\nu + \sum_2^\infty a_\nu z^\nu \sum_2^\infty \bar{a}_\nu \bar{\zeta}^\nu}{1 - b_1^2 z \bar{\zeta}} \right) .$$

Clearly, we find

$$(20)\quad B_{ik} = b_1^{2i}\,\frac{1}{i}\,\delta_{ik} + \left[b_1^2\,\frac{z\sum_2^n \bar a_\nu \bar\zeta^\nu + \bar\zeta \sum_2^n a_\nu z^\nu + \sum_2^n a_\nu z^\nu \sum_2^n \bar a_\nu \bar\zeta^\nu}{1 - b_1^2 z \bar\zeta} \right]_{i,k}$$

$$+ \frac{1}{2}\left[\left(b_1^2\,\frac{z\sum_2^n \bar a_\nu \bar\zeta^\nu + \bar\zeta \sum_2^n a_\nu z^\nu}{1 - b_1^2 z \bar\zeta} \right)^2 \right]_{i,k} + \dots$$

for $i, k \leqq n$. Here $[\Phi(z, \zeta)]_{i,k}$ denotes the coefficient of $z^i \zeta^k$ in the development $\Phi(z, \zeta)$. The terms not written down are at least cubic in $a_2, \dots, a_n$, that is,

$$(21)\quad B_{ik} = \frac{1}{i}\, b_1^{2i}\,\delta_{ik} + a_{i-k+1}\, b_1^{2k} + \bar a_{k-i+1}\, b_1^{2i}$$

$$+ \sum_{\mu=2}^{k} a_{\mu+i-k}\,\bar a_\mu\, b_1^{2(k-\mu+1)} + \frac{1}{2}\sum_{\alpha=1}^{i-1}\sum_{\beta=1}^{k-1} [a_{\alpha-\beta+1}\, b_1^{2\beta}$$

$$+ \bar a_{\beta-\alpha+1}\, b_1^{2\alpha}]\,[a_{i-k-(\alpha-\beta)+1}\, b_1^{2(k-\beta)} + \bar a_{k-i-(\beta-\alpha)+1}\, b_1^{2(i-\alpha)}] + \dots .$$

As before, the deleted terms are at least of third order in all a_k and quadratic in $a_2, \dots, a_n$. Again, a_1 is defined to be zero.

Let us display in particular the term

$$(22)\quad B_{nn} = \frac{1}{n}\, b_1^{2n} + \sum_{\mu=2}^{n} |a_\mu|^2\, b_1^{2(n-\mu+1)}$$

$$+ \frac{1}{2}\sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n-1} [a_{\alpha-\beta+1}\, b_1^{2\beta} + \bar a_{\beta-\alpha+1}\, b_1^{2\alpha}]$$

$$\cdot\, [a_{-(\alpha-\beta)+1}\, b_1^{2(n-\beta)} + \bar a_{-(\beta-\alpha)+1}\, b_1^{2(n-\alpha)}] + \dots .$$

Clearly, a_ϱ is non-zero only for $\varrho \geqq 2$. Hence, the last double sum reduces to

$$\sum_{\alpha \geqq \beta+1} |a_{\alpha-\beta+1}|^2\, b_1^{2(\beta+n-\alpha)}$$

and we arrive at the result

$$(23)\quad B_{nn} = \frac{1}{n}\, b_1^{2n} + \sum_{\mu=2}^{n} |a_\mu|^2\, b_1^{2(n-\mu+1)}$$

$$+ \sum_{\mu=2}^{n} (n-\mu)\, |a_\mu|^2\, b_1^{2(n-\mu+1)} + \dots .$$

3. The case $a_2 = \ldots = a_n = 0$.

We apply the above formulas to solve the following:

Problem. *The coefficient* $a_{2n+1} > 0$ *is to be estimated under the side conditions*

$$a_2 = \ldots = a_n = 0 . \tag{24}$$

The coefficients $a_{n+1}, \ldots, a_{2n}$ *are free parameters of the problem.*

Choose in the Nehari-inequalities (5) $N = n$, $x_n = 1$ and

$$x_0 = \frac{\operatorname{Re}\{\sum_1^n A_{0\mu} x_\mu\}}{\log b_1^{-1}},$$

which is the most favorable choice for x_0. We thus get

$$\operatorname{Re}\{\sum_{i,k=1}^n A_{ik} x_i x_k\} + \sum_{i,k=1}^n B_{ik} x_i \bar{x}_k \tag{25}$$

$$\leqq -\frac{[\operatorname{Re}\{\sum_{k=1}^n A_{0k} x_k\}]^2}{\log b_1^{-1}} + \sum_1^n \frac{|x_i|^2}{i} .$$

