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DEFICIENCY INDICES OF LINEAR MAPPING

BY

JUHA TIENARI

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Preface

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Oulu, April 1968.

JUHA TIENARI

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Introduction

In this paper we study linear mappings in Banach space, especially their deficiency index. My purpose is to prove that the deficiency index of a mapping-valued function with complex variable is constant in a connected subset of the complex plane when some natural continuity conditions are valid. This problem has been solved before in some cases. (See Krasnoselski [7], Krein—Krasnoselski—Milman [8], Kato [5], [6], Neubauer [11].) Further, we shall show by counter-examples that none of the assumptions that have been made can be omitted.

In Section 1, the deficiency index and regularity region are defined and the natural continuity conditions for a mapping-valued function are also given. In Section 2, complements and finite-dimensional manifolds are examined. The auxiliary theorems given here are needed in the proof of the main theorem. The main theorem is presented in Section 3. Section 4 deals with topological isomorphy and with two formal generalizations. In Section 5 three examples are presented.

The present paper will deal with vector spaces, and scalars are assumed to be complex numbers. The notations and definitions are the same as in Taylor: Introduction to Functional Analysis [12]. The notation »Taylor 4.2.-H» refers to Theorem 4.2.-H in this text-book.

1. Regularity region

1.1. Let T be a linear mapping, the domain of which is a linear manifold D in the Banach space B and the values of which are vectors in B. We denote

$$T(p) = T-pI,$$

where p is a complex number and I is the identical mapping. Now $T(p)$ is a linear mapping with domain D. The regularity region of T is said to be the set of all the complex numbers p with which $T(p)$ has a continuous inverse mapping. The regularity region is known to be an open subset of the complex plane (Taylor 5.1.-B). It will be denoted by the symbol $Z(T)$.

The deficiency index of the mapping T is defined as the dimension of the quotient space $B/\overline{T(D)}$, where $\overline{T(D)}$ is the closure of the set $\{Tx \mid x \in D\}$. (See Gohberg—Krein [3].) The abbreviation »def» is used for the deficiency index. Hence,

$$\text{def } T = \dim (B/\overline{T(D)}) . \tag{1}$$

If the vector space V is infinite-dimensional, we denote $\dim V = \infty$. Accordingly every infinite-dimensional vector space has the same dimension in this paper.

Krasnoselski [7] has proved the following theorem: *The deficiency index of the linear operator T-pI in a complex Hilbert space with $p \in C$ is constant in any connected subset of the regularity region of T.*

The result has subsequently been generalized to apply to linear operators in a Banach space (Krein—Krasnoselski—Milman [8]).

1.2. In the following this theorem will be proved in a new way, at the same time moderating the hypotheses. We examine a general mapping family $T(p)$ with complex parameter p. In order to find the natural continuity conditions, the operator $T(p) = T - pI$ defined above will first be examined.

According to the definition of the regularity region, $T(p)^{-1}$ exists and is continuous whenever $p \in Z(T)$.

Calculations show that

$$\begin{aligned} T(q)\, T(p)^{-1} &= (T-qI)\,(T-pI)^{-1} \\ &= (T-pI)\,(T-pI)^{-1} + (pI-qI)\,(T-pI)^{-1} \\ &= I_p + (p-q)\,(T-pI)^{-1} , \end{aligned}$$

where $p, q \in Z(T)$, while I_p is the identical mapping in the range of $T(p)$. Since I_p and $(T-pI)^{-1}$ are continuous, $T(q)\,(T(p)^{-1}$ is continuous, too. Furthermore,

$$\begin{aligned} \|T(q)\, T(p)^{-1} - I_p\| &= \|(p-q)\,(T-pI)^{-1}\| \\ &= |p-q|\, \|T(p)^{-1}\| . \end{aligned}$$

On the basis of the above, the mapping family $T(p) = T-pI$ with $p \in Z(T)$ can be concluded to satisfy the following continuity conditions:

(a) $T(p)^{-1}$ exists and is continuous whenever $p \in Z(T)$,

(b) $T(q)\, T(p)^{-1}$ is continuous whenever $p, q \in Z(T)$,

(c) $\lim\limits_{q \to p} T(q)\, T(p)^{-1} = T(p)\, T(p)^{-1} = I_p$ in the sense of norm convergence, that is,

$$\lim_{q \to p} \|T(q)\, T(p)^{-1} - I_p\| = 0 \text{ whenever } p \in Z(T) .$$

1.3. Let B and B_0 be Banach spaces, D a linear manifold in B_0, Z an open subset of the complex plane and $T(p) : D \to B$ a linear mapping whenever $p \in Z$. $T(p)$ can be interpreted to be a function of the parameter p with domain Z and its range a subset of the set $\{S : D \to B \mid S$ is a linear mapping$\}$. This function of the mapping value is called *a mapping family*. The same symbol $T(p)$ has been used here both for the function and for its value. This is unlikely to create confusion, since the expression »mapping family» will precede $T(p)$ whenever the function of the parameter p is meant. When $p \in Z$, $T(p)$ refers to a linear mapping with domain D.

This mapping family is said to have *the property* E if

(a) $T(p)^{-1}$ exists and is continuous whenever $p \in Z$,

(b) to each point $p \in Z$ there corresponds a positive number d such that $q \in Z$ and $T(q)\,T(p)^{-1}$ is continuous whenever $q \in C$ and $|p-q| < d$,

(c) $\lim_{q \to p} \|T(q)\,T(p)^{-1} - I_p\| = 0$ whenever $p \in Z$ and $I_p = T(p)T(p)^{-1}$ is the identical mapping in the range of $T(p)$.

If the mapping family $T(p)$ has the property E, then its domain Z is called *the regularity region* of the mapping family $T(p)$.

An example of a mapping family with the property E is, as we have seen, the operator $T(p) = T - pI$, where T is a linear mapping with its domain and range both in the same Banach space and the complex parameter p an element of the regularity region of T. The regularity region of the mapping family $T - pI$ is the same as that of the mapping T.

1.4. A general mapping family $T(p)$ with the property E is studied in the following. The deficiency index of the mapping $T(p)$ will be proved to be constant when p is an element in a connected subset of the regularity region of the mapping family $T(p)$.

It will also be proved by means of counter-examples that the assumptions associated with the property E cannot be reduced. Continuity of the mapping $T(p)^{-1}$ is found to be indispensable. The assumption (c) cannot be replaced by weak convergence, that is $\lim_{q \to p} T(q)\,T(p)^{-1}x = x$ whenever x is a vector in the range of $T(p)$. It will also be shown that the Banach space cannot be replaced by a general topological vector space. The assumption (b) is contained in the assumption (c), but it has been separately written for the sake of clarity.

1.5. An auxiliary theorem will first be proved. To avoid recurrent repetition of the same assumptions, the following notations shall be agreed:

B and B_0 are Banach spaces;

$T(p)$ is a mapping family with the property E;

Z is the regularity region of the mapping family $T(p)$;

D is a linear manifold in B_0, and $D \neq \{\bar{0}\}$;
when $p \in Z$, then $T(p) : D \to B$ is a linear mapping;
$R_p = \{T(p)x \mid x \in D\}$ is the range of $T(p)$.

Theorem 1. *When $p \in Z$, there are positive numbers d and m such that the mappings $T(q)\,T(p)^{-1}$ and $T(p)\,T(q)^{-1}$ do exist,*

$$\|T(q)\,T(p)^{-1}\| \leqq 2\,, \quad \|T(p)\,T(q)^{-1}\| \leqq 2 \tag{2}$$

and

$$|T(q)y| \geqq m|y|\,, \tag{3}$$

whenever $q \in C$, $|p-q| < d$ and $y \in D$.

Proof. Since the mapping family $T(p)$ has the property E and since $p \in Z$, such positive number d_1 exists that $q \in Z$ and $T(q)\,T(p)^{-1}$ exists and is continuous whenever $q \in C$ and $|p-q| < d_1$. According to the assumption (c) of the property E there exists a positive number d_2 such that $\|T(q)\,T(p)^{-1} - I_p\| < \frac{1}{2}$ whenever $q \in C$ and $|p-q| < d_2$.

We denote $d = \min(d_1, d_2)$. In the following q is an element in Z with $|p-q| < d$.

