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MAPS WITH CONTINUOUS CHARACTERISTICS AS A SUBCLASS OF QUASICONFORMAL MAPS

BY

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Preface

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Marjatta Näätänen

Introduction

The first definitions for quasiconformal maps were introduced by Grötzsch in 1928 and by Lavrentjev in 1935. These classes are not closed with respect to uniform limits, even if constants are excluded. A third definition without this drawback was given by Morrey in 1938, and in several other equivalent forms in the fifties. Today these maps are usually called quasiconformal, and in the present paper we use this terminology.

While Grötzsch maps are easily seen to be continuously differentiable quasiconformal maps, it is more difficult to see the position of Lavrentjev maps, often called maps with continuous characteristics p and Θ, in the hierarchy of quasiconformal maps. The fundamental result in this direction is due to Bojarski: Lavrentjev maps are generalized homeomorphic solutions of Beltrami equations with continuous coefficients. It follows, in particular, that Lavrentjev maps constitute a proper subclass of quasiconformal maps.

In the present paper we study once more the relationship between Lavrentjev maps and quasiconformal maps. A new proof, based on a method of Gehring, is given to the effect that the former maps are quasiconformal. Our main result concerns the local behaviour of Lavrentjev maps f. We prove that the characteristic $p(z)$ coincides at all points with the circular dilatation $H(z)$, defined as the upper limit of $\max_\alpha |f(z + re^{i\alpha}) - f(z)| / \min_\alpha |f(z + re^{i\alpha}) - f(z)|$ as $r \to 0$. From this it follows that p is also everywhere equal to the local maximal dilatation of the map. We conclude the paper with some examples: a Lavrentjev map non-differentiable at a point, a Lavrentjev map whose inverse is not a Lavrentjev map, and two Lavrentjev maps whose composition is not a Lavrentjev map.

1. Definitions of quasiconformality

1.1. *Ahlfors-Pfluger definition.* A *quadrilateral* Q is a Jordan domain with four specified boundary points z_1, z_2, z_3, z_4, whose order coincides with the positive ordering with respect to the Jordan domain. For every

Q there exists a class of rectangles with vertices at $0, \alpha, \alpha + i\beta, i\beta$, $\alpha > 0$, $\beta > 0$, which can be conformally mapped onto Q such that the vertices correspond to z_1, z_2, z_3, and z_4, respectively. The ratio

$$M(Q) = \alpha/\beta$$

depends only on Q and is called the *modulus of the quadrilateral* Q.

Definition 1. Let f be a sense-preserving homeomorphism of a domain G. If for all quadrilaterals Q, $\bar{Q} \subset G$,

$$M(f(Q)) \leq KM(Q),$$

$1 \leq K < \infty$, f is *K-quasiconformal.*

1.2. *Morrey definition.* A continuous function $f: G \to G'$ is ACL or *absolutely continuous on lines* in a domain $D \subset G$, if for every rectangle $R = \{z = x + iy \mid a < x < b,\ c < y < d\}$, $\bar{R} \subset D$, $f(x + iy_0)$ is absolutely continuous in $a < x < b$ for almost every y_0 in $c < y_0 < d$, and $f(x_0 + iy)$ is absolutely continuous in $c < y < d$ for almost all x_0 in $a < x_0 < b$.

The following theorem, often stated as the analytic definition, yields another characterization for K-quasiconformal maps:

Theorem. *A sense-preserving homeomorphism f of a domain G is K-quasiconformal if and only if*

(i) f *is* ACL *in* G

(ii) $\max\limits_{\alpha} |\partial_\alpha f(z)|^2 \leq KJ(z)$ *a.e. in* G,

where J denotes the Jacobian of f and $\partial_\alpha f$ the derivative in the direction α.

1.3. *Lavrentjev definition.* Given an ellipse, we define its characteristics as the ratio $p \geq 1$ of its semiaxes and, if $p > 1$, the angle Θ (mod. π) between its majoraxis and the positive x-axis. They define the ellipse up to a similarity transformation. The ellipse with centre at z, semi minoraxis of length h and characteristics p, Θ, is denoted by $E_h(p, \Theta; z)$ and the open point set bounded by the curve $E_h(p, \Theta; z)$ and containing the point z by $E_h^0(p, \Theta; z)$.