In the case (24) in question, the higher order corrections in (17), (21) and (23) vanish. Thus,

$$\begin{cases} A_{ik} = a_{i+k+1}; \quad 0 \leqq i \leqq k \leqq n, \quad i < n, \\ A_{nn} = a_{2n+1} - \dfrac{n+1}{2} a_{n+1}^2 ; \end{cases}$$

$$\begin{cases} B_{kk} = \dfrac{1}{k} b_1^{2k}, \\ k = 1, \ldots, n ; \end{cases}$$

$$\begin{cases} B_{ik} = 0, \\ 1 \leqq i < k \leqq n . \end{cases}$$

In view of (24), the inequality (25) assumes the form

$$\mathrm{Re}\left\{x_1 \sum_{k=n-1}^{n} a_{k+2}\, x_k + x_2 \sum_{k=n-2}^{n} a_{k+3} + \ldots\right.$$

$$+ x_q \sum_{k=n-q}^{n} a_{k+q+1}\, x_k + \ldots + x_{n-1} \sum_{k=1}^{n} a_{k+n}\, x_k$$

$$\left. + x_n \left(\sum_{k=1}^{n-1} a_{k+n+1}\, x_k + a_{2n+1} - \frac{n+1}{2}\, a_{n+1}^2\right)\right\}$$

$$+ \sum_{k=1}^{n} \frac{1}{k}\, b_1^{2k}\, |x^k|^2$$

$$\leqq - \frac{[\mathrm{Re}\,\{a_{n+1}\}]^2}{\log b_1^{-1}} + \sum_{k=1}^{n} \frac{|x_k|^2}{k},$$

that is,

$$a_{2n+1} - \frac{1}{n}\,(1 - b_1^{2n}) \tag{26}$$

$$\leqq \left(\frac{n+1}{2} - \frac{1}{\log b_1^{-1}}\right) [\mathrm{Re}\,\{a_{n+1}\}]^2 - \frac{n+1}{2}\,[\mathrm{Im}\,\{a_{n+1}\}]^2$$

$$- \mathrm{Re}\,\{x_1 \sum_{n-1}^{n} a_{k+2}\, x_k + x_2 \sum_{n-2}^{n} a_{k+3}\, x_k + \ldots$$

$$+ x_q \sum_{n-q}^{n} a_{k+q+1}\, x_k + \ldots + x_{n-1} \sum_{1}^{n} a_{k+n}\, x_k$$

$$+ \sum_{1}^{n-1} a_{k+n+1}\, x_k\} + \sum_{1}^{n-1} \frac{1}{k}\,(1 - b_1^{2k})\, |x_k|^2 .$$

We will study the possible maximum of the left side by choosing the parameters $x_1, \ldots, x_{n-1}$ properly. The result is obtained step by step as follows.

1° $x_1 = \ldots = x_{n-1} = 0$

$$a_{2n+1} - \frac{1}{n}\,(1 - b_1^{2n}) \tag{27}$$

$$\leqq \left(\frac{n+1}{2} - \frac{1}{\log b_1^{-1}}\right) [\mathrm{Re}\,\{a_{n+1}\}]^2 - \frac{n+1}{2}\,[\mathrm{Im}\,\{a_{n+1}\}]^2 .$$

Here $\dfrac{n+1}{2} - \dfrac{1}{\log b_1^{-1}} \leqq 0$ for

$$e^{-\frac{2}{n+1}} \leqq b_1 < 1 .$$

Equality occurs only in the case $b_1 = e^{-\frac{2}{n+1}}$. Consider the general case where $\frac{n+1}{2} - \frac{1}{\log b_1^{-1}} < 0$, which means that

$$e^{-\frac{2}{n+1}} < b_1 < 1 . \tag{28}$$

The left-hand side in (27) is non-positive and zero only if

$$a_{n+1} = 0 .$$

We continue to discuss the case of equality in the estimate for $a_{2n+1} - \frac{1}{n}(1 - b_1^{2n})$. In the interval (28) the condition (26) can be utilized by putting $a_{n+1} = 0$ on its right side.