Now $\|T(q)\,T(p)^{-1}\| \leqq \|T(q)\,T(p)^{-1} - I_p\| + \|I_p\| < \frac{1}{2} + 1$,

and for every vector $x \in R_p$,

$$\begin{aligned}|T(q)\,T(p)^{-1}\,x| &= |(T(q)\,T(p)^{-1} - I_p)\,x + x|\\ &\geqq |x| - \|T(q)\,T(p)^{-1} - I_p\|\,|x| \geqq \tfrac{1}{2}|x|\,.\end{aligned}$$

It follows from the above inequalities that

$$\|T(q)\,T(p)^{-1}\| \leqq 2\,,$$

the mapping $(T(q)\,T(p)^{-1})^{-1} = T(p)\,T(q)^{-1}$ exists, and $\|T(p)\,T(q)^{-1}\| \leqq 2$. (See Taylor 3.1.—B.)

Let y be an arbitrary element in D. We denote

$$x = T(p)\,y \quad \text{or} \quad y = T(p)^{-1}\,x\,.$$

By using the inequalities

$$|y| = |T(p)^{-1}\,T(p)y| \leqq \|T(p)^{-1}\|\,|T(p)y|$$

and

$$|T(q)\,T(p)^{-1}\,x| \geqq \tfrac{1}{2}|x|\,,$$

we obtain

$$\begin{aligned}|T(q)y| &= |T(q)\,T(p)^{-1}\,x| \geqq \tfrac{1}{2}|x| = \tfrac{1}{2}|T(p)y|\\ &\geqq \tfrac{1}{2}(\|T(p)^{-1}\|^{-1})\,|y|\,.\end{aligned}$$

We denote $m = \frac{1}{2}\|T(p)^{-1}\|^{-1}$.
Now $|T(q)y| \geqq m\,|y|$ whenever $y \in D$ and $|p-q| < d$. The theorem is proved.

2. Auxiliary theorems

2.1. Linear manifolds and their complements will be discussed in the following. The theorems presented are primarily auxiliary theorems required for later proofs. Theorem 4 is the same as Exercise VI.9.16. in the textbook of Dunford—Schwartz [2].

Two closed linear manifolds M and N in the Banach space B are said to be complements if $M \cap N = \{\bar{0}\}$ and $M + N = B$. Now every vector z in B can be uniquely presented in the form $z = x + y$ with $x \in M$ and $y \in N$ (Taylor: p. 240). We define the operator P by writing $Pz = x$. The domain of P is B and the range is M. We call the mapping P a projection of B onto M, and we denote $P = \text{proj}\,(M, N)$.

The following theorem deals with some important properties of a projection (Taylor: pp. 241—242):

Theorem 2. *If the closed linear manifolds M and N in the Banach space B are complements and if $P = \text{proj}\,(M, N)$, then P is a continuous linear mapping, $P^2 = P$,*

$$M = P(B) = \{Pz \mid z \in B\}$$

and

$$N = P^{-1}\{\bar{0}\} = \{y \in B \mid Py = \bar{0}\}\,.$$

In a Hilbert space every closed linear manifold has a complement (Taylor 4.82.—A). This does not hold for a general Banach space. Murray [10] has constructed a closed linear manifold which has no complements. Since a complement does not always exist for a closed linear manifold, we cannot define the deficiency index of a linear mapping T with the domain D as the dimension of a complement of $\overline{T(D)}$.

2.2. When M and N are closed linear manifolds, $M + N$ is not necessarily closed, not even in a Hilbert space. (See Halmos [4]: pp. 28—29.) In the following theorem we present a condition under which $M + N$ is closed.

Theorem 3. *If M and N are closed linear manifolds in the Banach space B, then the following two conditions are equivalent:*
(a) *$M \cap N = \{\bar{0}\}$ and $M + N$ is closed,*
(b) *there exists a constant $k > 0$ such that $|x + y| \geqq k$ whenever $x \in M$, $y \in N$ and $|y| = 1$.*

Proof. We suppose, first, that $|x + y| \geqq k > 0$ whenever $x \in M$, $y \in N$ and $|y| = 1$. We proceed to show that $M \cap N = \{\bar{0}\}$ and that $M + N$ is closed.

If there exists a vector $z \in M \cap N$ and $z \neq \bar{0}$, then $-\frac{z}{|z|} \in M$, $\frac{z}{|z|} \in N$ and $\left|\frac{z}{|z|}\right| = 1$. Now according to the assumption, $\left|-\frac{z}{|z|} + \frac{z}{|z|}\right| \geqq k$. This is impossible for $-\frac{z}{|z|} + \frac{z}{|z|} = \bar{0}$. Since there is in $M \cap N$ no vector differing from zero, $M \cap N = \{\bar{0}\}$.

Let z_0 be an arbitrary element in $\overline{M + N}$. According to the definition of a cluster point there exists a vector sequence $\{z_n\} \subset M + N$ such that $\lim z_n = z_0$. (We use the following brief notations: $\{z_n\} = \{z_n | n = 1, 2, 3, \ldots\}$, and $\lim z_n = \lim\limits_{n \to \infty} z_n$.) Since $z_n \in M + N$, it can be presented in the form $z_n = x_n + y_n$ with $x_n \in M$ and $y_n \in N$. When $|y_n - y_m| > 0$, then, according to the assumption

$$|z_n - z_m| = |x_n + y_n - x_m - y_m| = |y_n - y_m| \left|\frac{x_n - x_m}{|y_n - y_m|} + \frac{y_n - y_m}{|y_n - y_m|}\right| \geqq |y_n - y_m|\, k\,.$$

We find that for all values of n and m,

$$|y_n - y_m| \leqq \frac{1}{k}\, |z_n - z_m|\,. \tag{4}$$

Since $\{z_n\}$ is a convergent sequence, it is a Cauchy sequence. From the inequality (4) it follows that $\{y_n\}$ also is a Cauchy sequence. Since B is a Banach space, $\lim y_n = y_0$ exists. On the other hand, $x_n = z_n - y_n$ and hence, $\lim x_n = z_0 - y_0$. Since M and N are closed and $\{x_n\} \subset M$, $\{y_n\} \subset N$, necessarily $z_0 - y_0 \in M$ and $y_0 \in N$. Now $M + N$ contains its arbitrary cluster point z_0. Accordingly, $M + N$ is closed.

We now suppose that $M \cap N = \{\bar{0}\}$ and that $M + N$ is closed. We prove that a positive constant k exists such that $|x + y| \geqq k$ whenever $x \in M$, $y \in N$ and $|y| = 1$.

Now $M + N$ is a Banach space (Taylor 3.13.—B). The manifolds M and N are complements in the Banach space $M + N$. We denote $P = \text{proj}\,(N, M)$. According to Theorem 2, the projection P is a continuous linear mapping.

In order to prove the claim we assume the contrary. If the claim is not true, such vector sequences $\{x_n\} \subset M$ and $\{y_n\} \subset N$ would exist that $\lim |x_n + y_n| = 0$ and $|y_n| = 1$ for every value of n. Since P is continuous and $P(x_n + y_n) = y_n$, obviously $\lim y_n = P\,(\lim(x_n + y_n)) = \bar{0}$. This is impossible for $|y_n| = 1$ $(n = 1, 2, 3, \ldots)$. Since the contradiction is wrong, the claim is true. This completes the proof.

Theorem 4. *Supposing M is a closed linear manifold in the Banach space B, and N is a finite-dimensional linear manifold in B, then $M + N$ is closed.*

Proof. Now $M \cap N$ is a linear manifold in the finite-dimensional vector space N. Hence there exists a linear manifold N_0 in N such that $N_0 \cap (M \cap N) = \{\bar{0}\}$ and $N_0 + (M \cap N) = N$. Furthermore $N + M = N_0 + M$ and $N_0 \cap M = \{\bar{0}\}$. If $N_0 = \{\bar{0}\}$ or $M + N = M$, there is nothing to prove.

According to the above we can assume that $M \cap N = \{\bar{0}\}$ and $\dim N > 0$. The claim is proved by using Theorem 3. Because N is finite-dimensional, N is closed (Taylor 3.12.—C). The existence of a positive constant k meeting the following condition must be shown: $|x + y| \geqq k$ whenever $x \in M, y \in N$ and $|y| = 1$.

