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COMPACT AND FINITE-DIMENSIONAL ELEMENTS OF NORMED ALGEBRAS

BY

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Preface

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Contents

Introduction

Completely continuous elements of a commutative normed algebra have been studied by M. Freundlich [4], who defined them as those elements of the algebra for which the corresponding regular representations are compact operators. A different definition of a compact element in a normed algebra was given by K. Vala in [9]. According to his definition u is a compact element of the normed algebra $\mathfrak{A}$, if the linear transformation $x \to uxu$ is a precompact operator on $\mathfrak{A}$. This definition generalizes the notion of a compact operator, for if we take as $\mathfrak{A}$ the full operator algebra $L(E)$ on a Banach space E, theorem 3 in [8] shows that the compact elements of $\mathfrak{A}$ are precisely the compact operators on E. A special class of compact elements is formed by the elements $u \in \mathfrak{A}$ for which the operator $x \to uxu$ has a finite-dimensional range. We call these elements finite-dimensional (»éléments de rang fini» in the terminology of [9]).

In this paper we study compact and finite-dimensional elements in the sense of Vala. The most decisive results are reached for C^*-algebras, i.e. uniformly closed operator algebras on a Hilbert space that are closed with respect to the involution $T \to T^*$. An example in [9] shows that if the last condition is omitted, the sum of two compact elements may fail to be compact. However, the compact elements of a C^*-algebra form a two-sided ideal, which is the closure of the ideal of the finite-dimensional elements (theorem 3.10). This is turn coincides with the socle of the algebra (theorem 5.1). Certain other, notably spectral, properties of compact operators extend to compact elements (theorems 1.6 and 3.11). For a very special class of C^*-algebras, namely the factors of von Neumann, the two-sided ideal of compact elements — if non-zero — is minimal-closed.

The existence of non-zero compact elements imposes rather severe restrictions on the algebra (cf. e.g. corollary 2 of theorem 2.2, and theorem 4.3). In particular, a factor containing non-zero compact elements is isomorphic to the full operator algebra on some Hilbert space. Therefore results concerning the structure of the set of compact elements in a factor follow from the classical theory of compact operators on a Hilbert space. However, in section 7 we prove directly the above mentioned minimality property of the ideal of the compact elements in a factor, for in this way a new proof is obtained for theorem 3 in [8] in the case of a Hilbert space.

1. General properties of compact and finite-dimensional elements

1.1. Let E and F be real or complex normed spaces. Recall that $T \in L(E, F)$ is a *compact* (resp. *precompact*) *operator*, if it maps the closed unit ball of E onto a relatively compact (resp. precompact or, synonymously, totally bounded) set. The following definition is due to Vala [9]:

Definition. An element u of a real or complex normed algebra $\mathfrak{A}$ is called *compact*, if the mapping $x \to uxu$ is a precompact operator on $\mathfrak{A}$. An element u of an arbitrary algebra $\mathfrak{A}$ is called *finite-dimensional*, if the range of the mapping $x \to uxu$ on $\mathfrak{A}$ is finite-dimensional.

Every finite-dimensional element of a normed algebra $\mathfrak{A}$ is a compact element of $\mathfrak{A}$. It is also clear that a compact (resp. finite-dimensional) element of $\mathfrak{A}$ is a compact (resp. finite-dimensional) element of every subalgebra of $\mathfrak{A}$. This fact serves to justify the definition of a compact element in terms of precompact instead of compact operators. In the case of a Banach algebra this distinction is immaterial, since on a Banach space precompact operators are the same as compact operators.

For the simple proofs of the following results see [9].

Theorem 1.1. (i) *If u is a compact (resp. finite-dimensional) element of $\mathfrak{A}$ and $v \in \mathfrak{A}$, uv and vu are compact (resp. finite-dimensional) elements of $\mathfrak{A}$.*

(ii) *Every compact idempotent is finite-dimensional.*

1.2. The next theorem shows that often $\mathfrak{A}$ may be assumed to be a Banach algebra, since any normed algebra can be viewed as a dense subalgebra of a Banach algebra (cf. e.g. [5] p. 176). We first give a simple lemma, from which the theorem immediately follows.

Lemma 1.1. *Let E_1 be a normed space having E as a dense subspace. Let φ be a bounded linear operator from E_1 into the normed space F. Then φ is a precompact operator (resp. an operator with finite-dimensional range), if its restriction to E is one.*

Proof: Since the closed unit ball of E is dense in that of E_1, the first statement follows from the continuity of φ and the fact that the closure of a precompact set is precompact. The second is a consequence of the fact that finite-dimensional subspaces of F are closed.

Theorem 1.2. *Let $\mathfrak{A}_1$ be a normed algebra and let $\mathfrak{A}$ be its dense subalgebra. Then $u \in \mathfrak{A}$ is a compact (resp. finite-dimensional) element of $\mathfrak{A}$ if and only if it is a compact (resp. finite-dimensional) element of $\mathfrak{A}_1$.*

1.3. If $\mathfrak{A}$ is a real normed algebra, it can be embedded (real) isomorphically in a complex algebra $\mathfrak{A}_C = \mathfrak{A} \times \mathfrak{A}$ called its *complexification,* in which the algebra operations are so defined that (x , y) behaves like $x + iy$. Furthermore, $\mathfrak{A}$ can be given a norm so that this embedding $x \to (x , 0)$ is an isometry, and $\mathfrak{A}_C$ will be a Banach algebra if and only if $\mathfrak{A}$ is a Banach algebra (cf. [6], p. 6 and p. 8 theorem 1.3.2).

Theorem 1.3. *Let $\mathfrak{A}$ be a real normed algebra and let $\mathfrak{A}_C$ be its complexification. Then $u \in \mathfrak{A}$ is a compact (resp. finite-dimensional) element of $\mathfrak{A}$ if and only if $(u , 0)$ is a compact (resp. finite-dimensional) element of $\mathfrak{A}_C$.*

Proof: It follows from the proof of theorem 1.3.2 in [6] that the norm of $\mathfrak{A}_C$ is equivalent to the norm $|(x , y)| = \|x\| + \|y\|$. Thus we may consider $\mathfrak{A}_C$ under this norm. Denote the closed unit ball of $\mathfrak{A}$ by B and that of $\mathfrak{A}_C$ by B_C. Since $B_C \subset B \times B$, we have for $u \in \mathfrak{A}$

$$(u , 0)\, B_C(u , 0) \subset (u , 0)\, B \times B(u , 0) = (uBu) \times (uBu)\,.$$

Since the Cartesian product of two precompact sets is precompact, the compactness of u in $\mathfrak{A}$ implies the compactness of $(u , 0)$ in $\mathfrak{A}_C$. Conversely, if $(u , 0)$ is a compact element of $\mathfrak{A}_C$, u is a compact element of $\mathfrak{A}$, for then $(u , 0)\,(B \times \{0\})\,(u , 0)$ is a subset of the precompact set $(u , 0)\, B_C(u , 0)$ so that also uBu is precompact. For finite-dimensional elements the theorem follows from the simple facts that subspaces and finite direct sums of finite-dimensional spaces are finite-dimensional.

1.4. If $\mathfrak{A}$ is an algebra without a multiplicative identity, an identity can be adjoined to $\mathfrak{A}$ by embedding $\mathfrak{A}$ via the canonical mapping $x \to (x , 0)$ in the Cartesian product $\mathfrak{A}_1$ of $\mathfrak{A}$ and the scalar field. In $\mathfrak{A}_1$ the algebra operations are so defined that the couple (x , λ) can be treated like a formal sum $x + \lambda$. If $\mathfrak{A}$ is a normed algebra, $\mathfrak{A}_1$ becomes a normed algebra under the norm $\|(x , \lambda)\| = \|x\| + |\lambda|$, and $\mathfrak{A}_1$ will be a Banach algebra if and only if $\mathfrak{A}$ is a Banach algebra (cf. [6] p. 2).

Theorem 1.4. *An element $u \in \mathfrak{A}$ is a compact (resp. finite-dimensional) element of $\mathfrak{A}$ if and only if $(u , 0)$ is a compact (resp. finite-dimensional) element of $\mathfrak{A}_1$.*

Proof: Denote $B = \{x \in \mathfrak{A} \mid \|x\| \leq 1\}$, $B_1 = \{x \in \mathfrak{A}_1 \mid \|x\| \leq 1\}$, $C = \{\lambda \in \mathbf{K} \mid |\lambda| \leq 1\}$, where $\mathbf{K}$ is the scalar field ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$). If $(x, \lambda) \in B_1$, then $\|x\| \leq 1$ and $|\lambda| \leq 1$. Hence we have

$$(u, 0)\, B_1(u, 0) \subset (u, 0)\, (B \times C)\, (u, 0) .$$

The latter set is equal to $(uBu + Cu^2) \times \{0\}$, which is precompact if u is a compact element of $\mathfrak{A}$. The converse is obvious. For finite-dimensional elements the proof is still simpler.

1.5. Generalizing a result of Freundlich ([4] p. 277) we obtain

Theorem 1.5. *Let $\mathfrak{A}$ be an infinite-dimensional algebra and u a finite-dimensional element of $\mathfrak{A}$. Then u is a (left or right) divisor of zero.*

Proof: Choose elements $x_i \in \mathfrak{A}$, $1 \leq i \leq n$, such that $\{ux_iu \mid 1 \leq i \leq n\}$ is a basis for $u\mathfrak{A}u$. For an arbitrary $x \in \mathfrak{A}$ we can write

$$uxu = \sum_{i=1}^{n} \lambda_i u x_i u$$

or

$$u\Big(x - \sum_{i=1}^{n} \lambda_i x_i\Big)\, u = 0 .$$

If $x - \sum_{i=1}^{n} \lambda_i x_i$ were zero for every x, $\mathfrak{A}$ would be finite-dimensional contrary to the hypothesis. Hence there is $a \neq 0$ such that $uau \neq 0$. Then u is a left or right divisor of zero according as $ua = 0$ or $ua \neq 0$.