2° $x_2 = \ldots = x_{n-1} = 0$

The inequality (26) assumes the form

$$a_{2n+1} - \frac{1}{n}(1 - b_1^{2n}) \leqq -2 \operatorname{Re}\{x_1 a_{n+2}\} + (1 - b_1^2)\, |x_1|^2 .$$

In case $x_1 = \bar{a}_{n+2}$, we have the estimate

$$a_{2n+1} - \frac{1}{n}(1 - b_1^{2n}) \leqq - |a_{n+2}|^2 (1 + b_1^2) .$$

Thus, the equality in this estimate requires that necessarily

$$a_{n+2} = 0 .$$

This process can be repeated without restriction n times, and we end up at the following result:

Theorem. *Suppose that* $a_2 = \ldots = a_n = 0$ *and let* $a_{n+1}, \ldots, a_{2n}$ *be free parameters. Then for*

$$e^{-\frac{2}{n+1}} \leqq b_1 \leqq 1$$

we have

$$0 \leqq a_{2n+1} \leqq \frac{1}{n}(1 - b_1^{2n}) . \tag{29}$$

Equality in the case

$$e^{-\frac{2}{n+1}} < b_1 \leqq 1$$

requires that necessarily

$$a_{n+1} = \ldots . = a_{2n} = 0 . \tag{30}$$

The conditions $a_2 = \ldots\ldots = a_{2n} = 0$, $a_{2n+1} = \frac{1}{n}(1 - b_1^{2n})$ can be applied to the extremal conditions of the generalized Nehari inequality [5] to determine the extremal function $f(z)$ completely. It has to satisfy the equation

$$\frac{f}{(1 - f^{k-1})^{\frac{2}{k-1}}} = b_1 \frac{z}{(1 - z^{k-1})^{\frac{2}{k-1}}}, \quad k = 2n + 1 . \tag{31}$$

The image consists of $2n$ radial slits with equal lengths and starting at points located at the corners of a regular $2n$-gon. At the point

$$b_1 = e^{-\frac{2}{n+1}}$$

there is a one-parametric freedom of choice; a_{n+1} can be taken as a parameter. This phenomenon was first observed in the case of a_3 (see [3], [8]).

4. a_{2n+1}; $\delta_n \leqq b_1 < 1$

We shall now drop the assumption (24) and use the fact that for $\delta_n \leqq b_1 < 1$ all a_ν are small. To illustrate the method of estimation to be used, take again a_7 as an example.

Write (10) in the form

$$\begin{aligned} (32) \qquad & a_7 - \frac{1}{3}(1 - b_1^6) \\ & \leqq \left(2 - \frac{1}{\log b_1^{-1}}\right) [\mathrm{Re}\,\{t\}]^2 - 2\,[\mathrm{Im}\,\{t\}]^2 + \varDelta\,; \\ & \varDelta = \varDelta_1 + \varDelta_2\,; \\ & \varDelta_1 = \frac{1}{2}(1 - 9\,b_1^4)\,a_2^2 + \frac{1}{4}(9 - 25\,b_1^2)\,a_3^2\,, \\ & \varDelta_2 = \mathrm{Re}\left\{- 9\,a_2\,a_3\,t - \frac{19}{12}\,a_3^3 - \frac{1}{2}(3 + 5\,b_1^2)\,\bar{a}_2^2\,a_3 \right. \\ & \quad + \frac{11}{3}\,a_2^3\,t + \frac{61}{4}\,a_2^2\,a_3^2 - \frac{17}{2}\,a_2^4\,a_3 + \frac{1}{4}(1 - b_1^2)\,|a_2|^4 \\ & \quad \left. + \frac{95}{36}\,a_2^6 \right\}. \end{aligned}$$

Denote $\max(|a_2|, |a_3|) = M$, and estimate Δ_2 by aid of M. We find immediately that

$$\Delta_2 \leqq M^2 \cdot \lambda\varepsilon ,$$

where

$$\varepsilon = 1 - b_1$$

and λ is a fixed positive constant which could easily be estimated by aid of (12). There are now two alternatives

1) $|a_2| = M$.

Take $9 - 25\, b_1^2 < 0$, which means that $b_1 > \frac{3}{5}$. Then

$$\Delta \leqq \left[\frac{1}{2}(1 - 9\, b_1^4) + \lambda\varepsilon\right] |a_2|^2 ;$$

$$[\ ] = -4 + (18 + \lambda)\,\varepsilon - 27\,\varepsilon^2 + 18\,\varepsilon^3 - 4{,}5\,\varepsilon^4 .$$

We get $[\ ] < 0$ for ε small enough. Hence, in this case

$$a_7 - \frac{1}{3}(1 - b_1^6) \leqq 0 \tag{33}$$

for $0 < 1 - b_1 \leqq \varepsilon_1$; $1 - \varepsilon_1 \leqq b_1 < 1$.

2) $|a_3| = M$

Take $1 - 9\, b_1^4 < 0$, $b_1 > \frac{1}{\sqrt{3}}$.

$$\Delta \leqq \left[\frac{1}{4}(9 - 25\, b_1^2) + \lambda\varepsilon\right] |a_3|^2 .$$

As before, we find (33) to hold for $1 - \varepsilon_2 \leqq b_1 < 1$. Denote $\varepsilon_3 = \max(\varepsilon_1, \varepsilon_2)$. The existence of an interval

$$\delta_3 = 1 - \varepsilon_3 \leqq b_1 < 1 , \tag{34}$$

for which (33) holds, is thus established.

