ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

425

ON HARMONIC QUASICONFORMAL MAPPINGS

BY

O. MARTIO

HELSINKI 1968
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1969.425

Communicated 9 April 1968 by K. I. VIRTANEN and LAURI MYRBERG

KESKUSKIRJAPAINO
HELSINKI 1968

On harmonic quasiconformal mappings

1. Introduction. A plane harmonic mapping

$$w(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)e^{if(x)}}{1-2r\cos(x-\varphi)+r^2}\, dx \tag{1}$$

is known (cf. [1]) to be a homeomorphism from the closed unit disk onto itself if $f:[0\,,2\pi]\to[0\,,2\pi]$ is a homeomorphism. In this paper the problem studied is: Under which conditions on f is the mapping w K-quasiconformal?

E. Heinz [2] has obtained estimates of type

$$J(w)(re^{i\varphi}) \geq C(r) > 0$$

with $C(r)\to 0$ when $r\to 1$ for the Jacobian $J(w)$ of the mapping w. In 2. some conditions are given for the strictly positiviness of the constant C as well for the boundedness of the Jacobian.

The results obtained can be generalized (cf. [2], [5]) for the situation

$$L_1(u) = 0$$
$$L_2(v) = 0\,,$$

$w = u + iv$, where L_1 and L_2 are certain second order elliptic differential operators.

2. Lemmas. Let D be the class of all absolutely continuous homeomorphism $f:\boldsymbol{R}\to\boldsymbol{R}$ such that $f(0)=0$, $f(2\pi)=2\pi$, f' is 2π periodic a.e. and

$$\int_0^{2\pi} e^{if(x)}dx = 0$$

(i.e. $w(0)=0$). The mapping $f\in D$ is said to belong to D^k if $\operatorname{ess\,sup} f' \leq k$ and to D_k if $\operatorname{ess\,inf} f' \geq 1/k$. By the set $D^k(M)$ we denote the class of all continuously differentiable functions $f\in D^k$ so that for the modulus of continuity of their derivative

$$\omega_{f'}(\Delta) = \sup_{|x-y| \leq \Delta} |f'(x) - f'(y)|$$

the inequality

$$\int_0^{2\pi} \frac{\omega_{f'}(\Delta)}{\Delta} \, d\Delta \leq M$$

holds. It may be noted that every Hölder-continuously differentiable $f \in D$ is in $D^k(M)$ for some k and M.

In the following, the meaning of w is always the harmonic mapping (1) with boundary values f, B the open unit disk and ∂B its boundary. The function $P(r, x)$ denotes the Poisson kernel

$$\frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(x)} .$$

Lemma 1. *If* $f \in D^k(M)$ *then*

$$\pi^{-2} \leq |w_z|^2 + |w_{\bar{z}}|^2 \leq C_1(M + k^2)^2 , \tag{2}$$

$C_1 =$ *constant.*

Proof: The left side of the inequality (2) has been proved by E. Heinz [3] (cf. also [6]).

Because w_z and $w_{\bar{z}}$ are analytic functions of z and $\bar{z}$ it suffices to prove

$$\limsup_{z \to \partial B} |w_z|^2 + \limsup_{z \to \partial B} |w_{\bar{z}}|^2 \leq \limsup_{z \to \partial B} (|w_\varphi|^2 + |w_r|^2) \leq C_1(M + k^2)^2$$

where $z = re^{i\varphi}$. By partial integration it follows from (1) that

$$w_\varphi(z) = \int_0^{2\pi} if'(x)\, e^{if(x)}\, P(r, x - \varphi)\, dx \tag{3}$$

and

$$\begin{aligned} w_r(z) &= - \frac{2i}{1 - r^2} \int_0^{2\pi} f'(x) e^{if(x)} \sin(x - \varphi)\, P(r, x - \varphi)\, dx \\ &= \frac{2i}{1 - r^2} \int_0^{\pi} \frac{F(x, \varphi)}{x}\, x \sin(x)\, P(r, x)\, dx \end{aligned} \tag{4}$$

where $F(x, \varphi) = \frac{\partial}{\partial x} (e^{if(\varphi + x)} + e^{if(\varphi - x)})$. We have

$$1 + r^2 - 2r\cos(x) \geq (x/\pi)^2\,, \quad x\sin(x) \leq x^2$$

if $0 \leq x \leq \pi$, and

$$|F(x\,,\varphi)| \leq |f'(\varphi + x) - f'(\varphi - x)| + 4x \sup_{x\in R} |f'(x)|^2 \leq \omega_{f'}(2x) + 4xk^2\,.$$