Definition 2. A homeomorphism f is said to *map the infinitesimal ellipse* $E(p, \Theta; z_0)$ *onto an infinitesimal circle* if

$$\lim_{h \to 0} \frac{\max\limits_{z \in E_h} |f(z) - f(z_0)|}{\min\limits_{z \in E_h} |f(z) - f(z_0)|} = 1, \quad E_h = E_h(p, \Theta; z_0).$$

Especially if $p = 1$, we say that the *infinitesimal circle is mapped onto an infinitesimal circle.*

We call p, Θ the characteristics of f at z_0.

Definition 3. In a domain D a *continuous distribution of characteristics* p, Θ is defined, if p is a continuous real-valued map of D such that $p(z) \geq 1$ for all $z \in D$ and Θ is a continuous real-valued map of the set $\{z \mid z \in D, \ p(z) > 1\}$ such that the values of Θ are defined mod. π.

Definition 4. Let p, Θ be a continuous distribution of characteristics in a domain D such that p is bounded. A sense-preserving homeomorphism f of D is said to be a *Lavrentjev map* with characteristics p, Θ if, for every $z \in D$, f maps the infinitesimal ellipse $E(p(z), \Theta(z); z)$ onto an infinitesimal circle.

Remark 1. By the definition, Lavrentjev maps are preserved under conformal maps. As for the characteristics, if $f = f_1 \circ f_2$, where f_2 is conformal and f_1 a Lavrentjev map with characteristics p, Θ, f has characteristics $p, \Theta - \arg f_2'$, while in the case $f = f_2 \circ f_1$ the characteristics remain invariant.

Remark 2. The class of Lavrentjev maps was extended by considering maps which transform »infinitesimal ellipses onto infinitesimal ellipses»:

Definition 5. A homeomorphism f is said to *map the infinitesimal ellipse* $E(p_1, \Theta_1; z)$ *onto the infinitesimal ellipse* $E(p_2, \Theta_2; f(z))$ if the image of $E_h(p_1, \Theta_1; z)$ under f lies between the curves $E_k(p_2, \Theta_2; f(z))$ and $E_l(p_2, \Theta_2; f(z))$, $k \leq l$, such that

$$\lim_{h \to 0} (l/k) = 1 .$$

Definition 6. Let $f: D_1 \to D_2$ be a sense-preserving homeomorphism and $p_1, \Theta_1, p_2, \Theta_2$ continuous distributions of characteristics in D_1 and D_2. The map f is called a *generalized Lavrentjev map* with characteristics p_1, Θ_1 and p_2, Θ_2 if, for every $z \in D_1$, f maps the infinitesimal ellipse $E(p_1(z), \Theta_1(z); z)$ onto the infinitesimal ellipse $E(p_2(f(z)), \Theta_2(f(z)); f(z))$.

1.4. *Maps of class* C^1 *and Lavrentjev maps.* Let $f: D \to D'$, $f \in C^1$ [1]. It follows that all points $z_0 \in D$ are regular for f [2] and f has the representation

$$f(z) = f(z_0) + f_z(z_0)(z - z_0) + f_{\bar{z}}(z_0)(\bar{z} - \bar{z}_0) + o(z - z_0) ,$$

where $o(z - z_0)/(z - z_0) \to 0$ as $z \to z_0$. Hence, f maps the infinitesimal ellipse

[1]) A homeomorphism f is said to belong to the class C^1, if it, together with its inverse, is continuously differentiable. We also assume f to be sense-preserving.

[2]) We call a point z regular for f if f is differentiable at z with $J(z) \neq 0$.

$$E\left(\frac{|f_z(z_0)| + |f_{\bar z}(z_0)|}{|f_z(z_0)| - |f_{\bar z}(z_0)|}, \frac{1}{2}\left(\arg\frac{f_{\bar z}(z_0)}{f_z(z_0)} + \pi\right); z_0\right)$$

onto an infinitesimal circle.

The corresponding result is valid for f^{-1}. The relations between the derivatives of f and f^{-1} imply the equation

$$p_f(z_0) = p_{f^{-1}}(f(z_0))\,.$$

Since the derivatives define continuous distributions of characteristics, f and f^{-1} are Lavrentjev maps provided that p is bounded.

If $f \in C^1$ and, for every $z \in D$, $p(z) \leq 1 + \varepsilon$, f is said to be an *ε-Lavrentjev map*. The same is then valid for f^{-1}.