1.6. Let $\mathfrak{A}$ be a complex algebra with identity e. The *spectrum* $Sp_{\mathfrak{A}}(a)$ of an element $a \in \mathfrak{A}$ is defined as the set of those $\lambda \in \mathbf{C}$ for which $\lambda e - a$ is not regular. If $\mathfrak{A}$ does not possess an identity, we define the spectrum $Sp_{\mathfrak{A}}(a)$ of $a \in \mathfrak{A}$ as the spectrum of $(a, 0)$ in the algebra obtained from $\mathfrak{A}$ by adjunction of an identity. It is well known that this definition is equivalent to the one given in [6], p. 28, in terms of quasi-inverses (cf. [6] p. 27 and p. 32, theorem 1.6.9). If $\mathfrak{A}$ is a real algebra, the spectrum of $a \in \mathfrak{A}$ is defined as the spectrum of $(a, 0)$ in the complexification $\mathfrak{A}_C$ of $\mathfrak{A}$.

Theorem 1.6. *Let $\mathfrak{A}$ be a Banach algebra. The spectrum of any compact element of $\mathfrak{A}$ consists of a countable number of points, which can accumulate only in the origin. The spectrum of a finite-dimensional element of $\mathfrak{A}$ has a finite number of points.*

Proof: In virtue of theorems 1.3, 1.4, and the remarks immediately preceding them $\mathcal{U}$ may be assumed to be a complex Banach algebra with identity e. Denote by Y_u the centralizer of $u \in \mathcal{U}$, that is

$$Y_u = \{x \in \mathcal{U} \mid ux = xu\}.$$

Then Y_u is a closed subalgebra of $\mathcal{U}$ containing the identity. If $(\lambda e - u)^{-1}$ exists, it belongs to Y_u, and since obviously

$$Sp_{\mathcal{U}}(u) \subset Sp_{Y_u}(u),$$

we have

$$Sp_{\mathcal{U}}(u) = Sp_{Y_u}(u).$$

If u is a compact element of $\mathcal{U}$, the mapping $x \to uxu$ restricted to the Banach space Y_u defines the compact operator $x \to u^2x$ on Y_u. The spectrum of this operator is by theorem 1.6.9 in [6] equal to $Sp_{Y_u}(u^2)$. But the spectrum of a compact operator is countable and it can accumulate only in the origin (cf. [7] p. 281). By theorem 1.6.10 in [6],

$$Sp_{Y_u}(u^2) = (Sp_{Y_u}(u))^2 = (Sp_{\mathcal{U}}(u))^2.$$

Hence also $Sp_{\mathcal{U}}(u)$ is of the type stated in the theorem. If, in the above, u is finite-dimensional, $Sp_{Y_u}(u^2)$ and hence also $Sp_{\mathcal{U}}(u)$ is finite. This follows from the fact that the non-zero portion of the spectrum of any compact operator consists of eigenvalues and from a standard argument showing that eigenvectors corresponding to distinct eigenvalues are linearly independent (cf. e.g. [7] p. 281).

Corollary. *Let $\mathcal{U}_1$ be a Banach algebra having $\mathcal{U}$ as a closed subalgebra. If u is a compact element of $\mathcal{U}$, then*

$$Sp_{\mathcal{U}_1}(u) \setminus \{0\} = Sp_{\mathcal{U}}(u) \setminus \{0\}.$$

If, furthermore, $\mathcal{U}_1$ has an identity e such that $e \in \mathcal{U}$, then $Sp_{\mathcal{U}_1}(u) = Sp_{\mathcal{U}}(u)$.

Proof: It is easily seen that

$$Sp_{\mathcal{U}_1}(u) \subset Sp_{\mathcal{U}}(u) \cup \{0\}. \tag{1}$$

It follows from theorem 1.6 and the fact that all spectra are closed that

$$\partial\, Sp_{\mathcal{U}_1}(u) = Sp_{\mathcal{U}_1}(u)$$

and

$$\partial\, Sp_{\mathcal{U}}(u) = Sp_{\mathcal{U}}(u).$$

Hence by theorem 1.6.12 in [6]

$$Sp_{\mathfrak{U}}(u) \subset Sp_{\mathfrak{U}_1}(u) .$$

Zero was included in (1) because $\mathfrak{U}$ may have an identity which is not an identity of $\mathfrak{U}_1$, but it may be omitted in case $\mathfrak{U}_1$ has an identity e such that $e \in \mathfrak{U}$.

1.7. We summarize in the next lemma three well-known properties of precompact operators, which we shall often use without explicit mention.

Lemma 1.2. (i) *Any finite linear combination of precompact operators on a normed space is a precompact operator.*

(ii) *The composition of a bounded linear operator and a precompact operator, in either order, is a precompact operator.*

(iii) *The set of the precompact operators on a normed space E is closed in the norm topology of $L(E)$.*

Theorem 1.7. *Let $\{u_k\}$ and $\{v_k\}$ be sequences in a normed algebra $\mathfrak{U}$ with*

$$\lim_{k \to \infty} u_k = u , \quad \lim_{k \to \infty} v_k = v .$$

Denote the mapping $x \to u_k x v_k$ by U_k and the mapping $x \to uxv$ by U. If U_k is a precompact operator for every k, U is a precompact operator.

Proof: Elementary properties of the norm give

$$\|U_k(x) - U(x)\| = \|u_k x v_k - uxv_k + uxv_k - uxv\| \leq$$

$$\|u_k - u\| \, \|v_k\| \, \|x\| + \|v_k - v\| \, \|u\| \, \|x\| \leq 2 \, \|v\| \, \|u_k - u\| + \|u\| \, \|v_k - v\|$$

for $\|x\| \leq 1$ and k large enough. Hence

$$\lim_{k \to \infty} \|U_k - U\| = \lim_{k \to \infty} \sup_{\|x\| \leq 1} \|(U_k - U)(x)\| = 0 ,$$

and the conclusion follows from lemma 1.2 (iii).

Corollary. *The set of the compact elements of a normed algebra is closed.*

2. Ascoli's theorem and irreducible operator algebras

2.1. The characterization of precompact operators given by Vala in [8], underlying his definition of a compact element, is based on a new formulation of Ascoli's theorem. For the sake of completeness we reproduce this theorem and its proof in a somewhat more symmetric form.

For this purpose, let A and B be arbitrary sets, (F, d) a metric space, and Φ a mapping from $A \times B$ into F. Consider the following four statements:

I For every $y \in B$ and every $\varepsilon > 0$ there is a finite covering $(A_i)_{i \in \mathcal{J}}$ of A such that $u, v \in A_i$ implies $d(\Phi(u, y), \Phi(v, y)) < \varepsilon$.

II For every $x \in A$ and every $\varepsilon > 0$ there is a finite covering $(B_j)_{j \in \mathcal{J}}$ of B such that $w, z \in B_j$ implies $d(\Phi(x, w), \Phi(x, z)) < \varepsilon$.

III For every $\varepsilon > 0$ there is a finite covering $(A_i)_{i \in \mathcal{J}}$ of A such that $u, v \in A_i$ implies $d(\Phi(u, y), \Phi(v, y)) < \varepsilon$ for every $y \in B$.

IV For every $\varepsilon > 0$ there is a finite covering $(B_j)_{j \in \mathcal{J}}$ of B such that $w, z \in B_j$ implies $d(\Phi(x, w), \Phi(x, z)) < \varepsilon$ for every $x \in A$.

Theorem 2.1. (i) III *implies* I.
(ii) IV *implies* II.
(iii) II *and* III *together imply* IV.
(iv) I *and* IV *together imply* III.

Proof: By symmetry it suffices to prove (i) and (iii). But (i) is immediate, so let us prove (iii). Given $\varepsilon > 0$, let $(A_i)_{i \in \mathcal{J}}$ be a finite covering of A such that $u, v \in A_i$ implies $d(\Phi(u, y), \Phi(v, y)) < \frac{\varepsilon}{3}$ for every $y \in B$ (cf. III). Clearly, A and each A_i may be assumed non-empty. For each $i \in \mathcal{J}$ choose $x_i \in A_i$. By II there is for each $i \in \mathcal{J}$ a finite covering $(B^i_j)_{j \in \mathcal{J}_i}$ such that $w, z \in B^i_j$ implies $d(\Phi(x_i, w), \Phi(x_i, z)) < \frac{\varepsilon}{3}$. Then all intersections of the type $\bigcap_{i \in \mathcal{J}} B^i_{j_i}$, where $j_i \in \mathcal{J}_i$, form the required finite covering of B. For if $w, z \in \bigcap_{i \in \mathcal{J}} B^i_{j_i}$ and $x \in A$, there is $k \in \mathcal{J}$ such that $x \in A_k$ and $w, z \in B^k_{j_k}$. Hence we have

$$d(\Phi(x, w), \Phi(x, z)) \leq d(\Phi(x, w), \Phi(x_k, w)) + d(\Phi(x_k, w), \Phi(x_k, z)) +$$

$$+ d(\Phi(x_k, z), \Phi(x, z)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which means that IV holds.

Remark. Theorem 1 in [8] follows from the above theorem, for if B is a set of mappings from A into F and we set

$$\Phi(x, y) = y(x),$$

then the conditions I—III have the following interpretations:

I′ The range of each mapping $y \in B$ is precompact.
II′ For every $x \in A$ the set $\{y(x) \mid y \in B\}$ is precompact.
III′ The set B has equal variation (cf. [8] p. 4).

If each mapping $y \in B$ is bounded, B becomes a metric space with the metric

$$\varrho(y_1, y_2) = \sup_{x \in A} d(y_1(x), \ y_2(x)),$$

and then IV means that

IV′ the function space (B, ϱ) is precompact.

We note that theorem 6 in [1] is also an immediate consequence of theorem 2.1. We reformulate it for not necessarily complete spaces:

Corollary. *Let E_1, E_2, E_3, E_4, Y be normed spaces, let $a : E_1 \to E_2$, $d : E_3 \to E_4$ be precompact and $b : Y \to L(E_2, E_4)$, $c : Y \to L(E_1, E_3)$ bounded linear operators and denote*

$$Z = \{\eta \in Y \mid (b\eta) \circ a = d \circ c(\eta)\}.$$

Then the mapping $\eta \to (b\eta) \circ a$ is a precompact operator from Z into $L(E_1, E_4)$

Proof: Denote $A = \{\xi \in E_1 \mid \|\xi\| \leq 1\}$ and $B = \{(b\eta) \circ a | A \mid \eta \in Z, \|\eta\| \leq 1\}$. Since a is precompact, there is for every $\varepsilon > 0$ a finite covering $(A_i)_{i \in J}$ of A such that $\xi_1, \xi_2 \in A_i$ implies $\|b\| \, \|a(\xi_1) - a(\xi_2)\| < \varepsilon$. Hence the condition III′ in the preceding remark holds for A and B. Since $(b\eta \circ)a = d \circ c(\eta)$ for $\eta \in Z$, the precompactness of d yields II′ and the conclusion follows from IV′, which holds by theorem 2.1.