From (4) we get

$$\limsup_{z\to\partial B} |w_r(z)| \leq \pi \int_0^{2\pi} \frac{\omega_{f'}(t)}{t}\, dt + 4(\pi k)^2 \leq (2\pi)^2\,(M + k^2)$$

and from (3)

$$\limsup_{z\to\partial B} |w_\varphi(z)| \leq k\,.$$

Hence $|w_z|^2 + |w_{\bar z}|^2 \leq C_1(M + k^2)^2$ and the lemma is proved.

Although it is not true that $|w_z|^2 + |w_{\bar z}|^2$ is bounded if $f \in D^k$ (see 3.) this fact holds for the Jacobian $J(w)$ of the mapping w.

Lemma 2. *If $f \in D^k$ then there exists a constant $C = C(k)$ such that $J(w)(z) \leq C$.*

Proof: If the formulas (3) and (4) are applied to the real part u and to the imaginary part v of the mapping w we get

$$J(w)(z) = \frac{1}{r}\,(u_r v_\varphi - u_\varphi v_r) = \frac{2}{r(1 - r^2)} \times$$

$$\int_0^{2\pi}\int_0^{2\pi} f'(x)\,f'(y)\sin(f(x) - f(y))\sin(x - \varphi)\,P(r\,, x - \varphi)\,P(r\,, y - \varphi)\,dxdy\,.$$

Let $t = 1 - r$ then

$$P(r\,, x) \leq C'\,\frac{t}{t^2 + x^2}\,, \qquad C' = \text{const.},$$

and so

$$\sup_{z\in B} J(w)(z) \leq \sup_{0<t\leq 1}\,(2k)^3\,C'^2 \int_0^{\pi}\int_0^{\pi} \frac{tx|x - y|}{(t^2 + x^2)(t^2 + y^2)}\,dxdy\,.$$

The simple estimation shows

$$\sup_{z\in B} J(w)(z) \leq C'' \sup_{0<t\leq 1} \int_0^{\pi}\int_0^{\pi} \frac{tx^2 + txy}{(t^2 + x^2)(t^2 + y^2)}\,dxdy$$

$$= C'' \sup_{0<t\leq 1} \left[\pi \ \overline{\text{arc}} \ \text{tg}\left(\frac{\pi}{t}\right) + \frac{t}{4}\left(\log \frac{\pi^2 + t^2}{t^2}\right)^2 \right] \leq C(k)$$

which proves the lemma.

Lemma 3. *Suppose that* f *belongs to* $D^p(M)$. *If* $f \in D_k$ *then*

$$\liminf_{z \to \partial B} J(w)(z) \geq (2k)^{-1}\,, \tag{5}$$

and if f *does not belong to any* D_k *then*

$$\liminf_{z \to \partial B} J(w)(z) = 0\,. \tag{6}$$

Proof: According to Kellogg's theorem [4] (cf. also [7]) $\lim_{z \to \xi} w_r(z)$ and $\lim_{z \to \xi} w_\varphi(z)$, $\xi \in \partial B$, $z \in B$, exist therefore

$$\liminf_{z \to \partial B} J(w)(z) = \inf_{\varphi} \lim_{r \to 1} \frac{1}{r}\,(u_r v_\varphi - u_\varphi v_r)$$

$$= \inf_{\varphi} \lim_{r \to 1} \frac{f'(\varphi)\cos f(\varphi)(\cos f(\varphi) - u(re^{i\varphi})) + f'(\varphi)\sin f(\varphi)(\sin f(\varphi) - v(re^{i\varphi}))}{1 - r}\,.$$

