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# TWO REMARKS CONCERNING DIRICHLET'S L-FUNCTIONS AND THE CLASS NUMBER OF THE CYCLOTOMIC FIELD

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## TWO REMARKS CONCERNING DIRICHLET'S $L$-FUNCTIONS AND THE CLASS NUMBER OF THE CYCLOTOMIC FIELD

**1.** In the following we denote by $K$ an algebraic number field of degree $n$ over the field $P$ of rational numbers. In addition let $d$ denote the discriminant, $R$ the regulator and $h$ the number of classes of ideals of $K$. BRAUER (cf. [3], p. 745) has proved the following[1])

**Theorem 1.** *If $K$ ranges over a sequence of fields normal over $P$ for which $n/\log|d| \to 0$, then*

$$\log(Rh) \sim \log\sqrt{|d|}\ .$$

This theorem treats very general cases. If we restrict ourselves to the cyclotomic field $K = P(\zeta)$, where $\zeta$ is a primitive $p$th ($p$ is an odd prime.) root of unity, we have

**Theorem 2.** *For any positive $\varepsilon$,*

$$2p^{\frac{1}{2}p}(2\pi)^{\frac{1}{2}(1-p)}c(\varepsilon)p^{-\varepsilon} < Rh < 2p^{\frac{1}{2}p}(2\pi)^{\frac{1}{2}(1-p)}\log^{c}p\ .$$

*where $c$ and $c(\varepsilon)$ denote respectively an absolute positive constant and a positive constant depending on parameter $\varepsilon$ alone.*

This result introduced by TATUZAWA (cf. [8], p. 111) gives in this special case a sharper estimation than the one of theorem 1.

In this paper we consider the cyclotomic field $K = P(\zeta)$, where $\zeta$ is a primitive $m$th root of unity. We suppose that the natural number $m$ is $>1$ and in addition we exclude those even values of $m$, which are not divisible by 4. This restriction is not essential because both the primitive $m$th and $(m/2)$th roots of unity generate the same field if $m$ has some excluded value. We are able to extend theorem 2 and prove

**Theorem 3.** *If $c$ and $c(\varepsilon)$ are defined as above then*

$$c(\varepsilon)m^{-\varepsilon} < Rh/G < \exp(c(\log\log m + \omega(m)))\ ,$$

[1]) The result of theorem 1 is an improvement of BRAUER's earlier result (cf. [2], p. 243). This estimation cannot, however, be applied in the case of the cyclotomic field, because it treats only the cases, where $d$ tends to infinity in such a way that the degree of the corresponding fields is fixed. It should be noted that VINOGRADOV (cf. [9], p. 562) has presented estimations, which complete the result of BRAUER, but which cannot be applied here, because they also presuppose that the degree of the fields under examination is fixed.

*where*

$$G = (2\,\pi)^{-\frac{1}{2}\varphi(m)}\, w \sqrt{|d|}$$

(*As usual,* $\varphi$ *denotes Euler's function.*). *Here*

$$w = \begin{cases} 2m \text{ if } 2 \nmid m\,, \\ m \text{ if } 2 | m \end{cases}$$

*is the number of roots of* 1 *contained in* $P(\zeta)$.

It should be mentioned that it is possible to apply theorem 1 in the case of the cyclotomic field. It is known that this field is normal over $P$ and in addition we have

$$\log |d| \geqq \tfrac{1}{2}\, \varphi(m) \log m$$

(cf. [6], p. 27 and [5], p. 508). This yields

$$0 < n/\log |d| \leqq 2/\log m \to 0\,,$$

when $m$ tends to infinity. It is, however, easy to see that in the case of the cyclotomic field the result of theorem 3 is sharper than the one of theorem 1.