Let $f : D \to D'$, $g : D' \to D''$ be Lavrentjev maps of the class C^1. The combined map $g \circ f \in C^1$ and the characteristics satisfy

$$p_{g\circ f}(z) \leq p_g(f(z))\, p_f(z)\,,$$

since the same is valid for the composition of the affine maps consisting of the differentials of g and f.

2. Lavrentjev maps are quasiconformal

With a proof modified from that of Gehring [3], Theorem 2, we show in 2.2 that Lavrentjev maps form a subset of quasiconformal maps. In 2.3 we present an example to show that the subclass is proper.

This relationship was proved in different ways by Pesin and by Bojarski [2]. The former extends in [7] the class of generalized Lavrentjev maps and in [8] shows the equivalence of the new definition with the Ahlfors-Pfluger one. The result was already mentioned by Belinskij and Pesin in [1]. Bojarski has established the proof by studying the generalized homeomorphic solutions of Beltrami differential equations.

2.1. For our proof we list some properties of maps carrying infinitesimal ellipses onto infinitesimal circles.

Lemma 1. *If a map f carries the infinitesimal ellipse $E(p\,, \Theta\,; z)$ onto an infinitesimal circle and the derivative $|\partial_s f(z)|$ exists for some direction s_0, it exists in every direction s and*

$$\frac{1}{p}\,|\partial_{s_0} f(z)| \leq |\partial_s f(z)| \leq p\,|\partial_{s_0} f(z)|\,. \tag{1}$$

Proof. Let $z = f(z) = 0$ and let $|\Delta_h z_s|$ denote the modulus of the point on the curve $E_h(p , \Theta ; 0)$ which lies in the direction s. The modulus of the image point is denoted by $|\Delta_h f_s|$. Then, for any s

$$\lim_{h\to 0} |\Delta_h f_s| / |\Delta_h f_{s_0}| = 1 .$$

Furthermore,

$$\lim_{h\to 0} |\Delta_h f_{s_0}| / |\Delta_h z_{s_0}| = |\partial_{s_0} f(0)|$$

and $|\Delta_h z_{s_0}| / |\Delta_h z_s| = c_{s_0, s}$, $1/p \leq c_{s_0, s} \leq p$, where $c_{s_0, s}$, does not depend on h. Hence, there exists the limit

$$|\partial_s f(0)| = \lim_{h\to 0} |\Delta_h f_s| / |\Delta_h z_s| = |\partial_{s_0} f(0)|\, c_{s_0, s} .$$

In the case of $|\partial_{s_0} f(z)| = 0$ it follows

Lemma 2. *Let f carry the infinitesimal ellipse $E(p , \Theta ; z)$ onto an infinitesimal circle. If $|\partial_s f(z)|$ vanishes for some direction s_0, it vanishes for every direction and there exists a derivative $f'(z) = 0$.*

For a differentiable map Lemmas 1 and 2 yield

Lemma 3. *If f is differentiable at z and carries the infinitesimal ellipse $E(p , \Theta ; z)$ onto an infinitesimal circle, then either z is a regular point for f and*

$$\max_s |\partial_s f(z)|^2 \leq |J(z)|\, p(z) ,$$

or f has a vanishing derivative $f'(z)$.

Proof. The inequality (1) yields $\max_s |\partial_s f(z)| \leq p \min_s |\partial_s f(z)|$. Combined with the relation $|J(z)| = \max_s |\partial_s f(z)| \min_s |\partial_s f(z)|$, this concludes the proof.

2.2. After these remarks we are ready to prove

Theorem 1. *Lavrentjev maps are quasiconformal.*

Proof. Let f be a Lavrentjev map of a domain G and $K = \sup p(z)$, $z \in G$. We show that f satisfies the conditions in the Morrey definition:

(i) f is ACL in G

(ii) $\max_s |\partial_s f(z)|^2 \leq K J(z)$ a.e. in G.

We can presume that neither G nor its image G' contains the point at infinity, because an isolated singularity can be removed. Let $R = \{x + iy | a < x < b , c < y < d\}$, $\bar{R} \subset G$, $I_{y_0} = R \cap \{x + iy | y = y_0\}$, and let $T(y_0)$ denote the Lebesgue plane measure of the f-image of

$\{x + iy \mid a < x < b\,,\ c < y < y_0\}$. Since T is monotonic and finite, it has a finite derivative T' for almost every y_0, $c < y_0 < d$. Accordingly, for (i), it is sufficient to prove the absolute continuity of f on I_{y_0} when $T'(y_0)$ exists.