If the Poisson-representation is used for u and v, the above expression takes the form

$$\inf_{\varphi} \lim_{r \to 1} \frac{f'(\varphi)}{1 - r} \int_0^{2\pi} (1 - \cos(f(x) - f(\varphi)))\, P(r\,, x - \varphi)\, dx\,.$$

From the above formula we conclude that (6) holds if f does not belong to any D_k. On the other hand, the integrand is not negative and if $f \in D_k$ then $f' \geq 1/k$ and $P(r\,, x)(1 - r)^{-1} \geq (4\pi)^{-1}$ therefore

$$\liminf_{z \to \partial B} J(w)(z) \geq \frac{1}{4\pi k} \int_0^{2\pi} (1 - \cos(f(x) - f(\varphi)))\, dx = (2k)^{-1}$$

since

$$\int_0^{2\pi} e^{if(x)}\, dx = 0\,.$$

Thus the lemma is proved.

3. Results. The condition for the K-quasiconformality of the harmonic mapping w is

Theorem 1. *If* $f \in D^p(M)$ *then* w *is a quasiconformal mapping from the closed unit disk onto itself if, and only if,* $f \in D_k$ *for some* k. *If* $f \in D_k$ *then* w *is* K*-quasiconformal with*

$$K = K(p, M, k) \le C_2(M + p^2)^2 k \tag{7}$$

where C_2 *is an absolute constant.*

Proof: Suppose $f \in D^p(M) \cap D_k$. Let h be a harmonic function defined by

$$h = au + bv , \quad w = u + iv ,$$

where a and b are two real numbers and $a^2 + b^2 > 0$. Then h_z is an analytic function of z in B. According to Lewy's theorem [5] $J(w) > 0$ in B. Therefore h_z has no zeros. By the minimum principle

$$|h_z(z)| \ge \liminf_{\xi \to \partial B} |h_z(\xi)| . \tag{8}$$

If we set $z = x + iy$ then

$$\begin{aligned} 4|h_z(z)|^2 &= (u_x^2 + u_y^2)\, a^2 + 2(u_x v_x + u_y v_y)\, ab + (v_x^2 + v_y^2)\, b^2 \\ &\ge \frac{J(w)^2}{u_x^2 + u_y^2 + v_x^2 + v_y^2}\, (a^2 + b^2) \end{aligned} \tag{9}$$

by the well-known properties of positive definite forms. From (9) we obtain

$$\liminf_{\xi \to \partial B} |h_z(\xi)|^2 \ge (32 C_1)^{-1} (M + p^2)^{-2} k^{-2} (a^2 + b^2) \tag{10}$$

by Lemma 1 and 3. The estimates (8) and (10) show that

$$\begin{aligned} 4|h_z(z)|^2 &= (au_x + bv_x)^2 + (au_y + bv_y)^2 \\ &\ge (8C_1)^{-1} (M + p^2)^{-2} k^{-2} (a^2 + b^2) . \end{aligned} \tag{11}$$

If we take in (11) $a = v_y$, $b = - u_y$ and $a = - v_x$, $b = u_x$ and add these inequalities we get

$$J(w) \ge (8C_1)^{-1/2} (M + p^2)^{-1} k^{-1} (|w_z|^2 + |w_{\bar z}|^2)^{1/2} . \tag{12}$$

For the maximum dilatation of the mapping w we have by (12) and Lemma 1 the estimate

$$K = \sup_{z \in B} \frac{(|w_z| + |w_{\bar z}|)^2}{J(w)} \le 8C_1 (M + p^2)^2 k$$

which leads to (7).

If $f \in D^p(M)$ does not belong to any D_k then Lemma 1 and 3 show that

$$\limsup_{z \to \partial B} \frac{(|w_z| + |w_{\bar z}|)^2}{J(w)} = +\infty$$

and w cannot be K-quasiconformal for any K since the dilatation is a continuous function in B.

Remark. *The above method shows that* $K < 0.7\ 10^3(M + p^2)^2\, k$.

