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THE COEFFICIENTS OF
QUASICONFORMALITY OF ELLIPSOIDS

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1. Introduction

The problem of extremal plane quasiconformal mappings was first treated by H. Grötzsch [12], who proved that the extremal quasiconformal mapping of one rectangle onto another with vertices corresponding in the same order is the affine mapping. As observed by Ahlfors [1], this actually shows that the extremal quasiconformal mapping of one quadrilateral onto another is the composition of a conformal mapping, an affine mapping, and a second conformal mapping. Moreover, the work of Teichmüller [17] shows that there is a large class of extremal quasiconformal mappings that are given locally by such a sequence of mappings.

Next, suppose that Z and Z' are finite right circular cylinders in R^3 with bases of equal area and with lengths L and L' , respectively. It is easy to show by means of an extremal lengths argument that the natural affine mapping of Z onto Z' is extremal among all quasiconformal maps of Z onto Z' that preserve the bases. Moreover, from this and Theorem 11.2 of [10] it follows that the affine map is extremal among all quasiconformal mappings of Z onto Z' when $2^{-\frac{1}{2}} \leq L'/L \leq 2^{\frac{1}{2}}$.

For such reasons it is then natural to ask if in 3-space the affine mapping is extremal in the class of all quasiconformal homeomorphisms of an ellipsoid onto the unit ball. In the present paper we disprove this conjecture in several cases by constructing mappings with lower dilatations than those of the affine mapping. In several instances our work also gives the correct order for the coefficients of quasiconformality. For the prolate ellipsoid these coefficients are bounded, while in another case the dilatation of the affine map has the same order as that of an extremal mapping.

Before attacking these problems, we introduce in the next section some basic terminology and results, most of which are contained in [10].

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2. Quasiconformal mappings

Suppose that f is a diffeomorphism of a 3-space domain Ω onto Ω' . Then f is locally affine — if $P \in \Omega$ the differential mapping $df(P)$ takes the unit ball onto an ellipsoid with semiaxes of lengths, say, $a \geq b \geq c$. Then each of the dilatation functions

$$(1) \quad H_I(P, f) = \frac{ab}{c^2}, H_O(P, f) = \frac{a^2}{bc}$$

is greater than or equal to unity, with equality at a point P if and only if f is conformal there. Moreover,

$$(2) \quad H_I(P, f) \leq H_O(P, f)^2, H_O(P, f) \leq H_I(P, f)^2$$

for each $P \in \Omega$.

Next, the functionals

$$(3) \quad K_I(f) = \sup_{P \in \Omega} H_I(P, f), K_O(f) = \sup_{P \in \Omega} H_O(P, f)$$

are called the *inner* and *outer dilatations*²⁾ of f , respectively; each is a natural measure of how far f is from being a Möbius transformation of Ω . Then

$$(4) \quad K_I(f) \leq K_O(f)^2, K_O(f) \leq K_I(f)^2,$$

so that these dilatations are both infinite or finite. If both are bounded by K , $1 \leq K < \infty$, then f is called a *K-quasiconformal mapping*. A *quasiconformal mapping* is one which is *K*-quasiconformal for some *K*.

The above definitions may be generalized, by means of extremal lengths, to include an arbitrary homeomorphism f of Ω onto Ω' . If Γ is a family of arcs in R^3 , we let $F(\Gamma)$ denote the family of density functions ϱ that are nonnegative and Borel-measurable in R^3 and for which

$$\int_{\gamma} \varrho ds \geq 1$$

for each $\gamma \in \Gamma$. If γ is not locally rectifiable, the integral is taken with respect to linear measure [16]. Then following Väisälä [18] we define the modulus $M(\Gamma)$ of the arc family Γ as

$$(5) \quad M(\Gamma) = \inf_{\varrho} \int_{R^3} \varrho^3 d\omega,$$

²⁾ Sometimes the inner and outer dilatations are defined as the square root of these expressions.

where the infimum is taken over all $\varrho \in F(\Gamma)$ (See also [6] and [10]). Given an arbitrary homeomorphism f of Ω onto Ω' , the inner and outer dilatations of f are defined by

$$(6) \quad K_I(f) = \sup_{\Gamma} \frac{M(f(\Gamma))}{M(\Gamma)}, \quad K_o(f) = \sup_{\Gamma} \frac{M(\Gamma)}{M(f(\Gamma))},$$

where the suprema are taken over all families Γ in Ω for which $M(\Gamma)$ and $M(f(\Gamma))$ are not both 0 or ∞^2 .