**2.** In addition we consider the so-called Dirichlet's $L$-functions

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}\,.$$

Here $\chi(n)$ denotes a character (mod $k$), where $k$ is an arbitrary natural number and $s$ is a real variable. For these functions we have (cf. [6], p. 31)

**Theorem 4.** *Let* $\varepsilon$ *be an arbitrary positive number,* $\delta$ *a positive number* $< \frac{1}{2}$ *and*

$$\tau = \tau(k) = \begin{cases} \omega(k) \text{ if } 2 \nmid k\,, \\ \omega(k) + 1 \text{ if } 2 | k\,, \end{cases}$$

*where* $\omega(k)$ *denotes the number of different prime factors of* $k$. *If the extended Riemann hypothesis is true, there exists for every given pair* $s$ *and* $k$ ($\geqq 3$), *where*

$$s \geqq (\tau + \delta)/(\tau + 1),\; k > k_0(\varepsilon, \delta)\,,$$

*a non-principal character* $\chi(n)$ (mod $k$) *so that*

$$(1) \qquad |L(s, \chi)| < 1 + \varepsilon\,.$$

If $k$ is an odd prime, the above theorem gives the result of Ankeny and Chowla (cf. [1], p. 487).

In the following we denote a character by $\chi^*$, when it is a primitive character and we especially want to lay stress on this fact. In this paper we restrict ourselves to the case $k = p^u$ ($p$ is a prime). In this special case we are able to reach a slightly deeper result than the one of theorem 4. We prove

**Theorem 5.** *Let $\varepsilon$ and $\delta$ be defined in the same way as in theorem 4. If the extended Riemann hypothesis is true, there exists for every given pair $s$ and $p^u$, where*

$$s \geqq \begin{cases} (1+\delta)/2 \text{ if } p > 2 \text{ and } u \geqq u_0 = u_0(\varepsilon, \delta, p), \\ (2+\delta)/3 \text{ if } p = 2 \text{ and } u \geqq u_0 = u_0(\varepsilon, \delta), \end{cases} \tag{2}$$

*an odd primitive character $\chi^*$ (mod $p^u$), which satisfies the inequality* (1).

As a consequence of this we get

**Theorem 6.** *If the conditions of theorem 5 are satisfied, then there exist for every given pair $s$ and $p^u$ at least $u - u_0 + 1$ odd characters $\chi$ (mod $p^u$), which satisfy the inequality* (1).

**3.** Consider theorem 3. It is known that there exists for each character $\chi$(mod $m$) a unique character $\chi^*$ (mod $f$) equivalent to $\chi$, where $f = f(\chi)$ is the conductor of $\chi$. We denote by $S$ the set of all these characters $\chi^*$ (mod $f(\chi)$). Obviously $S$ is also the set of all the primitive characters each of which is equivalent to a character (mod $m$). We need the following lemmas:

**Lemma 1.** *If $S'$ is the set $S$ excluding the principal character $\chi_0$, we have*

$$h = (2\pi)^{-\frac{1}{2}\varphi(m)} R^{-1} w \sqrt{|d|} \prod_{\chi \in S'} L(1, \chi)$$

(cf. [4], p. 402).

**Lemma 2.** *Let $\chi (\neq \chi_0)$ be a character* (mod $m$), *which has the conductor $f$. Then*

$$L(1, \chi^*) = L(1, \chi) \prod_{p \mid m} (1 - \chi^*(p)/p)^{-1},$$

*where $\chi^*$ is the corresponding primitive character* (mod $f$) *and $p$ runs through the prime factors of $m$* (cf. [7], p. 127).

**Lemma 3.** $\omega(m) = O(\log m / \log \log m)$
(cf. [8], p. 108).

**Lemma 4.** *Let $N(k)$ denote the set of the characters* (mod $k$). *Then*

$$c(\varepsilon) m^{-\varepsilon} < \left| \prod_{\chi \in N'} L(1, \chi) \right| < \exp(c(\log\log m + \omega(m))),$$

*where $N'$ is the set $N(m)$ excluding $\chi_0$* (cf. [8], p. 110).

**Lemma 5.** *If* $m_t$ *is the greatest divisor of* $m$ *prime to* $t$, *then*

$$\sum_{\chi \in S} \chi(t) = \sum_{\chi \in N(m_t)} \chi(t) .$$

The proof of this lemma is analogous to the proof of the corresponding result of lemma 7 in [6].