The question arises: Is it possible to weaken the assumption $f \in D^p(M)$ in Theorem 1? The essential hypotheses in Lemma 1 and Lemma 3 are:

(A) $f : \boldsymbol{R} \to \boldsymbol{R}$ is a homeomorphism, $f(0) = 0$, $f(2\pi) = 2\pi$, and f is 2π periodic, (f can be normalized so that

$$\int_0^{2\pi} e^{if(x)} dx = 0) \,,$$

(B) $\lim_{\xi \to z} w_r(\xi)$ and $\lim_{\xi \to z} w_\varphi(\xi)$, $z \in \partial B$, exist.

Under these conditions f is continuously differentiable. This can be seen from the following: The function $g(x) = \lim_{z \to x} w_\varphi(z)$, $x \in \partial B$, is continuous. Since

$$w_\varphi(re^{i\varphi}) = \int_0^{2\pi} g(x)\, P(r\,, x - \varphi)\, dx$$

we get integrating by parts

$$w(re^{i\varphi}) - w(r) = \int_0^{\varphi} w_\varphi(re^{i\varphi})\, d\varphi = \int_0^{2\pi} (g(0) + \int_0^{x} g(t)dt)\, P(r\,, x - \varphi)\, dx\,.$$

If we let $r \to 1$ in the above expression we have

$$e^{if(\varphi)} - 1 = g(0) + \int_0^{\varphi} g(t)\, dt$$

and the conclusion is immediate. Thus the argument of Lemma 3 remains unaltered and instead of (2) in Lemma 1 we have

$$\pi^{-2} \le |w_z|^2 + |w_{\bar z}|^2 \le C_1'$$

for some C_1' depending on f because $g(x)$ and $h(x) = \lim_{z \to x} w_r(z)$, $x \in \partial B$, are continuous and ∂B is compact. Therefore Theorem 1 is still valid if we replace $f \in D^p(M)$ by (A) and (B) and $K(p\,, M, k)$ by $K(f)$. However, it is not known (to the author) which conditions on f are necessary and sufficient to quarantee (B). On the other hand the assumption $f \in D^p(M)$ cannot be much weakened for if we take

$$f(x) = \overline{\text{arc}} \sin\left(x + \int_0^x \frac{|t|}{t(-\log|t|)^s}\, dt\right), \qquad 0 < s < 1$$

in a sufficiently small neighbourhood of 0 and continue it to all of $\boldsymbol{R}$ in such a way that f is a C^∞-function except at the points $2\pi n$ $(n = 0, \pm 1, \ldots)$ and f belongs to D_k for some $k \geq 1$ then $f : \boldsymbol{R} \to \boldsymbol{R}$ is continuously differentiable and it can be shown that

$$\lim_{z \to 1} (|w_z| + |w_{\bar{z}}|)^2 = +\infty$$

as z converges to 1 along the real axis. By Lemma 2 $J(w)$ is bounded therefore w cannot be K-quasiconformal for any $K < \infty$.

University of Helsinki
Finland

References

[1] CHOQUET, G.: Sur un type de transformation analytique generalisant la represén-tation conforme et définie au moyen fonctions harmoniques. - Bull. Sci. Math. 119 (1945), pp. 156—165.

[2] HEINZ, E.: On certain nonlinear elliptic differential equations and univalent mappings. - J. Analyse Math. 5 (1956/57), pp. 197—272.

[3] —»— One-to-one harmonic mappings. - Pacific J. Math. 9 (1959), pp. 101—105.

[4] KELLOGG, O.: On the derivatives of harmonic functions on the boundary. - Trans. Amer. Math. Soc. 33 (1931), pp. 486—510.

[5] LEWY, H.: On the non-vanishing of the Jacobian in certain one-to-one mappings. - Bull. Amer. Math. Soc. 42 (1936), pp. 689—692.

[6] NIETSCHE, J.: On harmonic mappings. - Proc. Amer. Math. Soc. 9 (1958), pp. 268—271.

[7] ZYGMUND, A.: Trigonometric series, vol. I. - Cambridge (1959).

Printed May 1968.