It is easily verified that these definitions agree with (3) when f is a diffeomorphism. Moreover, it follows from Lemma 1.2 of [10] and Theorem 5 of [9] that inequalities (4) hold also for definitions (6), hence that $K_I(f)$ and $K_o(f)$ are both infinite or finite. In the latter case, f is said to be quasiconformal; f is K -quasiconformal if

$$K(f) = \max(K_I(f), K_o(f)) \leq K.$$

Finally, given a domain Ω homeomorphic to the unit ball B^3 , let

$$(7) \quad K_I(\Omega) = \inf_f K_I(f), \quad K_o(\Omega) = \inf_f K_o(f),$$

where the infima are taken over all homeomorphisms f of Ω onto B^3 . If Ω is not homeomorphic to B^3 , set $K_I(\Omega) = K_o(\Omega) = \infty$. The numbers $K_I(\Omega)$ and $K_o(\Omega)$ are called the *inner* and *outer coefficients of quasiconformality* of Ω , respectively. From (4) we have

$$(8) \quad K_I(\Omega) \leq K_o(\Omega)^2, \quad K_o(\Omega) \leq K_I(\Omega)^2,$$

so that these coefficients are both infinite or finite. In the latter case, Ω is called a *quasiconformal ball*. By (8) clearly $K_I(\Omega) \geq 1$ and $K_o(\Omega) \geq 1$; by Liouville's Theorem (Theorem 15 of [9]) one, and hence both, of these coefficients is 1 if and only if Ω is a ball or half space.

3. The prolate ellipsoid

We consider first the prolate ellipsoid. We prove

Theorem 1. *Let E_p be any prolate ellipsoid. Then the affine mapping of E_p onto the unit ball is not extremal for either the inner or the outer coefficient. Both coefficients of a prolate ellipsoid are bounded.*

Proof. According to [13, p. 177] or [14, pp. 295, 296], the plane mapping

$$y_1 + iy_2 = f(x_1 + ix_2) = k^{\frac{1}{2}} \operatorname{sn} \left(\frac{2K}{\pi} \zeta, k \right)$$

maps the ellipse $x_1^2/a^2 + x_2^2/b^2 < 1$ conformally onto the unit disk, where $x_1 + ix_2 = \sin \zeta$ and $\zeta = \alpha + i\beta$. Here K and K' are the elliptic integrals defined by

$$(9) \quad K = K(k) = \int_0^1 [(1-x^2)(1-k^2x^2)]^{-\frac{1}{2}} dx,$$

$$K' = K(k'), \quad k' = (1-k^2)^{\frac{1}{2}},$$

sn denotes the Jacobian elliptic sine function, and

$$(10) \quad a = \cosh t, \quad b = \sinh t, \quad t = \frac{\pi K'}{4K}.$$

If the $x_1 + ix_2$ and $y_1 + iy_2$ planes are rotated about the real axes, there results a diffeomorphism $y = f_p(x)$ of the prolate ellipsoid $E_p: x_1^2/a^2 + x_2^2/b^2 + x_3^2/b^2 < 1$ onto the unit ball. That is, we let

$$y_1 + is = f(x_1 + ir), \quad \varphi = \theta,$$

where (r, θ) and (s, φ) are polar coordinates in the x_2x_3 and y_2y_3 planes, respectively.

In determining the dilatations of f_p it is convenient to make use of the hyperbolic density [15]. Suppose Ω is a simply-connected plane domain with more than one boundary point, in the ζ plane, and let $\omega(\zeta)$ be a conformal mapping of Ω onto the upper half plane $\text{Im } \omega > 0$. Then the hyperbolic density for Ω is defined to be

$$(11) \quad \varrho(\zeta) = \varrho(\zeta, \Omega) = \frac{\left| \frac{d\omega}{d\zeta} \right|}{2 \text{Im } \omega}.$$

Hence the hyperbolic density ϱ_1 of the upper half plane $\Omega_1: x_2 > 0, x_3 = 0$ is

$$(12) \quad \varrho_1 = \varrho(x_1 + ix_2, \Omega_1) = \frac{1}{2x_2}.$$

Next, by virtue of the reflection principle, the plane mapping $f(x_1 + ix_2)$ carries the semi-ellipse $\Omega_2: x_1^2/\cosh^2 2t + x_2^2/\sinh^2 2t < 1, x_2 > 0, x_3 = 0$ conformally onto the upper half plane $y_2 > 0, y_3 = 0$. Hence the hyperbolic density of Ω_2 is

$$(13) \quad \varrho_2 = \varrho(x_1 + ix_2, \Omega_2) = \frac{|f'(x_1 + ix_2)|}{2 y_2}.$$