The expansions (17), (21) and (23) allow us now to repeat the above procedure in the general case a_{2n+1}. Let us determine the various expressions occuring in inequality (25).

$$\mathrm{I} = \sum_{i,k=1}^{n} A_{ik}\, x_i\, x_k$$

Write I as follows:

$$\mathrm{I} = \sum_{i=1}^{n-1} x_i \sum_{k=1}^{n-1} A_{ik}\, x_k + 2 \sum_{i=1}^{n-1} A_{in}\, x_i + A_{nn} .$$

(17) and (17′) give

$$
(35)\qquad \begin{cases} A_{ik} = a_{i+k+1} - \sum_{\nu=2}^{k+1} C_\nu^{ik}\, a_\nu\, a_{i+k+2-\nu} + \dots\,, \\ 0 \leqq i \leqq k \leqq n\,, \quad i < n\,; \\ A_{nn} = a_{2n+1} - \sum_{\nu=2}^{n} \nu\, a_\nu\, a_{2n+2-\nu} - \dfrac{n+1}{2}\, a_{n+1}^2 + \dots\,. \end{cases}
$$

Here the C_ν^{ik} are fixed constants whose actual value is unimportant. Hence we obtain

$$
(35')\qquad \begin{aligned} \mathrm{I} &= \sum_{i=1}^{n-1} x_i \sum_{k=1}^{n-1} [a_{i+k+1} - \sum_{\nu=2}^{k+1} C_\nu^{ik}\, a_\nu\, a_{i+k+2-\nu}]\, x_k \\ &+ 2 \sum_{i=1}^{n-1} [a_{i+n+1} - \sum_{\nu=2}^{n+1} C_\nu^{in}\, a_\nu\, a_{i+n+2-\nu}]\, x_i \\ &+ a_{2n+1} - \frac{n+1}{2}\, a_{n+1}^2 - \sum_{\nu=2}^{n} \nu\, a_\nu\, a_{2n+2-\nu} + \dots\,. \end{aligned}
$$

Consider, in particular, the difference

$$
d = 2 \sum_{i=1}^{n-1} a_{i+n+1}\, x_i - \sum_{\nu=2}^{n} \nu\, a_\nu\, a_{2n+2-\nu}\,.
$$

Denote in the latter sum

$$
\nu = n + 1 - i
$$

and obtain for d

$$
\begin{aligned} d &= 2 \sum_{i=1}^{n-1} a_{i+n+1}\, x_i - \sum_{i=n-1}^{1} (n+1-i)\, a_{n+1-i}\, a_{n+1+i} \\ &= \sum_{i=1}^{n-1} [2 x_i - (n+1-i)\, a_{n+1-i}]\, a_{n+1+i}\,. \end{aligned}
$$

The parameters $x_1, \dots, x_{n-1}$, which are at our disposal, may be selected to eliminate d:

$$
(36)\qquad x_i = \frac{n+1-i}{2}\, a_{n+1-i}\,; \quad i = 1, \dots, n-1\,.
$$

The above choice implies that

$$
(37)\qquad \mathrm{I} = a_{2n+1} - \frac{n+1}{2}\, a_{n+1}^2 + E_1\,,
$$

where E_1 is at least quadratic in $a_2, \dots, a_n$ and of third order in all a_k.

$$\text{II} = \sum_{k=1}^{n} A_{0k}\, x_k$$

It follows from (17) that

$$\begin{cases} A_{0k} = a_{k+1} - \dfrac{1}{2} \sum\limits_{\nu=2}^{k} a_\nu\, a_{k+2-\nu} + \dots , \\ k = 1 , \dots , n , \end{cases}$$

and hence

$$\text{II} = a_{n+1} + T . \tag{38}$$

Here T is quadratic in $a_2 , \dots , a_n$ and contains only these coefficients.

$$\text{III} = \sum_{i,k=1}^{n} B_{ik}\, x_i\, \bar{x}_k$$

Again, divide III into the following parts

$$\text{III} = \sum_{i=1}^{n-1} x_i \sum_{k=1}^{n-1} B_{ik}\, \bar{x}_k + 2 \operatorname{Re} \{ \sum_{k=1}^{n-1} B_{nk}\, \bar{x}_k \} + B_{nn} .$$