To prove this, we denote by Q the set $Q = \{y \in N \,|\, |y| = 1\}$. The function h is defined by the formula

$$h(y) = \inf\{|y + x| \,|\, x \in M\} \text{ for every vector } y \in Q .$$

Now h is continuous at every point of Q (Taylor: p. 72). Moreover, Q is a closed and bounded subset in the finite-dimensionaln ormed vector space N. Consequently there exists a vector $y_0 \in Q$ with which $h(y_0)$ is the smallest value of h, that is, $h(y) \geqq h(y_0)$ whenever $y \in Q$ (Taylor: p. 100).

If $h(y_0) = 0$, it would follow that y_0 is a cluster point of M (Taylor: p. 73). According to the assumption M is closed, and y_0 would consequently be an element of M. This is impossible for $y_0 \in N$, $|y_0| = 1$ and $M \cap N = \{\bar{0}\}$. Hence necessarily $h(y_0) > 0$. We denote $k = h(y_0)$. According to the definition of h,

$$|y + x| \geqq h(y) \geqq h(y_0) = k > 0$$

whenever $x \in M, y \in N$ and $|y| = 1$. This completes the proof.

2.3. **Theorem 5.** *If M is a closed linear manifold in the Banach space B and the quotient space B/M of M is finite-dimensional, then the manifold M has a complement N, and* $\dim N = \dim B/M$.

Proof. Since B/M is finite-dimensional, it has a finite basis. Consequently there exist in B/M linearly independent elements $x_1 + M$, $x_2 + M, \ldots, x_n + M$ and every element of B/M can be presented as their linear combination. The integer n equals the dimension of B/M.

First, the vectors $x_1, x_2, \ldots, x_n$ are proved to be linearly independent. If $p_1, p_2, \ldots, p_n$ are complex numbers and

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n = \bar{0} ,$$

then

$$p_1(x_1 + M) + p_2(x_2 + M) + \cdots + p_n(x_n + M) = \overline{0} + M .$$

Since $x_1 + M,\ x_2 + M, \ldots, x_n + M$ are linearly independent in B/M, the numbers $p_1, p_2, \ldots, p_n$ equal zero.

The linear manifold generated by the vectors $x_1, x_2, \ldots, x_n$ is denoted by N, that is,

$$N = \{p_1x_1 + p_2x_2 + \cdots + p_nx_n \mid p_1, p_2, \ldots, p_n \in C\} .$$

The vectors $x_1, x_2, \ldots, x_n$ form a basis of N and hence the dimension of N is n. Being finite-dimensional, N is closed. We show now that N is a complement of M, that is, $M \cap N = \{\overline{0}\}$ and $M + N = B$.

Let x be an arbitrary vector in $M \cap N$. Since $x \in N$, x can be presented in the form $x = p_1x_1 + p_2x_2 + \cdots + p_nx_n$, where $p_1, p_2, \ldots, p_n \in C$. On the other hand, $x \in M$, and therefore

$$p_1(x_1 + M) + p_2(x_2 + M) + \cdots + p_n(x_n + M) = x + M = \overline{0} + M .$$

Since $x_1 + M, x_2 + M, \ldots, x_n + M$ are linearly independent, $p_1 = p_2 = \cdots = p_n = 0$. Hence $M \cap N = \{\overline{0}\}$.

Finally an arbitrary vector z in B is chosen. Since $z + M \in B/M$, it follows from the properties of the basis that there exist complex numbers $p_1, p_2, \ldots, p_n$ such that

$$z + M = p_1(x_1 + M) + p_2(x_2 + M) + \cdots + p_n(x_n + M) .$$

We denote

$$x = p_1x_1 + p_2x_2 + \cdots + p_nx_n .$$

Now $x \in N$ and $z - x \in M$. Hence z is an element in $M + N$. This completes the proof.

Theorem 6. *Let M be a closed linear manifold in the Banach space B and let the quotient space B/M be infinite-dimensional. For every positive integer n there exists a linear manifold N such that $M \cap N = \{\overline{0}\}$, $\dim N = n$, and $M + N$ is closed.*

Proof. Since $\dim B/M = \infty, B/M$ has linearly independent elements $x_1 + M, x_2 + M, \ldots, x_n + M$ for every positive integer n. We denote by N the manifold generated by $x_1, x_2, \ldots, x_n$. Now $M \cap N = \{\overline{0}\}$ and $\dim N = n$. This can be proved as for Theorem 5 above. According to Theorem 4, $M + N$ is closed. This completes the proof.

Theorem 7. *Let M be a closed linear manifold and N a finite-dimensional linear manifold in the Banach space B.*

(a) *If* $N \cap M = \{\overline{0}\}$, *then* $\dim N \leqq \dim B/M$.

(b) *If* N *is a complement of* M, *then* $\dim N = \dim B/M$.

Proof. Since N is finite-dimensional, N has a basis $x_1, x_2, \ldots, x_n$. If $M \cap N = \{\bar{0}\}$, the elements $x_1 + M, x_2 + M, \ldots, x_n + M$ are obviously linearly independent in B/M, and $\dim B/M \geqq n = \dim N$. If $M \cap N = \{\bar{0}\}$ and $M + N = B$, the elements $x_1 + M$, $x_2 + M, \ldots,$ $x_n + M$ form a basis of B/M, and $\dim N = \dim B/M$. This completes the proof.

3. Proof of the main theorem

3.1. In this section we use the same notations as in Section 1. First, we prove two auxiliary theorems. Then we shall present the main theorem.

Theorem 8. *It is supposed that* N *is a closed linear manifold in the Banach space* B, $p \in Z$, $\bar{R}_p \cap N = \{\bar{0}\}$ *and* $\bar{R}_p + N$ *is closed. There is now a number* $d > 0$ *such that* $q \in Z$, $\bar{R}_q \cap N = \{\bar{0}\}$ *and* $\bar{R}_q + N$ *is closed whenever* $q \in C$ *and* $|p-q| < d$.

Proof. If $N = \{\bar{0}\}$, there is nothing to prove. It is now supposed that N contains non-zero vectors.

According to Theorem 1 there is a number $d_1 > 0$ such that $q \in Z$, $\|T(q)\,T(p)^{-1}\| \leqq 2$ and $\|T(p)T(q)^{-1}\| \leqq 2$ whenever $|p-q| < d_1$.

Since $\bar{R}_p \cap N = \{\bar{0}\}$ and $\bar{R}_p + N$ is closed, there is according to Theorem 3 a number $k > 0$ such that if $x \in \bar{R}_p$, $a \in N$ and $|a| = 1$, then $|x + a| \geqq k$. Now $k \leqq 1$, because for every $a \in N$ with $|a| = 1$ necessarily $k \leqq |0 + a| = 1$.

In the following it will be shown that $|a + y| \geqq \frac{k}{2}$ whenever $y \in \bar{R}_q$, $a \in N$, $|a| = 1$ and $|p-q|$ is sufficiently small.

It is agreed that $q \in C$ and $|p-q| < d_1$.

If $y \in R_q$, $|y| \geqq 2$, $a \in N$ and $|a| = 1$, then

$$|a+y| \geqq |y| - |a| \geqq 2 - 1 \geqq k > \frac{k}{2}\,. \tag{5}$$

We now suppose that $y \in R_q$ and $|y| \leqq 2$. We set

$$x = T(p)\,T(q)^{-1}\,y$$

or

$$y = T(q)\,T(p)^{-1}\,x\,.$$

Now $x \in R_p$ and

$$|x| \leqq \|T(p)\,T(q)^{-1}\|\,|y| \leqq 4\,.$$

Furthermore, $y = T(q)\,T(p)^{-1}\,x = x + (T(q)\,T(p)^{-1} - I_p)\,x$

and
$$\begin{aligned} |y+a| &= |x+a + (T(q)\,T(p)^{-1} - I_p)\,x| \\ &\geqq |x+a| - \|T(q)\,T(p)^{-1} - I_p\|\,|x| \\ &\geqq |x+a| - 4\|T(q)\,T(p)^{-1} - I_p\|\,. \end{aligned}$$

If $a \in N$ and $|a| = 1$, then $|x+a| \geqq k$ and

$$|y+a| \geqq k - 4\,\|T(q)\,T(p)^{-1} - I_p\|\,. \tag{6}$$

Since the mapping family $T(p)$ has the property E,

$$\lim_{q \to p} \|T(q)\,T(p)^{-1} - I_p\| = 0\,.$$

Hence there is a number $d_2 > 0$ such that $\|T(q)\,T(p)^{-1} - I_p\| < \frac{k}{8}$ or

$$k - 4\,\|T(q)\,T(p)^{-1} - I_p\| > \tfrac{1}{2}k \tag{7}$$

whenever $q \in Z$ and $|p-q| < d_2$. We denote $d = \min(d_1, d_2)$. From the inequalities (5), (6) and (7) it follows that $|y+a| \geqq \frac{1}{2}k$ whenever $y \in R_q$, $a \in N$, $|a| = 1$ and $|p-q| < d$.