Assume that $T'(y_0)$ exists and denote $I_{y_0} = I$. Let $F \subset I$ be a closed set and F' its image under f. We shall first derive a majorant for the linear measure $l(F')$ of F' which vanishes with $l(F)$. The result is then extended to the Borel subsets of I. We use the shorter notation $E_h(z) = E_h(p(z)\,,\,\Theta(z)\,;\,z)$, where $p\,,\,\Theta$ are the characteristics of f.

Denote

$$M(h\,,\,z_0) = \max_{z \in E_h(z_0)} |f(z) - f(z_0)|$$

$$N(h\,,\,z_0) = \min_{z \in E_h(z_0)} |f(z) - f(z_0)|\,.$$

Take $\varepsilon > 0$ such that $K\varepsilon < d(F\,,\,-R)$, where $d(F\,,\,-R)$ denotes the distance of F and the complement of R. Let

$$F_\varepsilon = \{z \mid z \in F\,,\ h \leq \varepsilon \Rightarrow M/N \leq 2\}\,. \tag{2}$$

Then $hp(z) \leq \varepsilon K \leq d(F\,,\,-R)$ for $z \in F_\varepsilon$ and $h \leq \varepsilon$. Hence, R contains the ellipses $E_h(z)$ for $z \in F_\varepsilon$ and $h \leq \varepsilon$. In the following, we let ε only assume values $1/q$, where q is a natural number.

Let z be an accumulation point of the set F_ε. If z has a neighbourhood containing no points where $p = 1$, the functions $p\,,\,\Theta$, and f are all continuous, and $z \in F_\varepsilon$. The other possibility is that p takes the value 1 in every neighbourhood of z. By the continuity of p, the characteristic ellipses belonging to the points of a sufficiently small neighbourhood of z are arbitrarily close to circles. So also in this case $z \in F_\varepsilon$. Hence, the sets F_ε are closed. As $\varepsilon \to 0$, they form a non-decreasing sequence converging to F so that $F = \bigcup F_\varepsilon$. Hence

$$\lim_{\varepsilon \to 0} l(F'_\varepsilon) = l(F')\,. \tag{3}$$

As a bounded and closed set F_ε is compact. It has a finite cover of ellipses $E^0_\varepsilon(z_n)$, $n = 1\,,2\,,\ldots\,,n_\varepsilon$, such that $|z_{n+1} - z_n| \geq \varepsilon$, $n = 1\,,2\,,\ldots\,,n_\varepsilon - 1$, $z_n \in F_\varepsilon$. This is seen as follows. We assume $F_\varepsilon \neq \emptyset$. Let $z_1 = x_1 + iy_0$, where $x_1 = \inf \operatorname{re} z$, $z \in F_\varepsilon$. Then $z_1 \in F_\varepsilon$, because F_ε is compact. Denote the interval $\{z = x + iy_0 \mid x_1 - \varepsilon < x < x_1 + \varepsilon\}$ by I_1. The set $F_\varepsilon - I_1$ is also compact. If $F_\varepsilon - I_1 \neq \emptyset$, choose $z_2 = x_2 + iy_0$ where $x_2 = \inf \operatorname{re} z$, $z \in F_\varepsilon - I_1$. The points z_1 and z_2 satisfy $|z_2 - z_1| \geq \varepsilon$. Continue in this way and terminate when $F_\varepsilon - \bigcup I_n$ is empty. Since F_ε is bounded, this happens with a finite n.

The major axis of the ellipses $E_\varepsilon^0(z_n)$, $n = 1, 2, \ldots, n_\varepsilon$, is at most of length $2K$ and the distance of their centres at least ε. These ellipses cover R at most $2K$ times. The length of each interval $E_\varepsilon^0(z_n) \cap I$ is at least 2ε. For the union of the ellipses, $E_\varepsilon^0 = \bigcup E_\varepsilon^0(z)$, we have

$$l(E_\varepsilon^0 \cap I) \geq 2\varepsilon n_\varepsilon/(2K) . \tag{4}$$

Let $O \subset I$ be an open set, $F \subset O$. We can choose O such that $l(O - F) < \varepsilon'$, where $\varepsilon' > 0$ is arbitrarily small. Fix $\varepsilon > 0$ such that $K\varepsilon < d(F, I - O)$. This is possible since $d(F, I - O) > 0$. Then O contains also the set $E_\varepsilon^0 \cap I$ and

$$l(E_\varepsilon^0 \cap I) < l(O) \leq l(F) + l(O - F) < l(F) + \varepsilon' .$$