2.2. For the first corollary of the next theorem we need the trivial implication (ii) in our formulation of Ascoli's theorem. For the notion of the irreducibility of operator algebras see [6] p. 48.

Theorem 2.2. *Let E be a Banach space and $\mathfrak{A}$ a strictly irreducible subalgebra of $L(E)$. If $S \in L(E)$ and the operator $X \to SXz$ from $\mathfrak{A}$ into E is precompact for some non-zero $z \in E$, S is a compact operator.*

Proof: Denote by φ the mapping $X \to Xz$ from the closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ into E. Then φ is a bounded linear mapping from one Banach space onto another, for the strict irreducibility of $\overline{\mathfrak{A}}$ implies that the range of φ is all of E. Denote by B the closed unit ball of $\overline{\mathfrak{A}}$ and by B_1 that of E. In virtue of Banach's open mapping theorem $\varphi(B)$ contains a neighborhood of zero, say $B_r = \{x \in E \mid \|x\| \leq r\}$. It follows, in virtue of lemma 1.1, that the set

$$S(B_1) = \frac{1}{r} S(B_r) \subset \frac{1}{r} S(\varphi(B))$$

is precompact, and the theorem is proved.

As in example 3 in [1], p. 49, the corollary of theorem 2.1 yields in a special case the result, contained in theorem 3 of [8], that for precompact operators S and T on a normed space E the operator $X \to SXT$ on $L(E)$ and hence its restriction to any subalgebra of $L(E)$ is precompact. As a consequence of the above theorem we have the following partial converse:

Corollary 1. *Let E and $\mathfrak{A}$ be as in the preceding theorem. If $S, T \in L(E)$, $T \neq 0$, and the operator $X \to SXT$ from $\mathfrak{A}$ into $L(E)$ is precompact, then S is a compact operator.*

Proof: There is a unit vector $x \in E$ such that $z = Tx \neq 0$. Considering the restrictions of the operators SXT, where $X \in \mathfrak{A}$ and $\|X\| \leq 1$, to the closed unit ball of E it is seen that theorem 2.1. (ii), the remark following it, and theorem 2.2. yield the conclusion at once.

Corollary 2. *Let E and $\mathfrak{A}$ be as in theorem 2.2. If S is a compact element of $\mathfrak{A}$, S is a compact operator on E.*

The above corollary and the remark preceding corollary 1 yield the following result:

Corollary 3. *Let $\mathfrak{A}$ be as in theorem 2.2. The compact elements of $\mathfrak{A}$ form a closed two-sided ideal of $\mathfrak{A}$.*

In a special case of corollary 1 T may also easily be shown to be a compact operator.

Corollary 4. *Let H be a Hilbert space and let $\mathfrak{U}$ be a strictly irreducible subalgebra of $L(H)$, closed with respect to the involution $T \to T^*$. If $S, T \in L(H)$, $S \neq 0$, and the operator $X \to SXT$ from $\mathfrak{U}$ into $L(H)$ is precompact, then T is a compact operator.*

Proof: The continuity of the involution implies that the operator $X \to X^* \to SX^*T \to T^*XS^*$ is precompact. By corollary 1, T^* and hence also T is a compact operator.

In the next theorem we summarize some results for finite-dimensional elements and operators with finite-dimensional range corresponding to theorem 3 in [8] and the corollaries of theorem 2.2.

Theorem 2.3. *Let E be a real or complex linear space, S and T linear operators on E, and $\mathfrak{U}$ an algebra of linear operators on E.*

(i) *If S and T have finite-dimensional ranges, the mapping $X \to SXT$ on $\mathfrak{U}$ has a finite-dimensional range.*

(ii) *If $\mathfrak{U}$ is strictly irreducible, $T \neq 0$, and the mapping $X \to SXT$ on $\mathfrak{U}$ has a finite-dimensional range, then S has a finite-dimensional range. In particular, if $\mathfrak{U}$ is strictly irreducible and S is a finite-dimensional element of $\mathfrak{U}$, S is an operator with finite-dimensional range.*

(iii) *If E is complex, $\mathfrak{U}$ a complex strictly irreducible Banach algebra, $S \neq 0$, and the mapping $X \to SXT$ on $\mathfrak{U}$ has a finite-dimensional range, then T is an operator with finite-dimensional range.*

Proof: (i) The image of $\mathfrak{U}$ under the mapping $X \to SXT$ is isomorphic to a subspace of the algebra of all linear operators from $T(E)$ into $S(E)$, hence finite-dimensional.

(ii) Choose $z \in E$ such that $Tz \neq 0$. Since Tz is a strictly cyclic vector, the set $\{SXTz \mid X \in \mathfrak{U}\}$ coincides with the entire range $S(E)$ of S. Hence, if the subspace $\{SXT \mid X \in \mathfrak{U}\}$ is finite-dimensional, $S(E)$ must be finite-dimensional.

(iii) Choose $z \in E$ such that $Sz \neq 0$. By theorem 2.4.6 in [6] the algebra $\mathfrak{U}$ is strictly dense on E ([6] p. 60). Hence, if $T(E)$ were infinite-dimensional, there would exist, for an arbitrarily large n, linearly independent vectors $y_1, \ldots, y_n \in T(E)$ and operators $X_1, \ldots, X_n \in \mathfrak{U}$ such that $X_i y_j = \delta_{ij} z$. But then the set $\{SX_iT \mid 1 \leq i \leq n\}$ would be linearly independent, and since n can be chosen arbitrarily, this contradicts the finite-dimensionality of the subspace $\{SXT \mid X \in \mathfrak{U}\}$.

3. Compact elements in algebras of operators on a Hilbert space

3.1. In this paper H always denotes a complex Hilbert space ($\neq \{0\}$) and $L(H)$ the algebra of bounded linear operators on H. A subalgebra of $L(H)$ that is closed with respect to the involution $T \to T^*$ is called a **-subalgebra* of $L(H)$. A *C^*-algebra* is a uniformly closed *-subalgebra of $L(H)$. According to an equivalent abstract definition a *C^*-algebra* (or a B^*-algebra, cf. [6] p. 180) is a Banach algebra with involution such that its norm satisfies the condition $\|x\|^2 = \|x^*x\|$ (cf. [2] pp. 6 and 39).

For a set $\mathfrak{N} \subset L(H)$ we denote

$$\mathfrak{N}' = \{T \subset L(H) \mid TS = ST \text{ for every } S \in \mathfrak{N}\},$$

and $\mathfrak{N}'' = (\mathfrak{N}')'$. A *-subalgebra $\mathfrak{U}$ of $L(H)$ is called a *von Neumann algebra,* if $\mathfrak{U} = \mathfrak{U}''$. Every von Neumann algebra is a *C^*-algebra* (cf. e.g. [5] p. 170).

3.2. Let $\mathfrak{U}$ be a subalgebra of $L(H)$ and E a projection in $\mathfrak{U}$ or in $\mathfrak{U}'$. We denote $T_E = ET \mid E(H)$ and $\mathfrak{U}_E = \{T_E \mid T \in \mathfrak{U}\}$. It is readily seen that $\mathfrak{U}_E$ is a subalgebra of $L(E(H))$ and a *-subalgebra if $\mathfrak{U}$ is one. If $\mathfrak{U}$ is a von Neumann algebra, $\mathfrak{U}_E$ is a von Neumann algebra ([2] p. 18).

Let $(H_i)_{i \in \mathcal{J}}$ be a family of Hilbert spaces and, for each $i \in \mathcal{J}$, let $\mathfrak{U}_i$ be a subalgebra of $L(H_i)$. Denote by H the Hilbert sum of the spaces H_i. If $T_i \in \mathfrak{U}_i$ and $\sup_{i \in \mathcal{J}} \|T_i\| < \infty$, we define $T = (T_i) \in L(H)$ by setting

$$T((x_i)_{i \in \mathcal{J}}) = (T_i x_i)_{i \in \mathcal{J}} .$$

The set of operators $T \in L(H)$ of this type is an algebra called the *product* of the algebras $\mathfrak{U}_i$ and denoted by $\prod_{i \in \mathcal{J}} \mathfrak{U}_i$. If $\mathcal{J} = \{1, \ldots, n\}$, we also denote

$$\prod_{i \in \mathcal{J}} \mathfrak{U}_i = \mathfrak{U}_1 \times \ldots \times \mathfrak{U}_n .$$

If each $\mathfrak{U}_i$ is a von Neumann algebra, $\prod_{i \in \mathcal{J}} \mathfrak{U}_i$ is also a von Neumann algebra (cf. [2] p. 21).

3.3. In this and the next subsection we record a few simple properties of compact and finite-dimensional elements in subalgebras of $L(H)$.

Theorem 3.1. *Let $\mathfrak{U}$ be a subalgebra of $L(H)$, $T \in \mathfrak{U}$ and E a projection.*

(i) *If $E \in \mathfrak{U}$ and T is a compact element of $\mathfrak{U}$, T_E is a compact element of $\mathfrak{U}_E$.*

(ii) *If $E \in \mathfrak{U}$ or $E \in \mathfrak{U}'$, $T(H) \subset E(H)$, $Ker(T) \supset (I - E)(H)$ and*

T_E is a compact element of $\mathfrak{U}_E$, then T is a compact element of $\mathfrak{U}$. Analogous statements hold for finite-dimensional elements.

Proof: The mapping $ESE \to S_E$ is a linear surjection from $E\mathfrak{U}E$ onto $\mathfrak{U}_E$. It is isometric, hence injective, for

$$\|S_E\| = \sup_{\|x\| \leq 1\,,\, x = Ex} \|ESx\| = \sup_{\|x\| \leq 1} \|ESEx\| = \|ESE\|\,.$$

Denote this (vector space) isomorphism by φ. Then the operator

$$X_E \xrightarrow{\varphi^{-1}} EXE \to TEXET \to ETEXETE \xrightarrow{\varphi} (TEXET)_E = T_E X_E T_E$$

s precompact. This proves (i). To prove (ii) we note that the operator

$$X \to EXE \xrightarrow{\varphi} X_E \to T_E X_E T_E = ETEXET \mid E(H) =$$

$$= (TXT)_E \xrightarrow{\varphi^{-1}} ETXTE = TXT$$

is precompact. The proof for finite-dimensional elements is analogous.