(21) and (23) give now

$$B_{nk} = a_{n-k+1}\, b_1^{2k} + \dots ; \quad k = 1 , \dots , n - 1 ,$$

$$B_{nn} = \frac{1}{n}\, b_1^{2n} + \sum_{\mu=2}^{n} (n - \mu + 1)\, |a_\mu|^2\, b_1^{2(n-\mu+1)} + \dots ,$$

which allows us to write III in the form

$$\begin{aligned} \text{III} &= \sum_{i=1}^{n-1} |x_i|^2 \frac{1}{i}\, b_1^{2i} + 2 \operatorname{Re} \{ \sum_{k=1}^{n-1} B_{nk}\, \bar{x}_k \} + B_{nn} + \dots \\ &= \sum_{i=1}^{n-1} \frac{1}{i} \left(\frac{n + 1 - i}{2} \right)^2 b_1^{2i} + 2 \sum_{i=1}^{n-1} \frac{n + 1 - i}{2}\, |a_{n-i+1}|^2\, b_1^{2i} \\ &+ \sum_{\mu=2}^{n} (n - \mu + 1)\, |a_\mu|^2\, b_1^{2(n-\mu+1)} + \frac{1}{n}\, b_1^{2n} + \dots . \end{aligned} \tag{38'}$$

By denoting

$$i = n + 1 - \mu ,$$

one obtains:

$$\mathrm{III} = \frac{1}{n}\, b_1^{2n} + \sum_{\mu=2}^{n} \frac{1}{n+1-\mu} \left(\frac{\mu}{2}\right)^2 b_1^{2(n+1-\mu)}$$

$$+ \sum_{\mu=2}^{n} \mu\, |a_\mu|^2\, b_1^{2(n+1-\mu)} + \sum_{\mu=2}^{n} (n+1-\mu)\, |a_\mu|^2\, b_1^{2(n+1-\mu)} + \ldots,$$

which simplifies to

$$\mathrm{III} = \frac{1}{n}\, b_1^{2n} + \sum_{\mu=2}^{n} \frac{\left(n+1-\frac{\mu}{2}\right)^2}{n+1-\mu}\, |a_\mu|^2\, b_1^{2(n+1-\mu)} + E_2\,. \tag{39}$$

E_2 is again of the same nature as E_1.

Combining I, II and III we arrive at the following form of (25)

$$a_{2n+1} - \frac{1}{n}\,(1 - b_1^{2n}) \tag{40}$$

$$\leqq \frac{n+1}{2}\, \mathrm{Re}\,\{t^2\} - \frac{[\mathrm{Re}\,\{t\}]^2}{\log b_1^{-1}}$$

$$+ \sum_{\mu=2}^{n} \frac{1}{n+1-\mu} \left[\left(\frac{\mu}{2}\right)^2 - \left(n+1-\frac{\mu}{2}\right)^2 b_1^{2(n+1-\mu)}\right] |a_\mu|^2 + E\,,$$

where

$$t = a_{n+1} + T\,.$$

Here E is quadratic in $a_2\,, \ldots, a_n$, is of third order in all a_k and consists only of a finite number of terms.

We are now in the position to arrive again at the conclusion drawn in the case of a_7. Notice first that

$$\left(\frac{\mu}{2}\right)^2 - \left(n+1-\frac{\mu}{2}\right)^2 b_1^{2(n+1-\mu)} < 0 \tag{41}$$

if

$$\left(\frac{\frac{\mu}{2}}{n+1-\frac{\mu}{2}}\right)^{\frac{1}{n+1-\mu}} < b_1 < 1\,.$$

Since $1 \leqq \frac{\mu}{2} \leqq \frac{n}{2}$, we see that (41) is always valid if

$$\left(\frac{n}{n+2}\right)^{\frac{2}{n+2}} < b_1 < 1\,. \tag{42}$$

Further,

$$\frac{n+1}{2} \operatorname{Re} \{t^2\} - \frac{[\operatorname{Re} \{t\}]^2}{\log b_1^{-1}}$$

$$= \left(\frac{n+1}{2} - \frac{1}{\log b_1^{-1}}\right) [\operatorname{Re} \{t\}]^2 - \frac{n+1}{2} [\operatorname{Im} \{t\}]^2 < 0$$

for

$$e^{-\frac{2}{n+1}} < b_1 < 1 \,. \tag{43}$$

Finally, take into account that

$$\begin{cases} |a_\nu| \leqq k_\nu \, \varepsilon & (\nu = 2\,,3\,,\ldots)\,, \\ \varepsilon = 1 - b_1\,, \end{cases} \tag{44}$$

where k_ν is a constant. This can be deduced for example from Löwner's coefficient representation for bounded functions by a rough estimation [9]. The nature of E allows now to establish that

$$|E| \leqq M^2 \lambda_n \varepsilon \,, \tag{45}$$

where

$$M = \max (|a_2|\,, \ldots\,, |a_n|)$$

and λ_n is a positive constant. Hence we read off from (40) that again the right side of (40) is negative if $0 < \varepsilon \leqq \varepsilon_n$, i.e.,

$$\delta_n = 1 - \varepsilon_n \leqq b_1 < 1 \,. \tag{46}$$

Thus we established the following result which also has been found by SIEWIERSKI by aid of a variational method [6].