Hence

$$\limsup_{\varepsilon \to 0} l(E_\varepsilon^0 \cap I) \leq l(F) .$$

On the other hand, $F_\varepsilon \subset E_\varepsilon^0 \cap I$ for every $\varepsilon > 0$. Since $l(F_\varepsilon) \to l(F)$ it follows that

$$\lim l(E_\varepsilon^0 \cap I) = l(F) . \tag{5}$$

The ellipses $E_\varepsilon^0(z_n)$, $n = 1, 2, \ldots, n_\varepsilon$, are contained in the rectangle $\{x + iy | \, a < x < b \, , y_0 - K\varepsilon < y < y_0 + K\varepsilon\}$ and cover it at most $2K$ times. Similarly, in the image plane

$$2K[T(y_0 + K\varepsilon) - T(y_0 - K\varepsilon)] \geq \sum_{n=1}^{n_\varepsilon} m(E_\varepsilon^{0\prime}(z_n)) , \tag{6}$$

where $E_\varepsilon^{0\prime}(z_n)$ is the f-image of $E_\varepsilon^0(z_n)$. By (2),

$$\sum_{n=1}^{n_\varepsilon} m(E_\varepsilon^{0\prime}(z_n)) \geq \pi \sum_{n=1}^{n_\varepsilon} [N(\varepsilon, z)]^2 \geq \frac{\pi}{4} \sum_{n=1}^{n_\varepsilon} [M(\varepsilon, z_n)]^2 . \tag{7}$$

The Schwarz inequality, combined with (6) and (7), yields

$$2K\,[T(y_0 + K\varepsilon) - T(y_0 - K\varepsilon)] \geq \frac{\pi}{4n_\varepsilon} \Big[\sum_{n=1}^{n_\varepsilon} M(\varepsilon, z_n)\Big]^2 .$$

From (4) it follows that

$$l\,(E_\varepsilon^0 \cap I)\, \frac{T(y_0 + K\varepsilon) - T(y_0 - K\varepsilon)}{2K\varepsilon} \geq \frac{\pi \Big[\sum_{n=1}^{n_\varepsilon} M(\varepsilon, z_n)\Big]^2}{16K^3} . \tag{8}$$

The image of $E_\varepsilon^0(z_n)$ is contained in a disc with radius $M(\varepsilon, z_n)$. For $\varepsilon_0 \geq \varepsilon$,

$$F'_{\varepsilon_0} \subset \bigcup_{n=1}^{n_\varepsilon} E^{0'}_{\varepsilon} (z_n) .$$

For every $\varepsilon \leq \varepsilon_0$, F'_{ε_0} has a cover of discs, whose diameter is at most $2 \sum_{n=1}^{n_\varepsilon} M(\varepsilon , z_n)$. The function f is uniformly continuous in R, hence $\sup M(\varepsilon , z_n) \to 0$ as $\varepsilon \to 0$. Therefore, by the definition of the linear measure,

$$l(F'_{\varepsilon_0}) \leq 2 \liminf_{\varepsilon \to 0} \sum_{n=1}^{n_\varepsilon} M(\varepsilon , z_n) \tag{9}$$

for every $\varepsilon_0 > 0$. The same result holds for the length of F', because of (3). By (5), the left hand side of (8) tends to the limit $l(F)\, T'(y_0)$ as $\varepsilon \to 0$. Together with (9) this yields

$$l(F)\, T'(y_0) \geq \pi\, l(F')^2/(64\, K^3) .$$

This is the desired majorant for $l(F')$:

$$l(F')^2 \leq 64\, K^3\, l(F)\, T'(y_0)/\pi . \tag{10}$$

Here $F \subset I$ is any closed set. The following step is to generalize the result for all Borel sets $B \subset I$.

We show first that the image I' of I is σ-finite with respect to linear measure. Let E be a closed subinterval of I. By (10), its image E' has a finite linear measure. The set I is a countable union of closed intervals, therefore, I' is σ-finite.