Now we must prove that $\bar{R}_q \cap N = \{\bar{0}\}$ and that $\bar{R}_q + N$ is closed whenever $|p-q| < d$. We choose a fixed element $q \in C$ with $|p-q| < d$. Let y_0 be an arbitrary element of $\bar{R}_q$. Now there is a vector sequence $\{y_n\}$ in R_q such that $\lim y_n = y_0$. Since $y_n \in R_q$ for every integer n, $|y_n + a| \geqq \frac{1}{2}k$ whenever $a \in N$ and $|a| = 1$. Hence $|y_0+a| = \lim |y_n+a| \geqq \frac{1}{2}k$. As $y_0 \in \bar{R}_q$ is arbitrary, it follows from the above that $|y+a| \geqq \frac{k}{2}$ whenever $y \in \bar{R}_q$, $a \in N$ and $|a| = 1$. According to Theorem 3, $\bar{R}_q \cap N = \{\bar{0}\}$, and $\bar{R}_q + N$ is closed. The theorem is proved.

3.2. **Theorem 9.** *If $p \in Z$ and the manifold $\bar{R}_p$ has a complement N, there exists a number $d > 0$ such that N is a complement of $\bar{R}_q$ whenever $q \in C$ and $|p-q| < d$.*

Proof. Since $\bar{R}_p \cap N = \{\bar{0}\}$ and $\bar{R}_p + N = B$ is closed, there exists, according to the preceding theorem, a number $d_1 > 0$ such that $q \in Z$, $\bar{R}_q \cap N = \{\bar{0}\}$ and $\bar{R}_q + N$ is closed whenever $q \in C$ and $|p-q| < d_1$.

We choose a fixed element $q \in Z$ with $|q-p| < d_1$. We now suppose that $\bar{R}_q + N$ is different from B, in other words, $\bar{R}_q + N$ is a proper subset of B. This is proved to be impossible if $|p-q|$ is sufficiently small.

The following Riesz's lemma is first used (Taylor 3.12.—E): *If M is a closed linear manifold of the Banach space B and $M \neq B$, then for every real number t with $0 < t < 1$ there exists $y \in B$ such that $|y| = 1$ and $|y+x| \geqq t$ whenever $x \in M$.*

According to this lemma there exists $y_0 \in B$ such that $|y_0| = 1$ and $|y_0 + y| \geqq \frac{1}{2}$ whenever $y \in \bar{R}_q + N$. Let P be the projection with range $\bar{R}_p$ and null space N, in other words,

$$P = \text{proj}\,(\bar{R}_p\,, N).$$

It follows from Theorem 2 that P is a continuous linear mapping. Since $y_0 \in B = \bar{R}_p + N$,

$$y_0 = x_0 + a$$

with $x_0 = Py_0 \in \bar{R}_p$ and $a \in N$. Now there exists a vector sequence $\{x_n\} \subset R_p$ such that $\lim x_n = x_0$. Being an element of R_p, x_n can be represented in the form

$$x_n = T(p)\, z_n \quad \text{with} \quad z_n \in D\,.$$

Since $T(q)\, z_n \in R_q$ and $T(q)\, z_n + a \in \bar{R}_q + N$,

$$\tfrac{1}{2} \leqq |y_0 - T(q)\, z_n - a| \leqq |y_0 - x_n - a| + |x_n - T(q)\, z_n|\,.$$

Now $\lim |y_0 - x_n - a| = |y_0 - x_0 - a| = 0$ and hence $|y_0 - x_n - a| < \frac{1}{4}$ when $n \geqq n_1$.
Hence

$$|T(q)\, z_n - x_n| \geqq \tfrac{1}{4} \quad \text{when} \quad n \geqq n_1\,. \tag{8}$$

On the other hand, $z_n = T(p)^{-1}\, x_n$ and

$$\begin{aligned} T(q)\, z_n - x_n &= T(q)\, T(p)^{-1}\, x_n - I_p x_n \\ &= (T(q)\, T(p)^{-1} - I_p)\, x_n\,. \end{aligned}$$

The projection P is continuous and $x_0 = Py_0$. Hence,

$$|x_0| \leqq \|P\|\, |y_0| = \|P\|\,.$$

Since $\lim x_n = x_0$, there exists an integer n_2 such that

$$|x_n| \leqq |x_0| + 1 \leqq \|P\| + 1 \tag{9}$$

whenever $n \geqq n_2$.

Since the mapping family $T(p)$ has the property E, it follows that

$$\lim_{q \to p} \|T(q)\, T(p)^{-1} - I_p\| = 0\,.$$

Hence there exists a number $d_2 > 0$ such that

$$\|T(q)\, T(p)^{-1} - I_p\| < \frac{1}{8(\|P\| + 1)} \tag{10}$$

whenever $q \in C$ and $|p - q| < d_2$.

If now $|p-q| < d_2$, it follows from the inequalities (8), (9) and (10) that

$$|T(q)\, z_n - x_n| \geqq \tfrac{1}{4} \quad \text{whenever} \quad n \geqq n_1$$

and

$$\begin{aligned} |T(q)\, z_n - x_n| &\leqq \|T(q)\, T(p)^{-1} - I_p\|\, |x_n| \\ &< \frac{1}{8(\|P\| + 1)} (\|P\| + 1) = \frac{1}{8} \end{aligned}$$

whenever $n \geqq n_2$.

Since this is impossible, $\bar{R}_q + N$ cannot be a proper subset of B when $|p-q| < d = \min(d_1, d_2)$.

Hence it follows from the assumptions $q \in C$ and $|p-q| < d$ that

$$\bar{R}_q \cap N = \{\bar{0}\} \quad \text{and} \quad \bar{R}_q + N = B \, ,$$

that is, N is a complement of $\bar{R}_q$. The theorem is proved.

3.3. **Theorem 10.** *If G is an open connected subset of the regularity region of the mapping family $T(p)$, then* $\operatorname{def} T(p)$ *is the same for every $p \in G$.*

3.4. *Proof.* As a connected set, G is nonvacuous. We examine the deficiency index of the mapping $T(p)$ with $p \in G$. There are two possibilities:

1. $\operatorname{def} T(p) = \dim B/\bar{R}_p = \infty$ for every element $p \in G$,
2. there exists $p \in G$ such that $\dim B/\bar{R}_p = m < \infty$.

In the former case there is nothing to prove.

It is now supposed that there exists an element p in G with $\dim B/\bar{R}_p = m < \infty$. In the following the element p and the integer m are kept fixed. We will prove that, for every element $q \in G$, $\dim B/\bar{R}_q = m$, that is, $\operatorname{def} T(q) = \operatorname{def} T(p)$.

Let $F = \{q \in G \mid \dim B/\bar{R}_q = m\}$. We show that $G = F$. Since G is open and connected, then $G = F$ if the following conditions are true:

1. F is nonvacuous and open,
2. it always follows from the assumptions $\{q_n\} \subset F$ and $\lim q_n = q \in G$ that $q \in F$.