Corollary. *A projection $E \in \mathfrak{U}$ is a finite-dimensional element of $\mathfrak{U}$ if and only if $\mathfrak{U}_E$ is finite-dimensional.*

Theorem 3.2. *Let H_i be a Hilbert space and $\mathfrak{U}_i$ a subalgebra of $L(H_i)$ for $i = 1, \ldots, n$. Then $T = (T_i)_{1 \leq i \leq n}$ is a compact (resp. finite-dimensional) element of $\mathfrak{U} = \mathfrak{U}_1 \times \ldots \times \mathfrak{U}_n$ if and only if each T_i is a compact (resp. finite-dimensional) element of $\mathfrak{U}_i$.*

Proof: Let E_i denote the projection of H onto H_i. Then $E_i \in \mathfrak{U}'$ and $\mathfrak{U}_{E_i}$ can be identified with $\mathfrak{U}_i$. For $S = (S_i) \in \mathfrak{U}$, denote $S_j' = (\delta_{ij} S_j)_{1 \leq i \leq n}$. Then $\|S_j'\| \leq \|S\|$, so that the mapping $S \to S_j'$ is continuous. Furthermore, the correspondence $S_j' \to S_j$ is an isometry. Thus, if each T_i is a compact element of $\mathfrak{U}_i$, it follows from lemma 1.2. (i), (ii) that the operator

$$X \to \sum_{i=1}^{n} T_i' X_i' T_i' = TXT$$

is precompact. To prove the converse, note that if $T = (T_i)$ is a compact element of $\mathfrak{U}$, the operator

$$X \to X \circ E_i \to T(X \circ E_i)\, T = T_i X T_i \circ E_i \to T_i X T_i \quad \text{on } \mathfrak{U}_i$$

is precompact. The proof for finite-dimensional elements is analogous.

Theorem 3.3. *Let* T *be a compact (resp. finite-dimensional) element of the *-subalgebra* $\mathfrak{A}$ *of* $L(H)$. *Then* T^* *is a compact (resp. finite-dimensional) element of* $\mathfrak{A}$.

Proof: The operator $X \to X^* \to TX^*T \to T^*XT^*$ is precompact (resp. has a finite-dimensional range).

3.4. Let E and F be projections in the *-subalgebra $\mathfrak{A}$ of $L(H)$. If there exists a partially isometric operator $U \in \mathfrak{A}$ with initial projection E and final projection F (equivalently, such that $U^*U = E$ and $UU^* = F$, cf. [2] p. 333), then we say that E and F are *equivalent* (with respect to $\mathfrak{A}$) and write $E \sim F$. We write $E \leq F$, if $E(H) \subset F(H)$ (equivalently, $E = EF = FE$), and $E \prec F$, if there is a projection $F' \in \mathfrak{A}$ such that $E \sim F' \leq F$ (cf. [2] p. 225).

The term »*relatively finite-dimensional projection*» will refer to a projection that is a finite-dimensional (or, equivalently, compact, cf. theorem 1.1 (ii)) element of a subalgebra of $L(H)$ clear from the context.

Theorem 3.4. *Let* E *and* F *be projections in the *-subalgebra* $\mathfrak{A}$ *of* $L(H)$. *If* $E \prec F$ *and* F *is relatively finite-dimensional, then* E *is also relatively finite-dimensional.*

Proof: There is a projection $F' \in \mathfrak{A}$ such that $E \sim F' = F'F$. Hence there is a partially isometric operator $U \in \mathfrak{A}$ such that $F' = UU^*$ and

$$E = E^2 = U^*UU^*U = U^*F'U = U^*F'FU .$$

By theorem 1.1 (i) E is a compact element of $\mathfrak{A}$.

We call a projection E in a *-subalgebra $\mathfrak{A}$ of $L(H)$ *finite*, if there is no projection $F \in \mathfrak{A}$ such that $F \sim E$ and $F \lneqq E$, i.e. $F \leq E$, $F \neq E$. If $\mathfrak{A}$ is a von Neumann algebra, it is equivalent to require that $\mathfrak{A}_E$ be a so-called finite von Neumann algebra (cf. [2] pp. 241 and 318).

Theorem 3.5. *Every relatively finite-dimensional projection* E *in a *-subalgebra* $\mathfrak{A}$ *of* $L(H)$ *is finite.*

Proof: Suppose E is not finite. Then we have $E = E_1 \sim E_2$, where $E_2 \lneqq E_1$. Let $U_1 \in \mathfrak{A}$ be a partially isometric operator such that $U_1^*U_1 = E_1$ and $U_1U_1^* = E_2$. Then $U_2 = U_1E_2 \in \mathfrak{A}$ is a partially isometric operator, for which $U_2^*U_2 = E_2$ and $U_2U_2^*(H) \subset E_2(H)$. Since $E_2 \lneqq E_1$ and $U_1 \mid E_1(H)$ maps $E_1(H)$ onto $E_2(H)$ bijectively, the inclusion must be proper. By induction we obtain a sequence of projections $E_1 \gneqq E_2 \gneqq \dots$ such that $E_n = EE_nE \in E\mathfrak{U}E$. Furthermore, the projections E_n are linearly independent. For suppose

$$\sum_{i=1}^{n} \lambda_i E_i = 0 \,. \tag{1}$$

Choosing non-zero vectors from the subspaces $(E_i - E_{i+1})(H)$, $1 \leq i \leq n$, and applying the operator (1) to them it is seen successively that $\lambda_1 = \dots = \lambda_n = 0$. Hence the space $E\,\mathfrak{U}\,E$ is infinite-dimensional, which means that E is not a finite-dimensional element of $\mathfrak{U}$.

3.5. In the following we summarize some spectral theoretic facts needed in the study of the compact elements of C^*-algebras.

According to a well-known theorem of Gelfand and Naimark ([5], p. 232) any commutative C^*-algebra $\mathfrak{U}$ with identity may be realized as the algebra $C(\mathfrak{M})$ of all continuous complex-valued functions on a compact Hausdorff space $\mathfrak{M}$. The algebra operations in $C(\mathfrak{M})$ are defined pointwise, involution $\hat{T} \to \hat{T}^*$ by setting $\hat{T}^*(M) = \overline{\hat{T}(M)}$ for $M \in \mathfrak{M}$ and norm by

$$\|\hat{T}\| = \sup_{M \in \mathfrak{M}} |T(M)| \,,$$

where $\hat{T} \in C(\mathfrak{M})$ is the function corresponding to $T \in \mathfrak{U}$. For details we refer to [5].

In particular, let $A \in L(H)$ be a Hermitian operator and denote by $\mathfrak{U}_A$ the minimal closed commutative $*$-subalgebra of $L(H)$ containing A and the identity operator I. It follows from proposition 1.3.10 in [3] that

$$Sp_{L(H)}(A) = Sp_{\mathfrak{U}_A}(A) \,.$$

Thus we may in this connection speak unequivocally of the spectrum $Sp(A)$ of A. In virtue of proposition 1.4.3 in [3] we may, in representing $\mathfrak{U}_A$ as a function algebra $C(\mathfrak{M})$, assume that $\mathfrak{M} = Sp(A)$. Thus a unique element, denoted by $f(A)$ of $\mathfrak{U}_A$ corresponds to each continuous complex-valued function f on $Sp(A)$.

The following formulation of the spectral decomposition of a Hermitian operator is proved in [5], pp. 248—249:

Lemma 3.1. *For every Hermitian operator* $A \in L(H)$ *there exists a unique function* $P : \mathbf{R} \to L(H)$, *called the spectral function of* A, *with the following properties:*

1° *$P(\lambda)$ is a projection operator for every $\lambda \in \mathbf{R}$.*
2° *$P(\lambda)P(\mu) = P(\lambda)$ for $\lambda \leq \mu$.*
3° *$P(\lambda)T = TP(\lambda)$ for every $T \in L(H)$ such that $AT = TA$.*
4° *The function $\lambda \to P(\lambda)x$ is continuous from the left for every $x \in H$.*
5° *Let $[a, b]$ be a closed interval containing $Sp(A)$ and f a continuous complex-valued function on $[a, b]$. We define $f(A)$ as in the preceding discussion, where f will then be thought of as being restricted to $Sp(A)$. Then $P(\lambda) = 0$ for $\lambda < a$ and $P(\lambda) = I$ for $\lambda > b$ and we have*

$$f(A) = \int_a^b f(\lambda) dP(\lambda)\,, \quad \textit{in particular} \quad A = \int_a^b \lambda dP(\lambda)\,,$$

where the integrals are of Riemann-Stieltjes type and exist with respect to the norm of $L(H)$.

The next result is implicit in the proof of the preceding lemma given in [5] (see also [7] p. 352). However, we give it an independent proof.

Lemma 3.2. *Let $A \in L(H)$ be a Hermitian operator and $\Delta \subset \mathbf{R} \setminus Sp(A)$ an open interval. Then the spectral function P of A is constant in Δ.*

Proof: Choose $c, d \in \Delta$ and $[a, b] \supset Sp(A)$ such that $a < c < d < b$. Then the projections $E_1 = P(c) - P(a)$, $E_2 = P(d) - P(c)$, and $E_3 = P(b) - P(d)$ are mutually orthogonal with sum equal to I. If $c \leq \lambda_0 \leq d$, then there exists $(\lambda_0 I - A)^{-1} \in L(H)$, and the invariance of the subspaces $E_i(H)$ under A implies that also $\lambda_0 \in \mathbf{R} \setminus Sp(A')$, where $A' = A \mid E_2(H)$. Next, if $\lambda_0 \notin [c, d]$, set

$$\delta = \min \{ |\lambda_0 - \xi| \mid c \leq \xi \leq d \} > 0\,.$$

Using the spectral decomposition

$$A' = \int_a^b \lambda\, dP'(\lambda)\,,$$

where $P'(\lambda) = P(\lambda) \mid E_2(H)$, it is easily seen that $\|A'x - \lambda_0 x\| \geq \delta \|x\|$ for every $x \in E_2(H)$, which implies in a well-known manner that $\lambda_0 \in \mathbf{C} \setminus Sp(A)$. It follows that $Sp(A') = \emptyset$, which is possible only if $E_2 = 0$.