Let $B \subset I$ be a Borel set. The set I' is σ-finite, so we can find a sequence of closed sets $F'_k \subset B'$, $k = 1, 2, \ldots$, such that $\lim l(F'_k) = l(B')$. The preimages $F_k = f^{-1}(F'_k)$ are also closed and (10) holds for $F = F_k$. Hence, (10) is also valid for the limits $l(B)$ and $l(B')$. Therefore, any Borel set $B \subset I$ satisfies the inequality

$$l(B')^2 \leq 64\, K^3\, l(B)\, T'(y_0)/\pi .$$

This implies the absolute continuity of f on I with respect to the linear measure l. So f is absolutely continuous on I as a function of one variable. A similar method shows the absolute continuity of f on almost every vertical interval in R.

It remains to verify the condition (ii). By the first part, f is absolutely continuous on I_y for almost every y, $c < y < d$. From the Fubini theorem we conclude that f has a finite derivative f_x a.e. in R. A similar result is valid for f_y. The domain G is a countable union of rectangles with sides parallel to the co-ordinate axes. Consequently, f has finite partial deriavatives a.e. in G. By a theorem of Gehring and Lehto [4], this implies that f is differentiable a.e. in G.

Let z be a point of differentiability. The derivative $\partial_s f(z)$ exists in every direction s. By Lemma 3, either z is a regular point or f has a vanishing derivative $f'(z)$. In the latter case (ii) is trivially satisfied. In the former case we have

$$\max_{\alpha} |\partial_\alpha f(z)|^2 \leq p(z)\, J(z) \leq KJ(z) .$$

2.3. *A quasiconformal map with is not a Lavrentjev map.* The map f,

$$f(z) = \begin{cases} Kx + iy & \text{for} \quad x \geq 0 \\ \dfrac{1}{K}\, x + iy & \text{for} \quad x < 0 \end{cases} , \quad K > 1 ,$$

is quasiconformal. The characteristics of f are $p = K$, $\Theta = \pi/2$ for $x > 0$, and $p = K$, $\Theta = 0$ for $x < 0$. For $x = 0$, f carries no infinitesimal ellipses onto infinitesimal circles. Therefore it has no continuous distribution of characteristics in the whole plane.

The map f serves also as an example of a quasiconformal map which is not a generalized Lavrentjev map.

3. Lavrentjev maps and Beltrami equations

All maps considered in the following are homeomorphisms of the open unit disc D onto itself. The results are applicable to Lavrentjev maps of domains conformally equivalent with the unit disc, because auxiliary conformal maps do not affect the results.

Bojarski [2] has proved the equivalence of generalized Lavrentjev maps with generalized homeomorphic solutions[1]) of Beltrami equations

$$(1) \qquad f_{\bar{z}} - \varkappa_1 f_z - \varkappa_2 \overline{f_z} = 0 ,$$

where the functions $\varkappa_1$ and $\varkappa_2$ are continuous in D and $\sup\{|\varkappa_1(z)| + |\varkappa_2(z)|\} < 1$.

Furthermore, the class of Lavrentjev maps is proved to coincide with the class of generalized homeomorphic solutions of

$$(2) \qquad f_{\bar{z}} - \varkappa f_z = 0 , \quad \varkappa \text{ continuous in } D , \quad \sup|\varkappa(z)| < 1 .$$

In this connection Bojarski also establishes the fundamental existence theorem, proved by Lavrentjev in [5]:

[1]) In a domain G a map f is a generalized homeomorphic solution of the equation (1) or (2) respectively, if f is ACL in G and satisfies the equation a.e. in G.

The existence theorem for Lavrentjev maps: *Let* p , Θ *be a continuous distribution of characteristics in* D *such that* p *is bounded. Then there exists a homeomorphism* $f: D \to D$, *such that* f *is a Lavrentjev map with characteristics* p , Θ *and unique up to a linear transformation.*

From the proof it follows, cf. [2], [9]: The map f is in D the uniform limit of a sequence of Lavrentjev maps of the class C^1 ; moreover, their characteristics converge in D uniformly to p , Θ .

By using Lavrentjev's existence theorem we reach in a different manner Bojarski's result on the equivalence of Lavrentjev maps with generalized homeomorphic solutions of (2).

Theorem 2. *Lavrentjev maps coincide with the generalized homeomorphic solutions of* (2).