Theorem. *In the class* $S(b_1)$ *of bounded univalent functions the sharp inequality*

$$0 \leqq a_{2n+1} \leqq \frac{1}{n} (1 - b_1^{2n}) \qquad (n = 1\,, 2\,, \ldots\,) \tag{47}$$

is true at least for some interval

$$\delta_n \leqq b_1 \leqq 1 \,, \tag{48}$$

where δ_n *is a positive number* < 1.

5. The coefficient of $\sqrt{f(z^2)}$ and application to general a_{2m}.

Let us start by considering the expansion

$$\sqrt{f(z^2)} = b_1^{\frac{1}{2}} z \left[1 + \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(2i-2)!\, i}{2^{2i-1} (i!)^2} (a_2 z^2 + \ldots + a_{\mu+1} z^{2\mu} + \ldots)^i\right]$$

$$= b_1^{\frac{1}{2}} (z + A_3 z^3 + \ldots + A_{2\mu+1} z^{2\mu+1} + \ldots)\,.$$

We can express the coefficients $A_{2\mu+1}$ of the new odd function in terms of the coefficients a_ν of the original function $f(z) \in S(b_1)$. We write

$$A_{2\mu+1} = \frac{1}{2} a_{\mu+1} - \frac{1}{4} K_\mu + \Delta_\mu \tag{49}$$

and define the terms K_μ and Δ_μ differently in the cases $\mu =$ even and $\mu =$ odd.

1) If $\mu = 2p,\ p = 1, 2, 3, \ldots,$

we let

$$K_\mu = K_{2p} = a_2 a_{2p} + a_3 a_{2p-1} + \ldots + a_p a_{p+2} + \frac{1}{2} a_{p+1}^2 \tag{50}$$

and denote by $\Delta_\mu = \Delta_{2p}$ the remainder term which is cubic in $a_2, \ldots, a_{2p}$ and at least quadratic in $a_2, \ldots, a_p$.

2) If $\mu = 2q + 1,\ q = 1, 2, 3, \ldots,$

we define

$$K_\mu = K_{2q+1} = a_2 a_{2q+1} + a_3 a_{2q} + \ldots + a_{q+1} a_{q+2}\,; \tag{51}$$

$\Delta_\mu = \Delta_{2q+1}$ is quadratic in $a_2, \ldots, a_{q+1}$ and cubic in $a_2, \ldots, a_{2q+1}$.

Let us apply the generalised Nehari inequality (5) for

$$N = 2m - 1 \qquad (m = 2, 3, \ldots)$$

in the case of the function $\sqrt{f(z^2)}$. The highest order coefficient in the inequality is

$$A_{2N+1} = A_{4m-1} = \frac{1}{2} a_{2m} - \frac{1}{4} K_{2m-1} + \Delta_{2m-1}\,. \tag{52}$$

It will appear that we can omit the parameters x_ν with even index:

$$x_0 = x_2 = \ldots = x_{2m-2} = 0\,. \tag{53}$$

This means neglecting the parameter x_0; hence we actually utilize the original Nehari inequality. Moreover, when maximizing A_{2N+1} we have to take

$$x_N = x_{2m-1} = 1 .$$

In Section 4 the left side of (25) was decomposed in the expressions I, II and III. We have to rewrite these expressions for the present case as follows:

$$\underline{\mathrm{I} = \sum_{i,k=1}^{N} A_{ik}\, x_i\, x_k}$$

In (35′) we substitute

$$i = 2r + 1\,, \quad k = 2s + 1\,, \quad \nu = 2t + 1\,.$$

In view of

$$\begin{cases} i = 1, \ldots, & 2m - 3\,, \\ k = 1, \ldots, & 2m - 3\,, \\ \nu = 3, \ldots, & 2m - 1\,, \end{cases}$$

we get for r, s, t:

$$\begin{cases} r = 0, \ldots, m - 2\,, \\ s = 0, \ldots, m - 2\,, \\ t = 1, \ldots, m - 1\,. \end{cases}$$