3.6. Our next main target is to show that the compact elements of any *-subalgebra of $L(H)$ form an ideal. The way is via von Neumann algebras and the spectral decomposition of compact Hermitian elements.

The following fundamental result is proved in [2], p. 228.

Lemma 3.3. *Let* $\mathcal{A}$ *be a von Neumann algebra and* $E, F \in \mathcal{A}$ *projections. There exists a projection* $G \in \mathcal{A} \cap \mathcal{A}'$ *such that* $FG \prec EG$ *and* $E(I-G) \prec F(I-G)$.

Theorem 3.6. *Let* $\mathcal{A}$ *be a von Neumann algebra and* $E, F \in \mathcal{A}$ *projections. If* F *is a finite-dimensional element of* $\mathcal{A}$ *and* $E \prec F$, *then each of the mappings* $X \to EXF$ *and* $X \to FXE$ *has a finite-dimensional range.*

Proof: There is a projection $F' \in \mathcal{A}$ such that $E \sim F' \leq F$. Let $U \in \mathcal{A}$ be a partially isometric operator for which $F' = UU^*$ and

$$E = E^2 = U^*UU^*U = U^*F'FU .$$

Then each of the mappings

$$X \to EXF = U^*F'FUXF$$

and

$$X \to FXE = FXU^*F'FU$$

has a finite-dimensional range.

Theorem 3.7. *Let* $\mathfrak{U}$ *be a *-subalgebra of* $L(H)$ *and let* E *and* F *be relatively finite-dimensional projections in* $\mathfrak{U}$. *Then the mapping* $X \to EXF$ *on* $\mathfrak{U}$ *has a finite-dimensional range.*

Proof: If $\mathfrak{U}$ does not contain the identity operator I, let $\mathfrak{U}_1$ be the direct sum of $\mathfrak{U}$ and $\mathbf{C}I$. Clearly, $\mathfrak{U}_1$ is isomorphic to the algebra obtained from $\mathfrak{U}$ by the customary adjunction of identity. Thus by theorem 1.4 we may assume that $I \in \mathfrak{U}$. Let $\mathcal{A}$ be the von Neumann algebra generated by $\mathfrak{U}$, i.e. $\mathcal{A} = \mathfrak{U}''$ (cf. [2], p. 2). By corollary 1 in [2], p. 44, $\mathfrak{U} = \overline{\mathcal{A}}^1$ where "$\overline{\quad}^1$" denotes closure with respect to the weak operator topology, i.e. the locally convex Hausdorff topology generated by the seminorms $T \to |(Tx, y)|$, where $x, y \in H$. It is well known (cf. [2], p. 34 or [5], p. 441) that for fixed T the mappings $X \to TX$ and $X \to XT$ are weakly continuous. This along with the fact that a finite-dimensional subspace of any Hausdorff topological vector space is closed shows that

$$E\mathcal{A}E = E\overline{\mathfrak{U}}^1E \subset \overline{E\mathfrak{U}E}^1 = E\mathfrak{U}E .$$

Hence E (and analogously F) is a finite-dimensional element of $\mathcal{A}$. Now consider the mapping $X \to EXF$ on $\mathcal{A}$. By lemma 3.3 there is a projection $G \in \mathcal{A} \cap \mathcal{A}'$ such that $EG \prec FG$ and $F(I-G) \prec E(I-G)$. Then

$$EXF = GEXF + (I - G)\,EXF = EGXFG + E(I - G)\,XF(I - G)\,.$$

In virtue of theorem 3.6 the mappings

$$X \to EGXFG \text{ and } X \to E(I - G)\,XF(I - G)$$

on $\mathcal{A}$ have finite-dimensional ranges. It follows that the mapping $X \to EXF$ on $\mathcal{A}$ and hence also its restriction to $\mathfrak{A}$ has a finite-dimensional range.

Corollary. *Any finite linear combination of relatively finite-dimensional projections in a *-subalgebra* $\mathfrak{A}$ *of* $L(H)$ *is a finite-dimensional element of* $\mathfrak{A}$.

Theorem 3.8. *Let* $\mathfrak{A} \subset L(H)$ *be a* C^**-algebra and let the Hermitian operator* A *be a compact element of* $\mathfrak{A}$. *Then* A *can be represented as a series or finite sum of the form*

$$A = \sum_n \lambda_n E_n\,, \tag{1}$$

where each $E_n \in \mathfrak{A}$ *and* $\lambda_n \in \mathbf{R}$, *and the* E_n *are mutually orthogonal relatively finite-dimensional projections. The number of non-zero terms in* (1) *is finite if and only if* A *is a finite-dimensional element of* $\mathfrak{A}$. *If the series is infinite, it converges in norm.*

Proof: By theorem 1.6 $Sp_{\mathfrak{A}}(A)\setminus\{0\}$, which is the same as $Sp_{L(H)}(A)\setminus\{0\}$ (corollary of theorem 1.6), consists of isolated points. Let $[a\,,b]$ be a closed interval containing $Sp(A)$ and having zero as an interior point, and choose $\varepsilon > 0$ so that $\varepsilon < \min\{|a|\,,|b|\}$. Let P be the spectral function of A. It follows from lemma 3.2 that the integral

$$\int_a^{-\varepsilon} \lambda dP(\lambda) + \int_\varepsilon^b \lambda dP(\lambda)$$

degenerates into a sum $\sum_{n=1}^{p} \lambda_n E_n$, where E_n is the jump of P corresponding to the point $\lambda_n \in Sp(A)$. Since the integral

$$A = \int_a^b \lambda\, dP(\lambda)$$

exists with respect to the norm of $L(H)$, it is easily seen that as $\varepsilon \to 0$, we obtain (1). Clearly, we may assume each $\lambda_n E_n \neq 0$ (if $A = 0$, the sum is void). Next we show that $E_n \in \mathfrak{A}$. Choose a continuous function $f: \mathbf{R} \to \mathbf{C}$ such that $f(\lambda_n) = 1$ and $f(\lambda) = 0$ outside an open interval Δ with $\Delta \cap Sp(A) = \{\lambda_n\}$. Let $\mathfrak{A}_A$ be as in the discussion preceding lemma 3.1. Evidently,

$$E_n = \int_a^b f(\lambda)\, dP(\lambda)\,,$$

and on the other hand

$$\int_a^b f(\lambda) dP\,(\lambda) = f(A) \in \mathfrak{A}_A\,.$$

Thus E_n can be expressed as a limit of polynomials in A, and the equation

$$\frac{1}{\lambda_n} A E_n = E_n \tag{2}$$

along with the continuity of the mapping $T \to TE_n$ shows that these may multiplied by $\frac{1}{\lambda_n} A$ without changing the limit. Therefore E_n is a limit of polynomials in A having no constant term, which proves that $E_n \in \mathfrak{A}$. The compactness of the element E_n follows from (2) and theorem 1.1 (i). Finally, the equation

$$AE_nA = \lambda_n^2 E_n$$

combined with the fact that the projections E_n are linearly independent shows that the sum (1) must be finite, if A is a finite-dimensional element of $\mathfrak{A}$, and the converse follows from the corollary of theorem 3.7.

Theorem 3.9. *Let $\mathfrak{A}$ be a *-subalgebra of $L(H)$ and let S and T be compact (resp. finite-dimensional) elements of $\mathfrak{A}$. Then the operator $X \to SXT$ on $\mathfrak{A}$ is precompact (resp. has a finite-dimensional range).*

Proof: In virtue of theorem 1.2 we may assume that $\mathfrak{A}$ is a C^*-algebra. Since S^*S is a positively definite Hermitian operator and a compact element of $\mathfrak{A}$, its spectral decomposition degenerates into the series or finite sum

$$S^*S = \sum_n \lambda_n E_n\,,$$

where each $\lambda_n > 0$ and each E_n is a relatively finite-dimensional projection (theorem 3.8). It follows from the corollaries of theorems 3.7 and 1.7 that the operator

$$|S| = (S^*S)^{1/2} = \sum_n \sqrt{\lambda_n} E_n$$

is a compact element of $\mathfrak{A}$. Let $S = U|S|$ be the polar decomposition of S (cf. [2], p. 334). Then the operator $X \to SXS^* = U|S|X|S|U^*$ and similarly the operator $X \to S^*XS$ (cf. theorem 3.3) is precompact. Hence, if we write

$$S = \frac{1}{2}(S + S^*) + i\frac{1}{2i}(S - S^*) = A + iB ,$$

A and B are Hermitian operators and compact elements of $\mathfrak{A}$. Similar remarks hold for T. Thus there exist sequences $\{S_n\}$ and $\{T_n\}$ in $\mathfrak{A}$ converging to S and T such that the corresponding operators $X \to S_n X T_n$ on $\mathfrak{A}$ have finite-dimensional ranges (theorems 3.8 and 3.7). It then follows from theorem 1.7 that the operator $X \to SXT$ on $\mathfrak{A}$ is precompact. To prove the second statement, note that if S and T are finite-dimensional elements of $\mathfrak{A}$, all spectral decompositions involved in the above proof will be finite sums.

Corollary 1. *Let $\mathfrak{A}$ be a *-subalgebra of $L(H)$.*

(i) *Any finite linear combination of compact elements of $\mathfrak{A}$ is a compact element of $\mathfrak{A}$.*

(ii) *Any finite linear combination of finite-dimensional elements of $\mathfrak{A}$ is a finite-dimensional element of $\mathfrak{A}$.*

(iii) *If $\mathfrak{A}$ is a C^*-algebra, any compact element of $\mathfrak{A}$ is the limit of a sequence of finite-dimensional elements of $\mathfrak{A}$.*

Proof: Statements (i) and (ii) are immediate consequences of theorem 3.9 and (iii) is contained in its proof.