Proof. Let f be a Lavrentjev map. By Theorem 1, f is quasiconformal. Hence, almost all points of D are regular for f . At regular points there is a one-to-one correspondence between the complex dilatation $\varkappa_f = f_{\bar z}/f_z$ and the characteristics of f by

$$
\begin{aligned}
&(3) \qquad & |\varkappa_f(z)| &= (p(z) - 1)/(p(z) + 1)\\
& & \arg\, \varkappa_f(z) &= 2\Theta(z) - \pi .
\end{aligned}
$$

Denote

$$
\begin{aligned}
&(4) \qquad & |\varkappa(z)| &= (p(z) - 1)/(p(z) + 1)\\
& & \arg\, \varkappa(z) &= 2\Theta(z) - \pi .
\end{aligned}
$$

The function $\varkappa$ is defined for all $z \in D$. If $p(z) = 1$, $\Theta(z)$ is not defined but neither is $\arg \varkappa(z)$ defined, since $|\varkappa(z)| = 0$. Furthermore, $\varkappa$ is continuous in D . By (3) and (4) $\varkappa_f = \varkappa$ a.e. Since f is quasiconformal, it is a generalized homeomorphic solution of (2), where $\varkappa$ is defined by (4).

It still remains to show that every generalized homeomorphic solution of (2) is a Lavrentjev map. For this, let $\varkappa$ be a function satisfying the conditions in (2). It defines a continuous distribution of characteristics in D by the equations

$$
\begin{aligned}
&(5) \qquad & p(z) &= (1 + |\varkappa(z)|)/(1 - |\varkappa(z)|)\\
& & \Theta(z) &= 1/2\, (\arg \varkappa(z) + \pi) ,
\end{aligned}
$$

where p is bounded. By Lavrentjev's existence theorem, there exists a Lavrentjev map f with characteristics p , Θ , unique up to a linear map. Because of (3) and (5), $\varkappa_f = \varkappa$ at regular points. Theorem 1 yields that f is quasiconformal. Hence, $\varkappa_f = \varkappa$ a.e. and f is a generalized homeomorphic solution of

$$f_{\bar z} - \varkappa f_z = 0 .$$

Let f_1 be another generalized homeomorphic solution of the same equation. The combined map $f_1 \circ f^{-1}$ is conformal. Since $f_1 = (f_1 \circ f^{-1}) \circ f$, it is a Lavrentjev map with characteristics (5).

Theorem 3. *If f is a generalized homeomorphic solution of* (2), *then its complex dilatation $\varkappa_f$ equals $\varkappa$ in its whole domain of definition.*

Proof. The dilatation $\varkappa_f$ is defined only at regular points. At such points, there is a one-to-one correspondence between $\varkappa_f$ and the characteristics p, Θ of f. Due to Theorem 2, the same is valid for p, Θ and $\varkappa$ in D. The equations (3) and (5) representing the correspondence are the same, therefore, $\varkappa_f = \varkappa$ at regular points.

Remark. As we have pointed out in the beginning of the section, Theorem 2 and Theorem 3 are valid for domains conformally equivalent with the unit disc.

4. Local dilatation measures

We list in 4.1 the definitions of the local dilatations H, F, D, $\varkappa$ and in 4.2 well-known connections between them. In 4.3 the local dilatations of a Lavrentjev map are examined. The main result established is that H and p are everywhere equal.

4.1. *Definitions of H, F, D, and $\varkappa$.* Let f be a homeomorphism of a domain G and $z \in G$. The *circular dilatation* H of f at z is defined as follows

$$H_f(z) = H(z) = \limsup_{r \to +0} \frac{\max_\alpha |f(z + re^{i\alpha}) - f(z)|}{\min_\alpha |f(z + re^{i\alpha}) - f(z)|} .$$

Let $U_z \subset G$ be a neighbourhood of a point z. The map f carries every quadrilateral Q, $\bar{Q} \subset G$, onto a quadrilateral Q'. Let

$$K(U_z) = \sup_{\bar{Q} \subset U_z} [M(Q')/M(Q)] ,$$

where M denotes the conformal modulus. The *maximal dilatation* of f at z is defined as follows

$$F_f(z) = F(z) = \inf_{U_z} K(U_z) .$$

Let $z \in G$ be a regular point for f. The ratio

$$D_f(z) = D(z) = \frac{\max_\alpha |\partial_\alpha f(z)|}{\min_\alpha |\partial_\alpha f(z)|}$$

is called the *dilatation quotient* of f at z. D is the ratio of the axes of the ellipses which the differential of f at z maps on circles.