3.5. Firstly, $F \neq \emptyset$ because $p \in F$. In order to prove that F is open, we choose an arbitrary element q in F. Now $q \in G$ and $\dim B/\bar{R}_q = m$. Since the quotient space of $\bar{R}_q$ is finite-dimensional then, according to Theorem 5, $\bar{R}_q$ has a complement N, and $\dim N = m$. Now, according to Theorem 9, there exists a number $d_1 > 0$ such that N is a complement of $\bar{R}_s$ whenever $s \in C$ and $|q-s| < d_1$. Since G is open and $q \in G$, there exists a number $d_2 > 0$ such that $\{s \in C \mid |q-s| < d_2\} \subset G$. We denote

$$d = \dim (d_1 , d_2) .$$

When $s \in C$ and $|q-s| < d$, it follows from the above that $s \in G$, N is a complement of $\bar{R}_s$, and $\dim B/\bar{R}_s = \dim N = m$. Hence $\{s \in C \mid |q-s| < d\} \subset F$. Since q is an arbitrary element of F, F is open.

3.6. It is secondly supposed that $\{q_n\} \subset F$ and $\lim q_n = q \in G$. We have now to show that $q \in F$. At first, we prove that the quotient space $B/\bar{R}_q$ is finite-dimensional.

We make contrahypothesis: $\dim B/\bar{R}_q = \infty$.

According to Theorem 6 there exists a linear manifold N so that $\bar{R}_q \cap N = \{\bar{0}\}$, $\bar{R}_q + N$ is closed and $\dim N = m+1$. According to Theorem 8 there exists a number $d > 0$ such that $\bar{R}_s \cap N = \{\bar{0}\}$ and $\bar{R}_s + N$ is closed whenever $|q-s| < d$. On the basis of Theorem 7 it is seen that $\dim B/\bar{R}_s \geqq m+1$ when $|q-s| < d$. Since $\lim |q_n-q| = 0$, necessarily $|q_n-q| < d$ and $\dim B/\bar{R}_{q_n} \geqq m+1$ for a sufficiently large value of the integer n. This is impossible since $q_n \in F$ and $\dim B/\bar{R}_{q_n} = m$ $(n = 1, 2, 3, \ldots)$. Since $B/\bar{R}_q$ cannot be infinite-dimensional, it must be finite-dimensional.

We denote $k = \dim B/\bar{R}_q$ and prove that $m = k$. According to Theorem 5, $\bar{R}_q$ has a complement N and $\dim N = \dim B/\bar{R}_q = k$. Now, according to Theorem 9, there exists a number $d > 0$ such that N is a complement of $\bar{R}_s$ whenever $s \in C$ and $|q-s| < d$. Hence $\dim B/\bar{R}_s = \dim N = k$ for every element $s \in C$ with $|q-s| < d$. Since $\lim |q_n-q| = 0$ there is, for a sufficiently large value of n, $|q_n-q| < d$ and $\dim B/\bar{R}_{q_n} = k$. On the other hand, $\dim B/\bar{R}_{q_n} = m$ for every value of the integer n because $q_n \in F$. From the above it follows that $m = k$. Now $q \in G$ and $\dim B/\bar{R}_q = m$. Hence $q \in F$.

We have proved that $F = G$ and hence, for every element $q \in G$, $\operatorname{def} T(q) = \operatorname{def} T(p) = m$, and the theorem is thus proved.

4. Comments

4.1. We firstly define the concept of topological isomorphism, and then we present two theorems which deal with R_p and $B/\bar{R}_p$. We use here same notations as in Section 1.

Two normed vector spaces V_1 and V_2 are said to be *topologically isomorphic* if there is a continuous linear mapping $T : V_1 \to V_2$ such that

$T(V_1) = V_2$, if the inverse mapping $T^{-1} : V_2 \to V_1$ exists and if T^{-1} is continuous and linear. (Taylor: p. 85).

Topological isomorphism is denoted by $V_1 \simeq V_2$. It follows from the properties of continuous mappings that topological isomorphism is an equivalence relation, whereby is understood that

$V \simeq V$ whenever V is a normed vector space;
whenever $V_1 \simeq V_2$ holds, so does $V_2 \simeq V_1$;
whenever $V_1 \simeq V_2$ and $V_2 \simeq V_3$ holds, so does $V_1 \simeq V_3$.

4.2. **Theorem 11.** *Let G be an open connected subset of the regularity region of the mapping family $T(p)$. If the manifold $\bar{R}_p$ has the complement N_p for every $p \in G$ then $N_p \simeq N_q$ and $B/\bar{R}_p \simeq B/\bar{R}_q$ whenever $p, q \in G$.*

Proof. The proof is similar to that of Theorem 10. We choose a fixed element $r \in G$ and denote $F = \{p \in G \mid N_p \simeq N_r\}$.

Since $r \in F, F \neq \emptyset$. Let p be an arbitrary element of F. According to Theorem 9 there is a number $d_1 > 0$ such that N_p is a complement of $\bar{R}_q$ whenever $q \in G$ and $|p-q| < d_1$. We use the following theorem: *If the closed linear manifolds M and N are complements in the Banach space B, then N and B/M are topologically isomorphic.* (See: Liusternik—Sobolev [9]: p. 58).

Hence $N_p \simeq B/\bar{R}_q$ and $N_q \simeq B/\bar{R}_q$. Since topological isomorphism is an equivalence relation and $N_p \simeq N_r$, it follows that $N_q \simeq N_r$ for every $q \in G$ with $|p-q| < d_1$. Since G is open and $p \in G$, there exists a number $d_2 > 0$ such that $q \in G$ whenever $|p-q| < d_2$. When $q \in C$ and $|p-q| < \min(d_1, d_2)$, then $q \in F$, and hence F is open.

It is now supposed that $\{p_n\} \subset F$ and $\lim p_n = q \in G$. It will be shown that $q \in F$. Since $q \in G \subset Z$ there exists, according to Theorem 9, a number $d > 0$ such that N_q is a complement of $\bar{R}_p$ whenever $p \in C$ and $|q-p| < d$. Since $\lim p_n = q$, there is $|p_n - q| < d$, provided that the values of n are sufficiently large. Hence,

$$N_{p_n} \simeq B/\bar{R}_{p_n} \quad \text{and} \quad N_q \simeq B/\bar{R}_{p_n}$$

for sufficiently large values of n. Since $p_n \in F$ and $N_{p_n} \simeq N_r$, there is $N_q \simeq N_r$ and $q \in F$. Since G is open and connected, it follows from the above that $F = G$. Hence $N_p \simeq N_r$, $N_q \simeq N_r$ and $N_p \simeq N_q$ whenever $p, q \in G$. Since $N_p \simeq B/\bar{R}_p$ and $N_q \simeq B/\bar{R}_q$, also $B/\bar{R}_p \simeq B/\bar{R}_q$. This completes the proof.

4.3. **Theorem 12.** *If G is an open connected subset of the regularity region Z, the manifolds R_p and R_q are topologically isomorphic whenever $p, q \in G$.*

Proof. When $p \in Z$, the domain of $T(p)^{-1}$ is R_p and the range of $T(p)^{-1}$ is D. Hence the domain of $T(q)T(p)^{-1}$ is R_p and the range of $T(q)T(p)^{-1}$ is R_q. Furthermore, the inverse mapping of $T(q)T(p)^{-1}$ is $T(p)T(q)^{-1}$. If we succeed in proving that $T(q)T(p)^{-1}$ is continuous whenever $p, q \in G$, it follows that R_p and R_q are topologically isomorphic, according to the definition of topological isomorphism.

We choose a fixed element $r \in G$ and denote $F = \{p \in G \mid T(r)T(p)^{-1}$ and $T(p)T(r)^{-1}$ are continuous$\}$.

The set F is nonvacuous because $r \in F$.

In order to prove that the set F is open, we examine an arbitrary element p in F. According to Theorem 1 there is a positive number $d_1 > 0$ such that $T(q)T(p)^{-1}$ and $T(p)T(q)^{-1}$ are continuous whenever $q \in C$ and $|p-q| < d_1$. Since G is open, there is a positive number $d_2 > 0$ such that $q \in G$ whenever $|p-q| < d_2$. We denote

$$d_3 = \min(d_1, d_2).$$

When $|p-q| < d_3$, the mappings

$$T(q)\, T(r)^{-1} = T(q)\, T(p)^{-1}\, T(p)\, T(r)^{-1}$$

and

$$T(r)\, T(q)^{-1} = T(r)\, T(p)^{-1}\, T(p)\, T(q)^{-1}$$

are continuous because they are composite mappings of continuous mappings. Since p is an arbitrary element of F and since $q \in F$ whenever $|p-q| < d_3$, F is open.