Corollary 2. *Every compact element T of an infinite-dimensional C^*-algebra $\mathfrak{A}$ is a (left or right) topological divisor of zero.*

Proof: Let $\{S_n\}$ be a sequence of finite-dimensional elements of $\mathfrak{A}$ such that $\lim_{n\to\infty} S_n = T$. Choosing a subsequence if necessary we may by theorem 1.5 assume that each S_n is, say, a left divisor of zero. Choose $Z_n \in \mathfrak{A}$ so that $\|Z_n\| = 1$ and $S_n Z_n = 0$. Then

$$\lim_{n\to\infty} TZ_n = \lim_{n\to\infty} (T - S_n) Z_n + \lim_{n\to\infty} S_n Z_n = 0 .$$

We summarize corollary 1 above, theorem 1.1 (i) and the corollary of theorem 1.7 in the following theorem:

Theorem 3.10. *Let $\mathfrak{A}$ be a *-subalgebra of $L(H)$. Denote by $\mathfrak{I}_c$ the set of the compact elements of $\mathfrak{A}$ and by $\mathfrak{I}_f$ the set of the finite-dimensional elements of $\mathfrak{A}$. Then $\mathfrak{I}_c$ and $\mathfrak{I}_f$ are two-sided ideals of $\mathfrak{A}$, and $\mathfrak{I}_c$ is closed. If $\mathfrak{A}$ is a C^*-algebra, $\mathfrak{I}_c$ is the closure of $\mathfrak{I}_f$.*

3.7. Next we apply the preceding theory to the spectral decomposition of compact normal elements of C^*-algebras. We show how the problem can be reduced to the Hermitian case without appealing to the general spectral theory of normal operators.

Theorem 3.11. *Let $\mathfrak{A} \subset L(H)$ be a C^*-algebra and let the normal operator T be a compact element of $\mathfrak{A}$. Then*

$$Sp_{L(H)}(T) \setminus \{0\} = Sp_{\mathfrak{A}}(T) \setminus \{0\}$$

consists of a countable number of eigenvalues $\lambda_1, \lambda_2, \ldots$. Furthermore, T can be represented as a series or finite sum of the form

$$T = \sum_n \lambda_n E_n \,, \tag{1}$$

where each non-zero eigenvalue λ_n of T occurs precisely once, E_n is the projection onto the eigenspace corresponding to the eigenvalue λ_n, the E_n are mutually orthogonal, and every E_n is a finite-dimensional element of $\mathfrak{A}$. If the series is infinite, it converges in norm, and $\lim_{n\to\infty} \lambda_n = 0$. The sum is finite if and only if T is a finite-dimensional element of $\mathfrak{A}$.

Proof: Write

$$T = \frac{1}{2}(T + T^*) + i\,\frac{1}{2i}(T - T^*) = A + iB \,,$$

where A and B are Hermitian and $AB = BA$. It was noted in the proof of theorem 3.9 that A and B are compact elements of $\mathfrak{A}$ (formally this follows from the statements of theorems 3.10 and 3.3). In virtue of theorem 3.8 we can write $A = \sum_j \xi_j F_j$. Setting $\xi_0 = 0$ and denoting by F_0 the projection onto the orthogonal complement of the Hilbert sum

$$\sum_{j \geq 1} \oplus\, F_j(H) = \overline{A(H)} \,,$$

we have $A = \sum_{j \geq 0} \xi_j F_j$, where H is the Hilbert sum of the subspaces $F_j(H)$. Similarly, $B = \sum_{k \geq 0} \eta_k G_k$. It is clear from lemma 3.1 3° and the definitions of F_0 and G_0 that $F_j G_k = G_k F_j$ for all $j, k \geq 0$, which means that each $F_j G_k$ is a projection. For $j \geq 1$, F_j is a finite-dimensional element of $\mathfrak{U}$, and since the G_k are mutually orthogonal, $F_j G_k = F_j G_k F_j \neq 0$ only for a finite number of indices $k_1^j, \ldots, k_{n_j}^j$. We get two sequences of projections

$$(2) \qquad F_0 G_0\,,\; F_0 G_1\,,\; F_0 G_2\,, \ldots$$

and

$$(3) \quad F_1 G_{k_1^1}\,,\; F_1 G_{k_2^1}\,, \ldots, F_1 G_{k_{n_1}^1}\,,\; F_2 G_{k_1^2}\,, \ldots, F_2 G_{k_{n_2}^2}\,,\; F_3 G_{k_1^3}\,, \ldots .$$

Since for $k \geq 1$ G_k is a finite-dimensional element of $\mathfrak{U}$, it can occur only a finite number of times in (3). Choosing alternately an element from each sequence we get the sequence

$$(4) \qquad E_0\,,\; E_1\,, \ldots, E_n\,, \ldots$$

of mutually orthogonal projections, which are finite-dimensional elements of $\mathfrak{U}$ for $n \geq 1$. If $E_n = F_j G_k$, we set $\lambda_n = \xi_j + i\eta_k$. Then

$$T = \sum_{k \geq 1} \lambda_n E_n$$

in the sense of the norm topology. This follows from a straightforward convergence argument based on the fact that the projections in (4) eventually build up any F_j or G_k, $j, k \geq 1$, and that for sets of pairwise orthogonal projections F_j and $F_j' \leq F_j$, $j = n, \ldots, n+p$,

$$\| \sum_{j=n}^{n+p} \xi_j F_j' \| \leq \| \sum_{j=n}^{n+p} \xi_j F_j \| .$$

Clearly, the λ_n can be assumed distinct and non-zero with the understanding that for $T = 0$ the sum will be void. Since

$$H = \sum_{n \geq 0} \oplus\; E_n(H)\,,$$

every $x \in H$ can be written as

$$x = \sum_{n \geq 0} E_n x\,.$$

If $\lambda_p x - Tx = 0$ we have

$$E_n(\lambda_p x - Tx) = (\lambda_n - \lambda_p)\, E_n x = 0\,,$$

so that $E_n x = 0$ for $n \neq p$. Hence $x = E_p x$ or $x \in E_p(H)$. Conversely, each $\lambda_n \neq 0$ is an eigenvalue of T such that the corresponding eigenspace contains $E_n(H)$. By the corollary of theorem 1.6 or by proposition 1.3.10 in [3],

$$Sp_{L(H)}(T)\setminus\{0\} = Sp_{\mathfrak{A}}(T)\setminus\{0\}\,,$$

and in virtue of theorem 1.6, or directly by the convergence in norm of the series (1), $\lim_{n\to\infty} \lambda_n = 0$ if the number of eigenvalues is infinite. Next we show that

$$Sp_{L(H)}(T) \subset \{\lambda_n \mid n \geq 0\}\,.$$

To this end, suppose $\lambda \neq \lambda_n$ for every $n \geq 0$ and set $\delta = \min_{n \geq 0} |\lambda - \lambda_n| > 0$ Then

$$\|\lambda x - Tx\|^2 = \sum_{n \geq 0} \|\lambda E_n x - \lambda_n E_n x\|^2 \geq \sum_{n \geq 0} \delta^2 \|E_n x\|^2 = \delta^2 \|x\|^2\,.$$

This inequality combined with the normality of T implies that $\lambda I - T$ has a bounded inverse. Thus $\lambda \notin Sp_{L(H)}(T)$. Finally, once the representation (1) is established, the last statement follows as in the proof of theorem 3.8.

4. Decomposition of relatively finite-dimensional projections

Rickart [6] calls an idempotent e in an arbitrary algebra $\mathfrak{A}$ minimal, if $e\mathfrak{A}e$ is a division algebra. We shall, however, adopt a different usage and call a projection E in a subalgebra $\mathfrak{A}$ of $L(H)$ *minimal*, if $E \neq 0$ and $\mathfrak{A}$ contains no non-zero projection $F \lneqq E$. If E is a projection in a subalgebra $\mathfrak{A}$ of $L(H)$ and $\dim (E\mathfrak{A}E) = 1$, we say that E is *relatively 1-dimensional.*

Obviously, a relatively 1-dimensional projection is minimal. The converse is true of every von Neumann algebra $\mathcal{A}$. For if E is a minimal projection in $\mathcal{A}$, E_E and 0 are the only projections in $\mathcal{A}_E$, and since a von Neumann algebra is generated by its projections ([2] p. 4), $\mathcal{A}_E$ consists of the scalar multiples of E_E. Since $\mathcal{A}_E$ is as a vector space isomorphic to $E\mathcal{A}E$ (see the proof of theorem 3.1), we have also $\dim (E\mathcal{A}E) = 1$.

In a general C^*-algebra, however, a minimal projection need not even be relatively finite-dimensional. This can be seen e.g. by considering the commutative C^*-algebra $\mathfrak{A}$ of all continuous complex-valued functions on the interval [0, 1] (viewed as a Hilbert space operator algebra, if so desired, cf. section 3.1). The only projections (i.e. Hermitian idempotents) in $\mathfrak{A}$ are the functions identically 1 or 0. The former is a minimal projection, but not a finite-dimensional element of $\mathfrak{A}$.

Theorem 4.1. *If $E \neq 0$ is a relatively finite-dimensional projection in the C^*-algebra $\mathfrak{A}$, there exists a relatively 1-dimensional projection F in $\mathfrak{A}$.*

Proof: Since the dimension of $E\mathfrak{A}E$ is a positive integer and $E \in E\mathfrak{A}E$, there is a non-zero projection $F \in E\mathfrak{A}E$ such that $\dim(F\mathfrak{A}F)$ is minimal. Suppose $F' \neq 0$ is a projection in $F\mathfrak{A}F$, $F' \neq F$. Then

$$\{0\} \neq F'\mathfrak{A}F' \subset F\mathfrak{A}F \subset E\mathfrak{A}E .$$

Since $F' = F'F$, but $F' \neq F$, we cannot have $F \in F'\mathfrak{A}F'$. Thus $\dim(F'\mathfrak{A}F') < \dim(F\mathfrak{A}F)$, contradicting the minimality of $\dim(F\mathfrak{A}F)$. It follows that F is the only non-zero projection in $F\mathfrak{A}F$. Clearly, $F\mathfrak{A}F$ is a C^*-algebra consisting of finite-dimensional elements, each of which can be expressed as a linear combination of projections belonging to $F\mathfrak{A}F$ (theorems 3.3, 3.10 and 3.8). Therefore $\dim(F\mathfrak{A}F) = \dim(\mathbf{C}F) = 1$.