Further, denote

$$C_\nu^{ik} = C_{2t+1}^{(2r+1)\,(2s+1)} = D_t^{rs}$$

and find

$$\text{(54)} \quad \mathrm{I} = \sum_{r=0}^{m-2} x_{2r+1} \sum_{s=0}^{m-2} [A_{2(r+s+1)+1} - \sum_{t=1}^{s} D_t^{rs}\, A_{2t+1}\, A_{2(r+s-t+1)+1}]\, x_{2s+1}$$
$$+ \; 2 \sum_{r=0}^{m-1} [A_{2(r+m)+1} - \sum_{t=1}^{m-1} D_t^{rs}\, A_{2t+1}\, A_{2(r+m-t)+1}]\, x_{2r+1}$$
$$+ \; \underline{A_{4m-1}} - \sum_{t=1}^{m-1} (2t + 1)\, \underline{A_{2t+1}\, A_{2(2m-t)-1}} + \cdots .$$

The underlined coefficients give the following effective contribution to I:

$$\mathrm{I} \ni \frac{1}{2}\, a_{2m} + \sum_{r=0}^{m-2} a_{r+m+1}\, x_{2r+1} - \sum_{t=1}^{m-1} \frac{t+1}{2}\, a_{t+1}\, a_{2m-t}\,.$$

By denoting

$$t = m - 1 - r\,; \quad r = m - 2, \ldots, 0\,,$$

we obtain for this contribution

$$\mathrm{I} \ni \frac{1}{2} a_{2m} + \sum_{r=0}^{m-2} \left[x_{2r+1} - \frac{m-r}{2} a_{m-r} \right] a_{m+r+1} \, .$$

This suggests the choice

$$x_{2r+1} = \frac{m-r}{2} a_{m-r} \qquad (r = 0, \ldots, m-2) \tag{55}$$

for the parameters $x_1, x_3, \ldots, x_{2m-3}$ at our disposal. Hence (54) is reduced to

$$\mathrm{I} = \frac{1}{2} a_{2m} + E_1 \, , \tag{56}$$

where E_1 is quadratic in $a_2, \ldots, a_m$ and cubic in $a_2, \ldots, a_{2m-1}$.

The number $\mathrm{II} = \sum_{k=1}^{N} A_{0k} x_k$ has the coefficient $x_0 = 0$ and therefore no effect in the present case. There remains the combination

$$\underline{\mathrm{III} = \sum_{i,k=1}^{N} B_{ik} x_i \bar{x}_k}$$

From (38′) we deduce by using (53)

$$\mathrm{III} = \sum_{r=0}^{m-2} |x_{2r+1}|^2 \frac{1}{2r+1} b_1^{2r+1} \tag{56′}$$

$$+ 2 \operatorname{Re} \{ \sum_{s=0}^{m-2} B_{(2m-1)(2s+1)} \bar{x}_{2s+1} \} + B_{(2m-1)(2m-1)} + \ldots ,$$

where

$$B_{(2m-1)(2s+1)} = A_{2(m-s-1)+1} b_1^{2s+1} + \ldots$$
$$= \frac{1}{2} a_{m-s} b_1^{2s+1} + \ldots ,$$

and

$$B_{(2m-1)(2m-1)} = \frac{1}{2m-1} b_1^{2m-1} + \sum_{t=1}^{m-1} (2m-2t-1) |A_{2t+1}|^2 b_1^{2m-2t-1} + \ldots$$

$$= \frac{1}{2m-1} b_1^{2m-1} + \sum_{\tau=0}^{m-2} \frac{2m-2\tau-3}{4} |a_{\tau+2}|^2 b_1^{2m-2\tau-3} + \ldots ,$$

where

$$\tau = t - 1 \, .$$

This gives

$$\mathrm{III} = \sum_{r=0}^{m-2} |x_{2r+1}|^2 \frac{1}{2r+1} b_1^{2r+1} + \mathrm{Re}\{\sum_{s=0}^{m-2} a_{m-s} b_1^{2s+1} \bar{x}_{2s+1}\}$$

$$+ \frac{1}{2m-1} b_1^{2m-1} + \sum_{\tau=0}^{m-2} \frac{2m-2\tau-3}{4} |a_{\tau+2}|^2 b_1^{2m-2\tau-3} + \ldots\ldots$$

Substituting

$$s = r, \quad \tau = m-2-r$$

into (55), and combining similar terms, we finally find

$$(57) \qquad \mathrm{III} = \frac{1}{2m-1} b_1^{2m-1} + \frac{1}{4} \sum_{r=0}^{m-2} \frac{(m+r+1)^2}{2r+1} |a_{m-r}|^2 b_1^{2r+1} + E_2 .$$

The omitted terms in E_2 are of the same nature as in E_1.