It is finally assumed that $\{p_n\} \subset F$ and $\lim p_n = q \in G$. We prove that $q \in F$. Since $q \in G$, there is, according to Theorem 1, a positive number d with which $T(p)T(q)^{-1}$ and $T(q)T(p)^{-1}$ are continuous whenever $p \in Z$ and $|q-p| < d$. Consequently, $T(p_n)T(q)^{-1}$ and $T(q)T(p_n)^{-1}$ are continuous for sufficiently large values of n. Now $p_n \in F$ for every integer n, and hence $T(p_n)T(r)^{-1}$ and $T(r)T(p_n)^{-1}$ are continuous. By forming composite mappings, we find that $T(q)T(r)^{-1}$ and $T(r)T(q)^{-1}$ are continuous, and hence $q \in F$.

From the above it follows that $F = G$. Consequently, whenever $p, q \in G$, the mappings

$$T(q)\, T(p)^{-1} = T(q)\, T(r)^{-1}\, T(r)\, T(p)^{-1}$$

and

$$T(p)\, T(q)^{-1} = T(p)\, T(r)^{-1}\, T(r)\, T(q)^{-1}$$

are continuous. Hence the manifolds R_p and R_q are topologically isomorphic. The theorem is proved.

4.4. Before proceeding to counter-examples, we shall examine some formal generalizations.

So far we have dealt with Banach spaces. On the other hand, every normed vector space V can be completed to a Banach space B_V so that $V \subset B_V$ and that the closure of V in B_V is B_V. (Taylor: pp. 98—99.)

When D and V are normed vector spaces and $T : D \to V$ is a linear mapping, we define

$$\operatorname{def} T = \dim T(D)^\circ = \dim \{f \in V' \mid f(y) = 0 \text{ whenever } y \in T(D)\},$$

where V' is the dual space of V.

Let B_V be the Banach space obtained by completing V. Since the dual spaces of V and B_V can be identified (Taylor 3.13.—A), since $T(D)^\circ = \overline{T(D)}^\circ$ (Taylor: p. 225) and since $\dim (\overline{T(D)}^\circ) = \dim (B_V/\overline{T(D)})$ (Taylor: p. 227),

$$\operatorname{def} T = \dim T(D)^\circ = \dim \overline{T(D)}^\circ = \dim B_V/\overline{T(D)}\ ,$$

where $\overline{T(D)}$ is the closure of $T(D)$ in the Banach space B_V.

In other words, when we change the value space of the mapping T to the Banach space B_V, the deficiency index of T remains unchanged. Hence Theorem 10 is still true if the value space of the mapping family $T(p)$ with the property E is a normed linear space and not a Banach space.

4.5. A second generalization can be made in respect of the parameter p. We have assumed that p is an element of an open set in the complex plane. This can be replaced by the assumption: p is an element in a metric space.

If we allow the parameter p to be an element of an open set Z in the metric space H, then the theorems presented and their proofs are true, if we change some of the notations as follows: The set Z is an open subset in the metric space H. The letters p, q, r and s with or without subindexes are symbols of elements of the metric space H. Furthermore, we write instead of the absolute value $|p-q|$ the distance $d(p, q)$ of the elements p and q.

5. Counter-examples

5.1. **Example 1.** In the first example it will be shown that in the property E all continuity assumptions cannot be omitted if we want Theorem 10 to be true, not even in the case that $T(p) = T-pI$.

We examine the Banach space

$$l^1 = \{(s_k) \mid (s_k) \text{ is a sequence of complex numbers, and } \sum_{k=1}^{\infty} |s_k| < \infty\}\,.$$

It is well-known that l^1 is a Banach space when

$$p(s_k) + q(t_k) = (ps_k + qt_k) \quad \text{and} \quad |(s_k)| = \sum_{k=1}^{\infty} |s_k| \,,$$

where $p, q \in C$ and $(s_k), (t_k) \in l^1$ (cf. Taylor: pp. 88—89).

We denote by e_k the sequence in which the k:th term is unity and the others are zero. Now $e_k \in l^1$, $|e_k| = 1$, and when $x = (s_k) \in l^1$,

$$x = \sum_{k=1}^{\infty} s_k e_k = \lim_{n \to \infty} \sum_{k=1}^{n} s_k e_k \,.$$

We define a mapping L: $l^1 \to l^1$ by the formula

$$L(\sum_{k=1}^{\infty} s_k e_k) = \sum_{k=1}^{\infty} s_k e_{k+1} \quad \text{with} \quad (s_k) \in l^1 \,. \tag{11}$$

Now

$$|L(\sum_{k=1}^{\infty} s_k e_k)| = |\sum_{k=1}^{\infty} s_k e_{k+1}| = \sum_{k=1}^{\infty} |s_k| \,.$$

Hence, $Lx \in l^1$ and $|Lx| = |x|$ whenever $x \in l^1$. The mapping L is obviously linear and, since it is bounded, it is continuous.

We set

$$T(p) = L - pI$$

and

$$Z = \{p \in C \mid T(p)^{-1} \text{ exists and is continuous}\} \,.$$

When $x = (s_k) \in l^1$, then $T(p)\, x = \sum_{k=1}^{\infty} (s_{k-1} - ps_k)\, e_k$ with $s_0 = 0$.

Now $T(p)x = \overline{0}$ if and only if $s_{k-1} - ps_k = 0$ for all $k = 1, 2, 3, \ldots$. Since $s_0 = 0$, we obtain $s_1 = 0$, $s_2 = 0, \ldots$, and hence it follows from the assumption $T(p)x = \overline{0}$ that $x = \overline{0}$. Consequently $T(p)^{-1}$ exists whenever $p \in C$.

Secondly, we prove that

$$Z = \{p \in C \mid |p| < 1\} \cup \{p \in C \mid |p| > 1\} \,.$$

When $p \in C$ and $|p| < 1$, then $|T(p)\, x| = |Lx - px| \geqq |Lx| - |px| \geqq (1 - |p|)\, |x|$ whenever $x \in l^1$. Hence $T(p)^{-1}$ is continuous and) $\|T(p)^{-1}\| \leqq (1 - |p|)^{-1}$. According to Theorem 10, $\operatorname{def} T(p) = \operatorname{def} T(0$ when $|p| < 1$. On the other hand,

$$\begin{aligned} R_0 &= T(0)\, (l^1) = \{Lx \mid x \in l^1\} \\ &= \{ \sum_{k=1}^{\infty} s_k e_{k+1} \mid (s_k) \in l^1\} \\ &= \{ (t_k) \in l^1 \mid t_1 = 0\} \,. \end{aligned}$$

The manifold $N_0 = \{ae_1 \mid a \in C\}$ is obviously a complement of R_0, and hence

$$\operatorname{def} L = \dim N_0 = 1 .$$

When $p \in C$ and $|p| > 1$, then $(L-pI)^{-1} = -\frac{1}{p}\sum_{n=0}^{\infty} p^{-n}L^n$, $(L-pI)^{-1}$ is continuous, and the range of $L - pI$ is the whole Banach space l^1 (cf. Taylor 5.2.—C). Hence $p \in Z$ and $\operatorname{def} T(p) = 0$.

Now $\{p \in C \mid |p| < 1 \text{ or } |p| > 1\} \subset Z$. If $\{p \in C \mid |p| = 1\} \cap Z \neq \emptyset$, Z would be a connected subset of C and hence, according to Theorem 10, $\operatorname{def} T(p)$ would be constant when $p \in Z$. Since this is not true, $p \notin Z$ if $p \in C$ and $|p| = 1$.

Now $\{p \in C \mid T(p)^{-1} \text{ exists}\} = C$. If all continuity conditions are omitted then we can study the mapping family $T(p) = L - pI$ with $p \in C$. But $\operatorname{def} T(p)$ is not constant in the complex plane C since

$$\operatorname{def} T(p) = \begin{cases} 0 & \text{when } |p| > 1 , \\ 1 & \text{when } |p| < 1 . \end{cases}$$

5.2. **Example 2.** We now show by means of an example:

If in the property E of the mapping family $T(p)$ the assumption

$$\lim_{q \to p} \|T(q)\, T(p)^{-1} - I_p\| = 0$$

is replaced by the weaker assumption $\lim_{q \to p} T(q)\, T(p)^{-1} x = x$ whenever $x \in R_p$, the deficiency index will not always be constant in a connected subset of the regularity region.