From (12) and (13) and the fact that the hyperbolic density $\varrho(\zeta, \Omega)$ is a monotonic decreasing function of domain we arrive at the following form of the Schwarz Lemma: For all points in Ω_2 ,

$$(14) \quad \frac{\varrho_2}{\varrho_1} = \frac{x_2}{y_2} |f'| \geq 1.$$

It is now easy to see, using (3) and (14), that

$$(15) \quad K_I(f_p) = \sup \left(\frac{x_2}{y_2} |f'| \right)^2, \quad K_O(f_p) = \sup \left(\frac{x_2}{y_2} |f'| \right),$$

where the suprema are taken over all points $(x_1, x_2, 0)$ in the ellipse $x_1^2/a^2 + x_2^2/b^2 < 1, x_3 = 0$, with a and b as in (10).

Next we use the fact that the hyperbolic density ϱ satisfies the differential equation $\Delta \log \varrho = 4\varrho^2$, where Δ is the Laplacian operator. From this fact and the inequality (14) it follows that

$$(16) \quad \Delta \log \frac{\varrho_2}{\varrho_1} = 4(\varrho_2^2 - \varrho_1^2) \geq 0$$

in Ω_2 and hence, by symmetry and continuity, throughout the ellipse $x_1^2/a^2 + x_2^2/b^2 < 1, x_3 = 0$. From (16) we conclude that $\log(\varrho_2/\varrho_1)$ is subharmonic in this ellipse as a function of x_1 and x_2 . Consequently the suprema in (15) must occur on the boundary. In fact, by symmetry, it is clear that the suprema will be achieved on the arc $\gamma: x_1^2/a^2 + x_2^2/b^2 = 1, x_2 \geq 0, x_3 = 0$.

Since $\beta = t = \pi K'/4K$ on γ , where $x_1 + ix_2 = \sin(x + i\beta)$,

$$(17) \quad f(x_1 + ix_2) = k^{\frac{1}{2}} \operatorname{sn} \left(\frac{2K}{\pi} \alpha + i \frac{K'}{2}, k \right)$$

there. Then use of the addition theorem for sn ([4, p. 38], [5, #125.01]), the special values $\operatorname{sn}(K'/2, k') = (1+k)^{-\frac{1}{2}}, \operatorname{cn}(K'/2, k') = [k/(1+k)]^{\frac{1}{2}}, \operatorname{dn}(K'/2, k') = k^{\frac{1}{2}}$ ([5, #122.10], [11, p. 120]), and the identity $k^2 \operatorname{sn}^2 u = 1 - \operatorname{dn}^2 u$ ([4, p. 9], [5, #121.00]) replaces (17) by

$$(18) \quad f(x_1 + ix_2) = \frac{(1+k)s + icd}{1 + ks^2}$$

on γ . Here we have employed the notation

$$(19) \quad s = \operatorname{sn} \left(\frac{2K}{\pi} \alpha, k \right), c = \operatorname{cn} \left(\frac{2K}{\pi} \alpha, k \right), d = \operatorname{dn} \left(\frac{2K}{\pi} \alpha, k \right).$$

Finally, employing the Gauss Transformation ([4, pp. 72, 73], [5, #164.01-.02]) we may reduce (19) to

$$(20) \quad f(x_1 + ix_2) = \operatorname{sn} \left(\frac{2A}{\pi} \alpha, \lambda \right) + i \operatorname{cn} \left(\frac{2A}{\pi} \alpha, \lambda \right)$$

on γ , where

$$(21) \quad \lambda = \frac{2k^{\frac{1}{2}}}{1+k}, \lambda' = \frac{1-k}{1+k},$$

$$A = K(\lambda) = (1+k)K, A' = K(\lambda') = (1+k)\frac{K'}{2}.$$

Then differentiation of (20) with respect to $x_1 + ix_2$ and use of the relations $(\operatorname{sn} u)' = \operatorname{cn} u \operatorname{dn} u$, $(\operatorname{cn} u)' = -\operatorname{sn} u \operatorname{dn} u$ and the identity $\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$ yields

$$(22) \quad |f'| = \frac{2A}{\pi} \frac{\operatorname{dn} \left(\frac{2A}{\pi} \alpha, \lambda \right)}{[\cos^2 \alpha + \sinh^2 t]^{\frac{1}{2}}}.$$