Applying now the formulas (55), (56) and (57) to the inequality (5), we obtain

$$\mathrm{I} + \mathrm{III} \leqq \frac{1}{2m-1} + \sum_{r=0}^{m-2} \frac{|x_{2r+1}|^2}{2r+1},$$

which implies that

$$(58) \qquad a_{2m} - \frac{2}{2m-1}(1-b_1^{2m-1})$$

$$\leqq \frac{1}{2} \sum_{r=0}^{m-2} \frac{1}{2r+1} [(m-r)^2 - (m+r+1)^2 b_1^{2r+1}] |a_{m-r}|^2 + E$$

$$(m = 2, 3, \ldots) .$$

The error term E is quadratic in $a_2, \ldots, a_m$ and cubic in all a_k.

The right side of (58) can be estimated as in (40). The estimation is based on the nature of E and the fact that

$$(m-r)^2 - (m+r+1)^2 b_1^{2r+1} < 0 \qquad (r = 0, \ldots, m-2)$$

for

$$\left(\frac{m}{m+1}\right)^2 < b_1 < 1 .$$

We end up with

$$(59) \qquad a_{2m} \leqq \frac{2}{2m-1}(1-b_1^{2m-1})$$

at least for some interval $\delta_m \leqq b_1 \leqq 1$, where $0 < \delta_m < 1$. Hence we proved:

Theorem. *In the class* $S(b_1)$ *of bounded univalent functions the sharp inequality*

(60) $$0 \leqq a_k \leqq \frac{2}{k-1}(1 - b_1^{k-1}) \qquad (k = 2, 3, \ldots)$$

is true at least for some interval

(61) $$\delta_k \leqq b_1 \leqq 1, \quad 0 < \delta_k < 1.$$

The result confirms the conjecture by Charzyński—Tammi [1], [10]. One extremal function is found by the solution of

(62) $$\frac{f}{(1 - f^{k-1})^{\frac{2}{k-1}}} = b_1 \frac{z}{(1 - z^{k-1})^{\frac{2}{k-1}}} \qquad (k = 2, 3, \ldots).$$

The proof of uniqueness of the extremal function requires a more detailed discussion of the error term. For $a_2 = \ldots = a_m = 0$ a result similar to that in Section 3 can be achieved.

7. a_{2m} with $a_2 = \ldots = a_m = 0$

Let us finally consider the following:

Problem. $a_{2m} > 0$ *is to be maximized with the side conditions*

(63) $$a_2 = \ldots = a_m = 0.$$

The coefficients $a_{m+1}, \ldots, a_{2m-1}$ *are free parameters of the problem.*

In view of (49), the side conditions (63) give for $A_{2\mu+1}$ the following values:

(64) $$\begin{cases} A_3 = \ldots = A_{2m-1} = 0, \\ A_{2\mu+1} = \dfrac{1}{2} a_{\mu+1} \qquad (\mu = m, \ldots, 2m-1). \end{cases}$$

The expressions (54) and (56′) simplify considerably in this case:

$$\mathrm{I} = \frac{1}{2} \sum_{r=1}^{m-2} x_{2r+1} \sum_{s=m-1-r}^{m-2} a_{r+s+2}\, x_{2s+1} + \sum_{r=0}^{m-2} a_{r+m+1}\, x_{2r+1} + \frac{1}{2} a_{2m},$$

$$\mathrm{III} = \sum_{r=0}^{m-2} |x_{2r+1}|^2 \frac{1}{2r+1} b_1^{2r+1} + \frac{1}{2m-1} b_1^{2m-1}.$$

The above expressions contain no remainder terms because of the side conditions (63).

The inequality (5) assumes now the form

$$(65)\qquad a_{2m} - \frac{2}{2m-1}(1 - b_1^{2m-1})$$

$$\leqq - \operatorname{Re}\{\sum_{r=1}^{m-2} x_{2r+1} \sum_{s=m-1-r}^{m-2} a_{r+s+2}\, x_{2s+1} + 2\sum_{r=0}^{m-2} a_{r+m+1}\, x_{2r+1}\}$$

$$+ \sum_{r=0}^{m-2} \frac{2}{2r+1}(1 - b_1^{2r+1})\,|x_{2r+1}|^2 .$$

Exactly as in Section 3, we conclude from this the following

Theorem. *Keep* $a_2 = \ldots = a_m = 0$ *and let* $a_{m+1}, \ldots, a_{2m-1}$ *be free parameters. Then for* $0 < b_1 \leqq 1$

$$(66)\qquad 0 < a_{2m} \leqq \frac{1}{2m-1}(1 - b_1^{2m-1}) \quad (m = 1, 2, \ldots) .$$

Equality is only possible if

$$(67)\qquad a_{m+1} = \ldots = a_{2m-1} = 0 .$$

Stanford University
Stanford, California, U.S.A.

University of Helsinki
Helsinki, Finland.

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