Theorem 4.1 combined with theorems 1.1 (i) and 3.8 yields the following result:

Corollary. *If the C^*-algebra $\mathfrak{A}$ contains a non-zero compact element, $\mathfrak{A}$ contains a relatively 1-dimensional projection.*

Theorem 4.2. *If $E \neq 0$ is a relatively finite-dimensional projection in the C^*-algebra $\mathfrak{A}$, E is the sum of a finite number of pairwise orthogonal relatively 1-dimensional projections of $\mathfrak{A}$.*

Proof: By theorem 4.1 there exists at least one relatively 1-dimensional projection $F \in E\mathfrak{A}E$. Since $FE = EF = F$, we have $F \leq E$. The number of pairwise orthogonal relatively 1-dimensional projections $F_i \in E\mathfrak{A}E$ (equivalently, $F_i \leq E$) is bounded by $\dim(E\mathfrak{A}E)$. Let $(F_i)_{1 \leq i \leq n}$ be a family of such projections, chosen so that n is maximal. Since

$$E' = E - \sum_{i=1}^{n} F_i$$

is a relatively finite-dimensional projection orthogonal to all the F_i and $E'\mathfrak{A}E' \subset E\mathfrak{A}E$, theorem 4.1 implies that $E' = 0$.

Remark. Using the above theorem we could decompose the relatively finite-dimensional projections appearing in the statements of theorems 3.8 and 3.11 into sums of pairwise orthogonal relatively 1-dimensional projections thus arriving at spectral representations even more closely reminiscent of the classical case of compact operators.

4.2. A von Neumann algebra $\mathcal{A}$ is called *discrete*, if it is isomorphic to a von Neumann algebra $\mathfrak{B}$ such that $\mathfrak{B}'$ is commutative. If $\mathcal{A}_E$ is not discrete for any non-zero projection $E \in \mathcal{A} \cap \mathcal{A}'$, $\mathcal{A}$ is called *continuous.*

Any von Neumann algebra is canonically isomorphic to the product of a discrete and a continuous von Neumann algebra (cf. [2] p. 122). It follows from corollary 3 in [2], p. 229, that no continuous von Neumann algebra contains a minimal projection. This fact together with theorem 3.2 and the corollary of theorem 4.1 yields the following result:

Theorem 4.3. *Let* $\mathcal{A}$ *be isomorphic to* $\mathcal{A}_1 \times \mathcal{A}_2$, *where* $\mathcal{A}_1$ *is a discrete and* $\mathcal{A}_2$ *a continuous von Neumann algebra. Let*

$$(T_i)_{i=1,2} \in \mathcal{A}_1 \times \mathcal{A}_2$$

correspond to the compact element T *of* $\mathcal{A}$. *Then* $T_2 = 0$. *In particular, no continuous von Neumann algebra contains a non-zero compact element.*

5. Characterization of the socle of a C^*-algebra

In an arbitrary algebra $\mathfrak{A}$ the sum of the minimal left (right) ideals is called the *left* (*right*) *socle* of $\mathfrak{A}$. If $\mathfrak{A}$ contains no minimal left (right) ideals, it is natural to define the left (right) socle of $\mathfrak{A}$ to equal $\{0\}$. (Note, however, that in Rickart's terminology, [6] p. 46, the corresponding socle in this case fails to exist.) If the left socle is equal to the right socle, it is called simply the *socle* of $\mathfrak{A}$. Lemma 2.1.12 in [6] combined with the next lemma shows that the socle of a C^*-algebra is always defined (possibly equal to $\{0\}$).

Lemma 5.1. *If* $\mathfrak{J} \neq \{0\}$ *is a left or right ideal of a* C^**-algebra, then* $\mathfrak{J}^2 \neq \{0\}$.

Proof: We give the proof for a left ideal. If $T \in \mathfrak{J} \setminus \{0\}$, then $T^*T \in \mathfrak{J}$ and $\|(T^*T)(T^*T)\| = \|(T^*T)^*(T^*T)\| = \|T\|^4 \neq 0$.

Theorem 5.1. *The socle of a* C^**-algebra* $\mathfrak{A}$ *coincides with the set of the finite-dimensional elements of* $\mathfrak{A}$.

Proof: Any minimal left ideal of $\mathfrak{A}$ has the form $\mathfrak{A}A$, where $\dim (A\mathfrak{A}A) = 1$ (cf. lemma 5.1 above, lemma 2.1.5 and its corollary in [6]). Thus every element in the socle of $\mathfrak{A}$ is a finite sum of finite-dimensional elements, hence a finite-dimensional element of $\mathfrak{A}$ (theorem 3.10). Conversely, every non-zero finite-dimensional element T of $\mathfrak{A}$ (and trivially 0) belongs to the socle of $\mathfrak{A}$, for it follows from theorems 3.3, 3.10, 3.8 and 4.2 that T can be expressed as a finite linear combination of relatively 1-dimensional projections, and any such projection E belongs to the minimal left ideal $\mathfrak{U}E$ of $\mathfrak{U}$ (cf. lemma 5.1 above and corollary 2.1.9 in [6]).

6. Compact elements and irreducible representations of C^*-algebras

6.1. In connection with C^*-algebras we mean by a *homomorphism* a mapping that preserves the $*$-algebra structure.

Theorem 6.1. *Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be C^*-algebras and $\Pi : \mathfrak{U}_1 \to \mathfrak{U}_2$ a homomorphism. If u is a compact element of $\mathfrak{U}_1$, $\Pi(u)$ is a compact element of $\Pi(\mathfrak{U}_1)$. The corresponding result holds for finite-dimensional elements.*

Proof: The algebra $\mathfrak{U}_0 = \Pi(\mathfrak{U}_1)$ is closed in $\mathfrak{U}_2$ (cf. [3] corollary 1.8.3, p. 18), hence complete. If B_1 denotes the closed unit ball of $\mathfrak{U}_1$, it follows from the open mapping theorem that there is an $r > 0$ such that

$$\Pi(B_1) \supset B'_r = \{x \in \mathfrak{U}_0 \mid \|x\| \leq r\} .$$

Since Π is continuous (cf. [3], p. 7), the set

$$\Pi(u)\, B'_1 \Pi(u) = \frac{1}{r}\, \Pi(u)\, B'_r \Pi(u) \subset \frac{1}{r}\, \Pi(u B_1 u)$$

is precompact. Hence $\Pi(u)$ is a compact element of $\mathfrak{U}_0$. The assertion concerning finite-dimensional elements is immediate.

6.2. Let $\mathfrak{U}$ be a C^*-algebra and Π a *representation* of $\mathfrak{U}$ on the Hilbert space H' (i.e. a homomorphism from $\mathfrak{U}$ into $L(H')$). The representation Π is said to be (*strictly*, resp. *topologically*) *irreducible*, if the algebra $\Pi(\mathfrak{U})$ is (strictly, resp. topologically) irreducible. By corollary 2.8.4 in [3], p. 45, the strict and topological irreducibility of $\mathfrak{U}$ are equivalent, so that we may without ambiguity speak simply of *irreducible representations* of C^*-algebras.

Theorem 6.2. *Let $\mathfrak{U}$ be a C^*-algebra, u a compact (resp. finite-dimensional) element of $\mathfrak{U}$, and Π an irreducible representation of $\mathfrak{U}$ on the Hilbert space H'. Then $\Pi(u)$ is a compact operator (resp. an operator with finite-dimensional range) on H'.*

Proof: By theorem 6.1 and the above remark $\Pi(u)$ is a compact (resp. finite-dimensional) element of a strictly irreducible operator algebra, and hence a compact operator by corollary 2 of theorem 2.2 (resp. an operator with finite-dimensional range by theorem 2.3 (ii)).

Theorem 6.3. *Let $\mathfrak{A}$ be a C^*-algebra and $u \neq 0$ a compact element of $\mathfrak{A}$. There exists a Hilbert space $H' \neq \{0\}$ and an irreducible representation Π of $\mathfrak{A}$ on H' such that Π maps the ideal $\mathfrak{I}_c$ of the compact elements of $\mathfrak{A}$ onto the ideal $LC(H')$ of the compact operators on H'.*

Proof: It follows from proposition 2.7.1 in [3] that there exists a Hilbert space H' and an irreducible representation $\Pi : \mathfrak{A} \to L(H')$ such that $\Pi(u) \neq 0$. By theorem 6.2, $\Pi(\mathfrak{I}_c) \subset LC(H')$. On the other hand, by the same theorem and corollary 4.1.10 in [3], $LC(H') \subset \Pi(\mathfrak{A})$. Since $\Pi(\mathfrak{I}_c)$ is a non-zero closed two-sided ideal of $\Pi(\mathfrak{A})$ ([3], corollary 1.8.3), and hence of $LC(H')$, corollary 4.1.7 in [3] shows that $\Pi(\mathfrak{I}_c) = LC(H')$.

Remark. It follows from theorem 6.2 that if every element of a C^*-algebra $\mathfrak{A}$ is compact, $\mathfrak{A}$ is a so-called CCR-algebra (»C^*-algèbre liminaire» in the terminology of Dixmier, cf. [3], p. 86). In any C^*-algebra $\mathfrak{A}$ the ideal of the compact elements of $\mathfrak{A}$ is contained in the maximal closed two-sided CCR-ideal of $\mathfrak{A}$ (cf. [3], proposition 4.2.6). In general this inclusion is proper, for example always in the case of an infinite-dimensional commutative C^*-algebra with identity (cf. [6] lemma 2.4.4).

7. Compact and finite-dimensional elements in factors

7.1. A von Neumann algebra $\mathcal{A} \subset L(H)$ is called a *factor*, if $\mathcal{A} \cap \mathcal{A}' = \{\lambda I \mid \lambda \in \mathbf{C}\}$. Since for a factor $\mathcal{A}$ the only projections in $\mathcal{A} \cap \mathcal{A}'$ are 0 and I, lemma 3.3 shows that for any pair of projections $E, F \in \mathcal{A}$, either $E \prec F$ or $F \prec E$. The following analogue of the Euclidean algorithm is a consequence of corollary 2 in [2], p. 228. It is given explicitly in [5], p. 456.

Lemma 7.1. *Let $\mathcal{A}$ be a factor and $E, F \in \mathcal{A}$ projections such that $E \prec F$. Then*

$$F = \sum_{i \in \mathcal{J}} E_i + E_0 ,$$

where $E_0 \prec E$, each $E_i \sim E$, and all projections on the right are mutually orthogonal.