Let $B = l^1$. We define the mapping $T(p) : l^1 \to l^1$ as follows:

$$T(p)\, x = \sum_{k=1}^{\infty} (2 \sqrt[k]{|p|}\, s_k - s_{k-1})\, e_k , \tag{12}$$

where $p \in C$, $|p| < 1$, $x = (s_k) = \sum_{k=1}^{\infty} s_k e_k \in l^1$ and $s_0 = 0$. Direct calculation yields

$$\begin{aligned} |T(p)\, x| &= \sum_{k=1}^{\infty} |2 \sqrt[k]{|p|}\, s_k - s_{k-1}| \\ &\leqq \sum_{k=1}^{\infty} (2\, |s_k| + |s_{k-1}|) \leqq 3\, |x| . \end{aligned}$$

Hence, $T(p)x \in l^1$ and $|T(p)x| \leqq 3|x|$ whenever $x \in l^1$, $p \in C$ and $|p| < 1$. Since $T(p)$ is obviously linear, $T(p)$ is a continuous linear mapping and its domain is the whole Banach space l^1. In particular,

$$T(p)\, e_k = 2 \sqrt[k]{|p|}\, e_k - e_{k+1} \quad \text{and} \quad T(0)\, e_k = -\, e_{k+1} .$$

We first notice that $T(0) = -L$, where L is the same operator as in Example 1. (See the equations (11) and (12).) Hence, $T(0)^{-1}$ exists and is continuous, and $\operatorname{def} T(0) = \operatorname{def} L = 1$.

Secondly, the mapping $T(p)$ is examined with $0 < |p| < 1$. When $x = (s_k) \in l^1$ and $T(p)x = \overline{0}$,

$$\sum_{k=1}^{\infty} (2 \sqrt[k]{|p|}\, s_k - s_{k-1})\, e_k = \overline{0}\,.$$

Now $2 \sqrt[k]{|p|}\, s_k = s_{k-1}$. Since $s_0 = 0$, $s_k = 0$ for every value of k. It follows from the assumption $T(p)x = \overline{0}$ that $x = \overline{0}$, and hence $T(p)^{-1}$ exists whenever $p \in C$ and $0 < |p| < 1$.

The range of $T(p)$ and the continuity of the inverse mapping must also be examined.

Let $y = \sum_{k=1}^{\infty} t_k e_k$ be an arbitrary element of l^1. We shall show the existence of such a vector $x = (s_k)$ in l^1 that $T(p)x = y$. Here p is a fixed complex number with $0 < |p| < 1$. The equation $T(p)x = y$ means the same as

$$(13) \qquad 2 \sqrt[k]{|p|}\, s_k - s_{k-1} = t_k \text{ for every value of } k\,.$$

Now $s_0 = 0$, $s_1 = \dfrac{t_1}{2|p|}$ and $s_k = \dfrac{t_k + s_{k-1}}{2\sqrt[k]{|p|}}$ $(k > 1)$. The above recursion formula defines the numbers s_k by means of the numbers t_k. It remains to be shown that $(s_k) \in l^1$ whenever $(t_k) \in l^1$.

Since $\lim_{k \to \infty} \sqrt[k]{|p|} = 1$, there exists an integer k_0 such that

$$\tfrac{1}{4} < \frac{1}{2\sqrt[k]{|p|}} < \tfrac{3}{4} \quad \text{whenever} \quad k \geqq k_0\,.$$

Therefore,

$$|s_k| \leqq \tfrac{3}{4}\, |t_k + s_{k-1}| \leqq \tfrac{3}{4}\, (|t_k| + |s_{k-1}|) \quad \text{whenever} \quad k \geqq k_0\,.$$

If n is an integer and $n \geqq k_0$,

$$\begin{aligned} \sum_{k=k_0}^{n} |s_k| &\leqq \tfrac{3}{4}\Big(\sum_{k=k_0}^{n} |t_k| + \sum_{k=k_0}^{n} |s_{k-1}|\Big) \\ &\leqq \tfrac{3}{4} \sum_{k=k_0}^{n} |t_k| + \tfrac{3}{4} \sum_{k=k_0}^{n} |s_k| + \tfrac{3}{4}|s_{k_0-1}| - \tfrac{3}{4}|s_n|\,. \end{aligned}$$

Hence,

$$\tfrac{1}{4} \sum_{k=k_0}^{n} |s_k| \leqq \tfrac{3}{4} \sum_{k=k_0}^{n} |t_k| + \tfrac{3}{4}\, |s_{k_0-1}|$$

and

$$(14) \qquad \sum_{k=k_0}^{n} |s_k| \leqq 3 \sum_{k=1}^{\infty} |t_k| + 3|s_{k_0-1}| \quad \text{whenever} \quad n \geqq k_0 .$$

It follows from the inequality (14) that $\sum_{k=1}^{\infty} |s_k|$ converges and $(s_k) \in l^1$.

Now the range of $T(p)$ is the whole Banach space l^1, $T(p)x = \overline{0}$ only if $x = \overline{0}$, and $T(p)$ is continuous and linear. Hence, $T(p)^{-1}$ exists and is continuous (Taylor 4.2.—H).

The above proof also contains the result:

$$\operatorname{def} T(p) = \begin{cases} 0 & \text{when} \quad 0 < |p| < 1 , \\ 1 & \text{when} \quad p = 0 . \end{cases}$$

In the following it will be proved that $\lim_{q \to p} T(q)T(p)^{-1}y = y$ whenever $|p| < 1$ and y is a vector in the range of $T(p)$. When $y \in R_p$, there exists $x \in l^1$ such that $T(p)x = y$. Obviously $\lim_{q \to p} T(q)T(p)^{-1}y = y$ if and only if $\lim_{q \to p} T(q)x = T(p)x$. When

$$x = \sum_{k=1}^{\infty} s_k e_k \in l^1 , \ |p| < 1 \quad \text{and} \quad |q| < 1 , \quad \text{then}$$

$$|T(p)\, x - T(q)\, x| = |\sum_{k=1}^{\infty} 2(\sqrt[k]{|p|} - \sqrt[k]{|q|})\, s_k e_k|$$

$$= 2 \sum_{k=1}^{\infty} |\sqrt[k]{|p|} - \sqrt[k]{|q|}\,|\, |s_k| .$$

Let $\varepsilon > 0$ be given. Since $\sum_{k=1}^{\infty} |s_k| < \infty$, there exists an integer k_0 with which $\sum_{k=k_0}^{\infty} |s_k| < \varepsilon$. On the other hand, according to operations with limits,

$$\lim_{q \to p} \sum_{k=1}^{k_0-1} |\sqrt[k]{|p|} - \sqrt[k]{|q|}\,|\, |s_k| = 0$$

and there exists $\delta > 0$ such that

$$\sum_{k=1}^{k_0-1} |\sqrt[k]{|p|} - \sqrt[k]{|q|}\,|\, |s_k| < \varepsilon \quad \text{whenever} \quad q \in C \quad \text{and} \quad |p - q| < \delta .$$

Now $|T(p)\, x - T(q)\, x| \leqq 2 \sum_{k=1}^{k_0-1} |\sqrt[k]{|p|} - \sqrt[k]{|q|}\,|\, |s_k| + 2 \sum_{k=k_0}^{\infty} (1 + 1)\, |s_k|$

$$< 2\varepsilon + 4\varepsilon = 6\varepsilon ,$$

whenever $q \in C$, $|q| < 1$ and $|p-q| < \delta$. It follows that $\lim_{q \to p} T(q)\, x = T(p)\, x$ whenever $x \in l^1$ and $|p| < 1$.