But on the arc γ , $x_2 = \operatorname{Im} \sin(\alpha + it) = \cos \alpha \sinh t$, and with (20) and (22) we achieve

$$(23) \quad \frac{x_2}{y_2} |f'| = \frac{2A}{\pi} \frac{\operatorname{dn} \left(\frac{2A}{\pi} \alpha, \lambda \right)}{\operatorname{cn} \left(\frac{2A}{\pi} \alpha, \lambda \right)} \frac{\cos \alpha \sinh t}{[\cos^2 \alpha + \sinh^2 t]^{\frac{1}{2}}}.$$

The value of (23) is easily seen to be $\frac{2A}{\pi} \tanh t$ when $\alpha = 0$. But since

$$t = \frac{\pi K'}{4K} = \frac{\pi A'}{2A}$$

by (10) and (21), it follows that the suprema in (15) are achieved for $\alpha = 0$ if and only if the following holds:

Lemma 1. For each λ , $0 \leq \lambda < 1$, and all α , $-\pi/2 \leq \alpha \leq \pi/2$,

$$(24) \quad \frac{\operatorname{dn}^2 \left(\frac{2A}{\pi} \alpha, \lambda \right) (\cos^2 \alpha) \left(\cosh^2 \frac{\pi A'}{2A} \right)}{\operatorname{cn}^2 \left(\frac{2A}{\pi} \alpha, \lambda \right) \cos^2 \alpha + \sinh^2 \frac{\pi A'}{2A}} \leq 1.$$

A proof of this lemma is given in [3]. Thus (15) becomes

$$(25) \quad K_I(f_p) = \left(\frac{2A}{\pi} \tanh \frac{\pi A'}{2A} \right)^2, K_O(f_p) = \frac{2A}{\pi} \tanh \frac{\pi A'}{2A}.$$

That these dilatations, and hence the coefficients of E_p , are bounded is a consequence of the limit

$$\lim_{\lambda \rightarrow 1} \frac{2A}{\pi} \tanh \frac{\pi A'}{2A} = [\lim_{\lambda \rightarrow 1} A'] \left[\lim_{\lambda \rightarrow 1} \frac{\tanh \frac{\pi A'}{2A}}{\frac{\pi A'}{2A}} \right] = \frac{\pi}{2}$$

(See also [3]).

We wish to compare the dilatations (25) with those of the natural affine mapping $g_p(x) = (x_1/a, x_2/b, x_3/b)$ of E_p onto the unit ball. Since g_p is a diffeomorphism it is easy to see that

$$(26) \quad \begin{aligned} K_I(g_p) &= \left(\frac{a}{b}\right)^2 = \left(\coth \frac{\pi A'}{2A}\right)^2, \\ K_O(g_p) &= \frac{a}{b} = \coth \frac{\pi A'}{2A}. \end{aligned}$$

But in [3] we have proved

Lemma 2. For $0 \leq \lambda < 1$,

$$\frac{1}{2} \left(1 + \frac{2A}{\pi}\right) \tanh \frac{\pi A'}{2A} \leq 1,$$

with strict inequality if $0 < \lambda < 1$.

But since trivially $\left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \leq \frac{1}{2} \left(1 + \frac{2A}{\pi}\right)$ it follows from this lemma that $K_I(f_p) < K_I(g_p)$ and $K_O(f_p) < K_O(g_p)$ when $0 < \lambda < 1$, so that the affine mapping is not extremal for either coefficient.

4. The outer coefficient of the oblate ellipsoid

We turn our attention now to the oblate ellipsoid. We begin by proving

Theorem 2. Let E_o be any oblate ellipsoid. Then the affine mapping of E_o onto the unit ball is not extremal for the outer coefficient.

Proof. The proof we present is based on a general mapping introduced by Gehring and Väisälä [10]. We define a radial mapping $y = h(x)$ from $E_o : x_1^2/a^2 + x_2^2/b^2 + x_3^2/a^2 < 1, a > b$ to the unit ball as follows: If $P \in E_o, P \neq 0$, then P has a unique representation of the form $P = rQ$, where $Q \in \partial E_o$ and $0 < r < 1$. For each such $P \in E_o$ define

$$(27) \quad h(P) = r^p h(Q), \quad h(Q) = \frac{Q}{|Q|},$$

where p is some fixed positive number, and let $h(0) = 0$.