The next lemma is proved in [5], p. 460.

Lemma 7.2. *Let $\mathcal{A}$ be a factor and $E, F \in \mathcal{A}$ projections such that E is finite and $0 \neq E \prec F$. The index set $\mathcal{J}$ in the preceding lemma is finite if and only if F is a finite projection.*

7.2. For non-zero elements of a factor the converse of theorem 3.9 is valid (theorem 7.2). We first prove an auxiliary result.

Theorem 7.1. *Let E and F be non-zero projections in the factor $\mathcal{A}$. If the operator $X \to EXF$ on $\mathcal{A}$ is precompact, E and F are finite-dimensional elements of $\mathcal{A}$.*

Proof: Suppose first $E \prec F$, which means that $E \sim F_1 \leq F$ for a projection $F_1 \in \mathcal{A}$. There is a partial isometry $U \in \mathcal{A}$ such that $F_1 = UU^*$ and

$$E = U^*UU^*U = U^*F_1FU .$$

Hence the operator $X \to EXE$ is precompact, and E is a finite-dimensional element of $\mathcal{A}$. By lemma 7.1 F may be written as the orthogonal sum

$$F = \sum_{i \in \mathcal{J}} E_i + E_0 ,$$

where $E_i \sim E$ and $E_0 \prec E$. Let $U_i \in \mathcal{A}$ be a partial isometry having E_i as initial and E as final projection. Every finite subset of the set $\{EU_iF \mid i \in \mathcal{J}\}$ is linearly independent. For if

$$\sum_{k=1}^{n} \lambda_k EU_kF = 0 , \tag{1}$$

it follows by successive applications of the operator (1) to non-zero vectors from the subspaces $E_k(H)$ that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Since the precompact operator $X \to EXF$ maps the unit ball of the subspace $E\mathcal{A}F$ onto itself, $E\mathcal{A}F$ must be finite-dimensional. By theorem 3.4 E_0 and all the E_i are relatively finite-dimensional projections, and since the above argument shows that the index set $\mathcal{J}$ is finite, F is in virtue of the corollary of theorem 3.7 a finite-dimensional element of $\mathcal{A}$. In the case $F \prec E$ the conclusion follows from the above proof and the precompactness of the operator $X \to X^* \to EX^*F \to FXE$.

Theorem 7.2. *Let S and T be non-zero elements of the factor $\mathcal{A}$. If the operator $X \to SXT$ on $\mathcal{A}$ is precompact (resp. has a finite-dimensional range), S and T are compact (resp. finite-dimensional) elements of $\mathcal{A}$.*

Proof: Suppose first that S and T are Hermitian. Since $S \neq 0$, there is a closed interval $[a, b]$, not containing zero, such that $E_1 = P_1(b) - P_1(a) \neq 0$ for the spectral function P_1 of S. Let P_2 be the spectral function of T, choose an arbitrary closed interval $[c, d]$ not containing zero, and denote $E_2 = P_2(d) - P_2(c)$, $P_2'(\lambda) = P_2(\lambda) \mid E_2(H)$, $T' = T \mid E_2(H)$. If $\delta = \min\{|c|, |d|\}$, an application of the spectral decomposition

$$T' = \int_c^d \lambda \, dP'(\lambda)$$

shows that $\|T'x\| \geq \delta \|x\|$ for all $x \in E_2(H)$. Since $\delta > 0$, this implies that T' has a bounded inverse. Since $T' \in \mathcal{A}_{E_2}$ and $(T')^{-1}$ commutes with every operator on $E_2(H)$ that commutes with T',

$$(T')^{-1} \in (\mathcal{A}_{E_2})'' = \mathcal{A}_{E_2} .$$

Thus there exists $T'' \in \mathcal{A}$ such that $(T')^{-1} = T''_{E_2}$. Therefore $E_2 = TE_2T''E_2$, and similarly we can find $S'' \in \mathcal{A}$ such that $E_1 = SE_1S''E_1$. Hence the operator

$$X \to E_1XE_2 = SE_1S''E_1XTE_2T''E_2$$

is precompact, and it follows from theorem 7.1 that E_2 is a finite-dimensional element of $\mathcal{A}$.

Given $\varepsilon > 0$, we can choose a Riemann—Stieltjes sum Σ approximating T such that $\|T - \Sigma\| < \frac{\varepsilon}{2}$, and that in the corresponding partition the lengths of the intervals adjacent to zero are less that $\frac{\varepsilon}{4}$. If Σ' is the sum obtained from Σ by discarding the terms corresponding to these intervals, $\|T - \Sigma'\| < \varepsilon$, and the foregoing argument shows that the projections appearing in Σ' are finite-dimensional elements of $\mathcal{A}$. Thus it follows from theorem 3.10 that T is a compact element of $\mathcal{A}$. Let

$$(1) \qquad T = \sum_n \lambda_n F_n$$

be the spectral decomposition T (cf. theorem 3.8). Since $E_1 \prec F_n$ or $F_n \prec E_1$, to each non-zero F_n corresponds a non-zero partial isometry $U_n \in \mathcal{A}$ whose initial projection is dominated by F_n and final projection by E_1. Since the F_n are mutually orthogonal, it is readily seen that the operators $E_1U_nF_n$ are linearly independent. Therefore, if the operator $X \to SXT$ and hence the operator

$$X \to E_1 XT = SE_1 S'' E_1 XT$$

has a finite-dimensional range, the number of non-zero terms in (1) must be finite, as otherwise the set consisting of the operators $\frac{1}{\lambda_n} U_n$ would be mapped onto an infinite linearly independent set. Since the operator

$$X \to X^* \to SX^* T \to T^* XS^* = TXS$$

is precompact (resp. has a finite-dimensional range), the above proof shows that S is also a compact (resp. finite-dimensional) element of $\mathcal{A}$. In the general case the operator $X \to S^* SXT^* T$ is precompact (resp. has a finite-dimensional range) and $S^* S \neq 0$, $T^* T \neq 0$ are Hermitian. It was noted in the proof of theorem 3.9 that $|S|$ and $|T|$ are compact (resp. finite-dimensional) elements of $\mathcal{A}$, if $S^* S$ and $T^* T$ are so. Since in the polar decompositions $S = U|S|$ and $T = V|T|$ the partial isometries U and V belong to $\mathcal{A}$ (cf. [2] p. 5), the conclusion follows from theorem 3.10.

Remark. Theorem 7.2 cannot be extended to an arbitrary von Neumann algebra. To see this, let H be an infinite-dimensional Hilbert space and $\mathcal{M} \subset L(H)$ an infinite set of mutually orthogonal nonzero projections, one of which, say E, has a finite-dimensional range. Then $\mathcal{A} = \mathcal{M}''$ is a commutative von Neumann algebra. The operator $X \to IXE = EXE$ on $\mathcal{A}$ has a finite-dimensional range, but I is not a compact element of the infinite-dimensional algebra $\mathcal{A}$.

7.3. An argument similar to the proof of theorem 3.4 yields

Theorem 7.3. *Let $\mathcal{J}$ be a two-sided ideal in the *-subalgebra $\mathcal{U}$ of $L(H)$. If E and F are projections in $\mathcal{U}$, $F \in \mathcal{J}$, and $E \prec F$, then $E \in \mathcal{J}$.*

Theorem 7.4. *If $\mathcal{A}$ is a factor, the ideal $\mathcal{J}_f$ of the finite-dimensional elements of $\mathcal{A}$ is contained in every two-sided ideal of $\mathcal{A}$.*

Proof: Let $\mathcal{J}$ be a two-sided ideal of $\mathcal{A}$ and $0 \neq T \in \mathcal{J}$. Since $T^* T \neq 0$, there is a closed interval $[c, d]$ not containing zero, such that $E = P(d) - P(c) \neq 0$ for the spectral function P of $T^* T$. As in the proof of theorem 7.2 we can find an operator $B \in \mathcal{A}$ such that $E = T^* TEBE$. Thus $E \in \mathcal{J}$. Let $F \in \mathcal{J}_f$ be a non-zero projection. Since $E \prec F$ or $F \prec E$, theorem 7.3 shows that $E \in \mathcal{J}_f$ or $F \in \mathcal{J}$. In any case, $\mathcal{J}$ contains a non-zero relatively finite-dimensional projection. Since every relatively finite-dimensional projection is finite by theorem 3.5, it follows from lemma 7.2 and theorem 7.3 that $\mathcal{J}$ contains every relatively finite-dimensional projection. Hence in virtue of theorem 3.8 we have $\mathcal{J}_f \subset \mathcal{J}$.

Combined with theorem 3.10 the above theorem gives

Corollary. *The ideal of the compact elements of the factor* $\mathcal{A}$ *is contained in every closed two-sided ideal of* $\mathcal{A}$.

7.4. It is well known that the operators with finite-dimensional range on a Hilbert space H form a minimal and the compact operators on H a minimal closed two-sided ideal of $L(H)$. Since $L(H)$ is a factor, theorem 7.4 and its corollary show that the former ideal coincides with the ideal of the finite-dimensional elements of $L(H)$ and the latter with the ideal of the compact elements of $L(H)$. Thus theorems 3.9 and 7.2 yield independently of Ascoli's theorem the following special case of theorem 3 in [8] (resp. of theorem 2.3):

Theorem 7.5. *Let* S *and* T *be non-zero operators on* H. *The operator* $X \to SXT$ *on* $L(H)$ *is compact* (*resp. has a finite-dimensional range*) *if and only if* S *and* T *are compact operators* (*resp. have finite-dimensional ranges*).

Remark. A factor is either discrete or continuous, and every discrete factor is isometrically isomorphic to the full operator algebra on some Hilbert space (cf. [2], pp. 121 and 8). Hence, in virtue of theorem 4.3, the existence of non-zero compact elements in a factor $\mathcal{A}$ implies the existence of a Hilbert space H' such that $\mathcal{A}$ is isometrically isomorphic to $L(H')$. Thus, if we take for granted the fact that the sets of the compact elements of $L(H')$ and the compact operators on H' coincide, the study of the compact elements of factors is reduced to the classical theory of compact operators on a Hilbert space. In particular, the corollary of theorem 7.4 is a consequence of this theory.

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