We summarize the properties of our mapping family found in the foregoing:

1° $T(p)$ is a continuous linear mapping from Banach space l^1 to Banach space l^1 whenever $p \in C$ and $|p| < 1$,

2° $T(p)^{-1}$ exists and is continuous whenever $|p| < 1$,

3° $T(q)\, T(p)^{-1}$ is a continuous linear mapping whenever $|p| < 1$ and $|q| < 1$,

4° $\lim_{q \to p} T(q)\, T(p)^{-1}\, x = x$ whenever $p \in C$, $|p| < 1$ and $x \in R_p$,

5° $\operatorname{def} T(p) = \begin{cases} 1 & \text{when } p = 0 , \\ 0 & \text{when } 0 < |p| < 1 . \end{cases}$

We show, finally, that this mapping family $T(p)$ has the property E when we restrict the values of the parameter p to the set $Z = \{p \in C \mid 0 < |p| < 1\}$. We choose an arbitrary complex number $p \in Z$. It follows from the above that $T(p)$ and $T(p)^{-1}$ are continuous linear mappings with domains l^1. Hence the assumptions (a) and (b) of the property E are obviously true.

We must prove that $\lim_{q \to p} \|T(q)T(p)^{-1} - I\| = 0$. When $x = \sum_{k=1}^{\infty} s_k e_k \in l^1$ and $q \in Z$,

$$|T(q)\, x - T(p)\, x| = 2 \sum_{k=1}^{\infty} \left|\sqrt[k]{|q|} - \sqrt[k]{|p|}\right| |s_k| .$$

By using the mean value theorem of differential calculus, we obtain

$$0 < \sqrt[k]{a} - \sqrt[k]{b} < \frac{a-b}{kb} \quad \text{when} \quad 0 < b < a < 1 \quad \text{and} \quad k \geqq 1 .$$

If $q \in Z$ and $|p - q| < \frac{1}{2}|p|$,

$$\left|\sqrt[k]{|q|} - \sqrt[k]{|p|}\right| \leqq \frac{|q-p|}{k \min(|q|, |p|)} \leqq \frac{|q-p|}{\frac{1}{2}|p|} \quad \text{and}$$

$$|T(q)\, x - T(p)\, x| \leqq 2 \sum_{k=1}^{\infty} \frac{2|q-p|}{|p|} |s_k| = \frac{4 \cdot |q-p|}{|p|} |x| .$$

Hence,

$$\begin{aligned} \|T(q)\, T(p)^{-1} - I\| &= \|(T(q) - T(p))\, T(p)^{-1}\| \leqq \|T(q) - T(p)\|\, \|T(p)^{-1}\| \\ &\leqq \frac{4}{|p|} \|T(p)^{-1}\|\, |q-p| , \end{aligned} \tag{15}$$

whenever $q \in Z$ and $|q - p| < \frac{1}{2}|p|$.

It follows from the inequality (15) that $\lim_{q \to p} \|T(q)\, T(p)^{-1} - I\| = 0$. This is an example of a mapping family $T(p)$ with the property E and which cannot be written in the form $T - pL$ with T and L denoting linear mappings.

5.3. **Example 3.** A counter-example shall be constructed to show that Theorem 10 is not true in a general topological vector space.

When D is a linear manifold in a topological complex vector space V and $T : D \to V$ is a linear mapping, the regularity region of T is defined to be the set of the complex numbers p with which $(T - pI)^{-1}$ exists and is continuous. Here I stands for the identical mapping of the space V.

We denote $T(p) = T - pI$ and $R_p = \{T(p)x \mid x \in D\}$ where $p \in C$. Since V is a topological vector space, the closure $\bar{R}_p$ of R_p is defined. It follows from the continuity of addition and multiplication that $\bar{R}_p$ is a closed linear manifold of V. We set

$$\operatorname{def} T(p) = \dim V/\bar{R}_p ,$$

where $V/\bar{R}_p$ is the algebraic quotient space of the linear manifold $\bar{R}_p$ (Liusternik—Sobolev [9]: pp. 57—58).

We examine the so-called Montel space M. It is the set of all analytical complex-valued functions with the set $Q = \{s \in C \mid |s| < 1\}$ for domain. The metrics for the space M are defined by the formula

$$d(x\,, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{N_k(x - y)}{1 + N_k(x - y)} ,$$

where $x, y \in M$ and

$$N_k(x - y) = \sup \left\{ |x(s) - y(s)| \;\middle|\; s \in C \quad \text{and} \quad |s| \leqq 1 - \frac{1}{k} \right\} .$$

Now M is a complete metric vector space, convergence of a function sequence in it meaning that the function sequence converges uniformly in every compact subset of Q (Taylor: p. 150). The space M is also separable and locally convex, but these properties are not required here.

We examine a linear mapping $T : M \to M$ defined by the formula

$$(Tx)\,(s) = s\, x(s) \tag{16}$$

whenever $x \in M$ and $s \in Q$. It follows from the basic properties of uniform convergence that T is continuous.

It will be shown, first, that $T(p)^{-1}$ exists whenever $p \in C$. When $x \in M$, then $T(p)\, x = \bar{0}$ means the same as $(s - p)\, x(s) = 0$ for every $s \in Q$. Since the analytical function x is continuous in the set Q, it follows from the assumption $T(p)\, x = \bar{0}$ that $x(s) = 0$ for every number $s \in Q$.

When $p \in C$ and $y \in R_p$, there exists $x \in M$ such that

$$y(s) = (s - p)\, x(s)$$

for every $s \in Q$.

If $|p| \geqq 1$, then $\dfrac{y(s)}{s - p}$ is an analytical function of s with $s \in Q$ for every $y \in M$. Hence $R_p = M$ when $p \in C$ and $|p| \geqq 1$.

The case $|p| < 1$ will be examined next. If $x \in M$ and $y(s) = (s-p)\, x(s)$ with $s \in Q$, then $y(p) = 0$. On the other hand, if $y \in M$ and $y(p) = 0$, the function x defined by the formula

$$x(s) = \begin{cases} \dfrac{y(s)}{s - p} & \text{when} \quad s \in Q \quad \text{and} \quad s \neq p, \\[2ex] y'(p) = \lim\limits_{s \to p} \dfrac{y(s) - y(p)}{s - p} & \text{when} \quad s = p, \end{cases}$$

is analytical in Q (cf. Ahlfors [1]: p. 100) and, moreover,

$$y(s) = (s - p)\, x(s) \quad \text{or} \quad y = T(p)\, x\, .$$

Hence $R_p = \{y \in M \mid y(p) = 0\}$ when $|p| < 1$.

We show next that R_p is closed and $\dim M/R_p = 1$, where $p \in C$ and $|p| < 1$. If $\{y_n\} \subset R_p$ and $\lim y_n(s) = y(s)$ uniformly in every compact subset of Q, then $y_n(p) = 0$ for every value of n and $y \in M$. Hence, $y(p) = 0$ and $y \in R_p$. Since y is the arbitrary cluster point of R_p, R_p is closed.

In order to show that $\dim M/R_p = 1$, we denote by h an analytical function the value of which always equals 1, that is, $h(s) = 1$ whenever $s \in Q$. When $y \in M$,

$$y = (y - y(p)\, h) + y(p)\, h$$

and the value of the function $y - y(p)h$ at a point p is $y(p) - y(p)h(p) = 0$. Hence every vector $y \in M$ can be represented in the form

$$y = y_1 + y_2\, ,$$

where $\quad y_1 = y - y(p)\, h \in R_p$

and $\quad y_2 = y(p)\, h \in \{qh \mid q \in C\}\, .$

Moreover, $\{qh \mid q \in C\} \cap R_p = \{\bar{0}\}$. Consequently $\{qh \mid q \in C\}$ is an algebraic complement of R_p, and $\dim B/R_p = \dim \{qh \mid q \in C\} = 1$ (cf. Liusternik—Sobolev [9]: p. 58).

It has been proved above that $T(p)$ is continuous, $T(p)^{-1}$ exists, and the range of $T(p)$ is a closed linear manifold of the complete metric vector space. Hence $T(p)^{-1}$ is continuous (Taylor 4.2.—H).

The regularity region of T is C and

$$\operatorname{def} T(p) = \begin{cases} \dim M/M = 0 & \text{when } |p| \geqq 1, \\ \dim M/R_p = 1 & \text{when } |p| < 1 . \end{cases}$$

The deficiency index, therefore, is not constant in the whole regularity region, which is an open connected set.

University of Oulu
Oulu, Finland

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