Next, for each $Q \in \partial E_o$ let $\psi(Q)$ denote the acute angle which the tangent plane at Q makes with the ray from 0 to Q , and let $\psi = \min \psi(Q)$. An elementary geometric argument shows that $\psi = \arcsin 2ab/(a^2 + b^2)$. Now take the exponent p in (27) to be

$$(28) \quad p = \sin \psi = \frac{2ab}{a^2 + b^2}.$$

Then by Theorem 5.1 of [10] it follows that

$$(29) \quad K_o(h) \leq 2^{\frac{1}{2}} \cot \frac{\psi}{2} \cos \frac{\psi}{2} = \frac{a}{b} \left(\frac{2a^2}{a^2 + b^2} \right)^{\frac{1}{2}}.$$

For the affine mapping g_o of E_o onto the unit ball we have

$$(30) \quad K_o(g_o) = \left(\frac{a}{b} \right)^2.$$

But comparison of (29) and (30) shows that $K_o(g_o) > K_o(h)$ because $a > b$. We conclude that the affine mapping is not extremal for the outer coefficient.

5. The inner coefficient of the oblate ellipsoid

In Theorem 1 we showed that for the prolate ellipsoid $E_p : x_1^2/a^2 + x_2^2/b^2 + x_3^2/b^2 < 1$ the affine mapping g_p has the dilatations $K_I(g_p) = (a/b)^2$ and $K_o(g_p) = a/b$, while the coefficients of E_p are of the order $O(1)$ as a/b tends to ∞ . Thus for large a/b the affine mapping is very far from being extremal.

The situation is similar for the outer coefficient of the oblate ellipsoid $E_o : x_1^2/a^2 + x_2^2/b^2 + x_3^2/a^2 < 1, a > b$. For the affine mapping g_o has outer dilatation $K_o(g_o) = (a/b)^2$, while in Theorem 2 there was given a mapping h of E_o onto the unit ball such that

$$K_o(h) \leq \frac{a}{b} \left(\frac{2a^2}{a^2 + b^2} \right)^{\frac{1}{2}} = O\left(\frac{a}{b}\right)$$

as a/b tends to ∞ .

The problem of determining whether the affine mapping of an oblate ellipsoid is extremal for the inner coefficient appears to be more difficult. For in this case Theorem 3 below shows that the inner dilatation of the affine map has the same order as $K_I(E_o)$ as a/b tends to ∞ .

Theorem 3. *Let E_o be the oblate ellipsoid $x_1^2/a^2 + x_2^2/b^2 + x_3^2/a^2 < 1$, $a > b$. Then $K_I(E_o)/(a/b)$ is monotonically decreasing as a/b tends to ∞ , and*

$$(31) \quad \lim_{a/b \rightarrow \infty} \frac{K_I(E_o)}{(a/b)} = A,$$

where $0.22 < A \leq 1$.

Proof. Since the affine mapping g_o of E_o onto the unit ball has inner dilatation $K_I(g_o) = a/b$, we see immediately that $K_I(E_o)/(a/b) \leq 1$ for all values of a/b .

For convenience we now assume, as we obviously may, that $b = 1$. To prove the monotonicity, let f_1 be any quasiconformal mapping of $E_o = E_o(a_1)$ onto the unit ball and let

$$x' = g(x) = (a_1x_1/a_2, x_2, a_1x_3/a_2)$$

be the natural affine mapping of $E_o(a_2)$ onto $E_o(a_1)$, where $a_2 > a_1$. Then $y = f_2(x)$ is a quasiconformal mapping of $E_o(a_2)$ onto the unit ball, where $f_2 = f_1 \circ g$. It is easy to see that $K_I(g) = a_2/a_1$, and by (6) there results

$$(32) \quad K_I(f_2) \leq K_I(f_1) K_I(g) = \frac{a_2}{a_1} K_I(f_1).$$

Taking the infimum first over all f_2 and then over all f_1 in (32) we achieve

$$K_I(E_o(a_2)) \leq \frac{a_2}{a_1} K_I(E_o(a_1)),$$

from which the asserted monotonicity follows.

We now obtain the lower bound in the theorem. Our method is similar to an unpublished argument of Gehring and Väisälä. Let f be any quasiconformal mapping of E_o onto the unit ball. Let E_i denote the arc of the circle $x_1^2 + x_3^2 = a^2, x_2 = 0$ that is contained in the closed i -th quadrant of $x_2 = 0, i = 1, 2, 3, 4$, and let Γ_1 and Γ_2 be the families of arcs that join E_1 to E_3 and E_2 to E_4 , respectively, in E_o .