We write, by lemma 2,

$$\prod_{\chi \in S'} L(1, \chi) = \prod\nolimits_1 \prod\nolimits_2 \prod\nolimits_3 ,$$

where

$$\prod\nolimits_1 = \prod_{\chi \in N'} L(1, \chi) , \quad \prod\nolimits_2 = \prod_{\chi \in S} \prod_{p|m} (1 - \chi(p)/p)^{-1} ,$$

$$\prod\nolimits_3 = \prod_{p|m} (1 - p^{-1}) .$$

We get an estimation for the product $\prod_1$ from lemma 4. Consider the product $\prod_2$ (cf. [6], p. 26). It can be expressed in the form

$$\prod\nolimits_1 = \exp(\sum_{\chi \in S} \sum_{p|m} \sum_{j=1}^{\infty} \chi(p^j) j^{-1} p^{-j}) .$$

Further we have, by lemma 5,

$$\sum_{\chi \in S} \chi(p^j) = \begin{cases} \varphi(m_p) \text{ if } p^j \equiv 1 \pmod{m_p} , \\ 0 \text{ otherwise,} \end{cases}$$

where $m_p$ denotes the greatest divisor of $m$ prime to $p$. Let $\sigma$ denote the least positive exponent so that

$$p^{\sigma} \equiv 1 \pmod{m_p} .$$

Now if

$$p^{\nu} \equiv 1 \pmod{m_p} , \tag{3}$$

then $\sigma | \nu$ and, on the other hand, if $\sigma | \nu$ then (3) holds. Since we can write

$$\sum_{j=1}^{\infty} (p^{-\sigma})^j = (p^{\sigma} - 1)^{-1} \leqq m_p^{-1} ,$$

we have

$$\sum_{\chi \in S} \sum_{p|m} \sum_{j=1}^{\infty} \chi(p^j) j^{-1} p^{-j} = O(\sum_{p|m} \varphi(m_p)/m_p) = O(\omega(m)) .$$

Hence

$$e^{-c\omega(m)} < \prod\nolimits_2 < e^{c\omega(m)} .$$

For $\prod_3$ we finally get

$$2^{-\omega(m)} < \prod_3 < \omega(m) .$$

The above results yield, by lemma 3,

$$c(\varepsilon)m^{-\varepsilon} < |\prod_{\chi \in S'} L(1, \chi)| < \exp(c(\log\log m + \omega(m))) , \tag{4}$$

which proves theorem 3. Here $c(\varepsilon)$ and $c$ are not necessarily the same in lemma 4 and in (4).

**4.** Consider now theorem 5. If $s$ satisfies the conditions (2) and the extended Riemann hypothesis is true, we can write (cf. [6], p. 35)

$$\prod_{\chi \in Q(p^u)} L(s, \chi) = \exp(\varphi(p^u)\, \psi(p^u)) ,$$

where $Q(p^u)$ is the set of the odd characters $\chi \pmod{p^u}$, and $\psi(p^u) \to 0$, when $p^u$ tends to infinity. Because $Q(p^{u-1}) \subset Q(p^u)$, we can decide that

$$T = T(p^u) = Q(p^u) - Q(p^{u-1})$$

is the set of the odd primitive characters $\chi^* \pmod{p^u}$. Further

$$\begin{aligned} \prod_{\chi \in T} L(s, \chi) &= \exp(\varphi(p^u)\, \psi(p^u) - \varphi(p^{u-1})\psi(p^{u-1})) \\ &= \exp(\varphi(p^{u-1})\psi_1(p^u)) , \end{aligned} \tag{5}$$

where

$$\psi_1(p^u) = p\psi(p^u) - \psi(p^{u-1}) .$$

If $p$ is fixed and $u \to \infty$, then $\psi_1(p^u) \to 0$. Let $g$ denote the number of the odd primitive characters $\pmod{p^u}$. We find that

$$g = \begin{cases} \tfrac{1}{2}(\varphi(p^u) - \varphi(p^{u-1})) \text{ if } u \geqq 2 , \\ \tfrac{1}{2}(p-1) \text{ if } u = 1 . \end{cases}$$

In both cases $g \geqq \tfrac{1}{2}\varphi(p^{u-1})$, and we can see, by (5), that theorem 5 is true.

Because $T(p^j) \cap T(p^{j-1}) = \Phi \ (u \geqq j \geqq 2)$ and

$$\bigcup_{j=u_0}^{u} T(p^j) \subset Q(p^u) ,$$

we can determine that there exist at least $u - u_0 + 1$ odd characters $\pmod{p^u}$ satisfying the inequality (1). This yields theorem 6.

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