Now let H be the portion of E_o that is included between the two planes $x_1 + x_3 = \pm a$, and let ϱ be the density function which is $2^{-\frac{1}{2}} a^{-1}$ in H and 0 elsewhere. Then $\varrho \in F(\Gamma_1)$. Moreover, an elementary integration shows that

$$m(H) = \frac{5\sqrt{2}}{6} \pi a^2.$$

Thus

$$(33) \quad M(\Gamma_1) = M(\Gamma_2) \leq \int_H \varrho^3 d\omega = \frac{m(H)}{2\sqrt{2} a^3} = \frac{5\pi}{12a}.$$

Next, by Corollary 3.3 of [10] we have either

$$(34) \quad M(f(\Gamma_1)) \geq v(1) \quad \text{or} \quad M(f(\Gamma_2)) \geq v(1),$$

where for $r > 0$, $v(r)$ denotes the modulus of the family of arcs which join the segment $-1 \leq x_1 \leq 0$, $x_2 = x_3 = 0$ to the ray $r \leq x_1 \leq \infty$, $x_2 = x_3 = 0$ in the half space $x_2 > 0$. By a simple estimate based on Theorem 8 of [2] it is easy to show that

$$(35) \quad v(1) \geq \frac{2\pi}{(4.6)^2}.$$

Hence by (33), (34), (35), and (6) we have

$$K_I(f) \geq \max_{i=1,2} \frac{M(f(\Gamma_i))}{M(\Gamma_i)} > 0.22 a,$$

so that $K_I(E_o) = \inf_f K_I(f) > 0.22 a$.

Remark. Let Γ denote the family of curves in E_o that join the axis $-1 \leq x_2 \leq 1$, $x_1 = x_3 = 0$ to the circle $x_1^2 + x_3^2 = a^2$, $x_2 = 0$, and let f be any quasiconformal mapping of E_o onto the unit ball. In (3.14) of [10] Gehring and Väisälä conjectured that $M(f(\Gamma)) \geq (\pi/q)^2$, where $q = 2^{\frac{1}{2}} K((\frac{1}{2})^{\frac{1}{2}}) = 2.62 \dots$. If this conjecture were true, then an argument similar to that given above shows that the lower bound for A in the theorem could be improved to $A > 0.39$.

6. The general ellipsoid

Finally we consider a general ellipsoid, that is, an ellipsoid which is not necessarily an ellipsoid of revolution. We prove

Theorem 4. *Let E be any ellipsoid. Then the affine mapping of E onto the unit ball is not extremal for the outer coefficient.*

Proof. By Theorems 1 and 2 we may assume that E is neither prolate nor oblate. Thus let $1 < b < a < \infty$, and let E denote the ellipsoid $x_1^2/a^2 + x_2^2 + x_3^2/b^2 < 1$. The affine mapping

$$y = g(x) = (x_1/a, x_2, x_3/b)$$

of E onto the unit ball may be regarded as the composition of two affine mappings — an affine mapping

$$x' = g_1(x) = (bx_1/a, x_2, x_3)$$

of E onto the oblate ellipsoid $E_o : x_1'^2/b^2 + x_2'^2 + x_3'^2/b^2 < 1$, followed by a second affine mapping

$$y = g_2(x') = (x'_1/b, x'_2, x'_3/b)$$

of E_o onto the unit ball.

It is easy to see that $K_o(g) = ab$, $K_o(g_1) = a/b$, and $K_o(g_2) = b^2$. Consequently

$$(36) \quad K_o(g) = K_o(g_1) K_o(g_2).$$

Now let f_2 be the quasiconformal mapping of the oblate ellipsoid E_o given in (27). Then the composition of g_1 with f_2 is a new quasiconformal mapping h of E onto the unit ball. Hence by (6), Theorem 2, and (36), we have

$$K_o(h) \leq K_o(g_1) K_o(f_2) < K_o(g_1) K_o(g_2) = K_o(g).$$

This completes the proof of Theorem 4.

We return in conclusion to the notation $x_1^2/a^2 + x_2^2/b^2 + x_3^2/c^2 < 1$, $a \geq b \geq c$ for the general ellipsoid E . It follows easily from Theorem 1 in §3 above that (i) if $a \rightarrow \infty$ with b and c fixed then $K_I(E)$ and $K_o(E)$ are bounded, while the methods of proof in Theorem 3 show that (ii) if $a \rightarrow \infty$ and $b \rightarrow \infty$ with c fixed then both coefficients become infinite. These observations were also made by Gehring in a recent treatment [7], his proof of (ii) being based on Theorem 3 of [19] and Theorem 2.3 of [10].

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