In Section 3 we have introduced the *complex dilatation* $\varkappa$, $\varkappa(z) = f_{\bar{z}}(z)/f_z(z)$. It is related to D by the equation

$$D(z) = (1 + |\varkappa(z)|)/(1 - |\varkappa(z)|) .$$

The function $\varkappa$ also defines the direction of the maximal distortion. Since $\partial_\alpha f(z) = f_z(z) + f_{\bar{z}}(z)\, e^{-2i\alpha}$, the function $|\partial_\alpha f(z)|$ assumes its maximal value $|f_z(z)|\,(1 + |\varkappa(z)|)$ for

$$\alpha = 1/2 \arg \varkappa(z) .$$

4.2. *Local dilatation measures of a quasiconformal map.* The local dilatations of a quasiconformal map satisfy the following relations

$$\begin{aligned} F(z_0) = \limsup_{z\to z_0} F(z) &= \lim_{z\to z_0} \operatorname{sup\,ess} F(z) = \limsup_{z\to z_0} D(z) \\ &= \lim_{z\to z_0} \operatorname{sup\,ess} D(z) = \lim_{z\to z_0} \operatorname{sup\,ess} H(z) . \end{aligned}$$

For proofs, see [6], p. 207.

Especially if z is a regular point for f, the definitions of the dilatations yield

$$\begin{aligned} H_{f^{-1}}(f(z)) = D_{f^{-1}}(f(z)) = p_{f^{-1}}(f(z)) &= [1 + |\varkappa_{f^{-1}}(f(z))|]/[1 - |\varkappa_{f^{-1}}(f(z))|] \\ = H_f(z) = D_f(z) = p_f(z) &= [1 + |\varkappa_f(z)|]/[1 - |\varkappa_f(z)|] . \end{aligned}$$

4.3. H *and* p *for a Lavrentjev map.* Let f be a quasiconformal map. Then p is not necessarily defined at every point, cf. 2.3. Hence, wanting to compare $H(z)$ and $p(z)$ at an arbitrary point z, we assume f to be a Lavrentjev map.

By the definition of p, it seems natural that $H_{f^{-1}}(f(z))$ is equal to $p(z)$. This is proved in Theorem 4. We then establish the less obvious result that for Lavrentjev maps p, H, and F coincide at every point. In the proof we use Lemma 5 on the distortion of ellipses. The corresponding lemma on circles was used by Lavrentjev in his proof for the existence theorem.

Theorem 4. *Let f be a Lavrentjev map with characteristics p, Θ. Then $H_{f^{-1}}(f(z)) = p(z)$ at all points.*

Proof. We assume f to be a homeomorphism of the unit disc onto itself. This is no essential restriction, since H and p are local properties and conformal invariants.

Fix a point $z_0 \in D$ and denote $E_h = E_h(p(z_0), \Theta(z_0); z_0)$. In the image plane we construct the closed annulus A_h with $f(z_0)$ as centre and with radii $\max|f(z)-f(z_0)|$ and $\min|f(z)-f(z_0)|$, $z\in E_h$. The ellipse E_h is contained in the preimage $f^{-1}(A_h)$. Let the closure of $E_k^0 = E_k^0(p(z_0), \Theta(z_0); z_0)$ contain $f^{-1}(A_h)$ and have a common boundary point with $f^{-1}(A_h)$. Then

$$\min_{z\in E_k} |f(z)-f(z_0)| = \max_{z\in E_h} |f(z)-f(z_0)| . \tag{2}$$

Combined with (2), the definition of Lavrentjev map yields

$$\lim_{h\to 0} \frac{\max\limits_{z\in E_k} |f(z)-f(z_0)|}{\min\limits_{z\in E_h} |f(z)-f(z_0)|} = 1 .$$

Hence, the module of the annulus with centre $f(z_0)$ and with the radii $\max\limits_{z\in E_k}|f(z)-f(z_0)|$ and $\min\limits_{z\in E_h}|f(z)-f(z_0)|$, tends to zero as $h\to 0$. The same is valid for its subdomain $f(C_h)$, bounded by the images of E_h and E_k. By the quasiconformality of f this also holds for the preimage C_h.

The affine transformation g which carries the ellipses E_h and E_k onto circles maps the ring C_h onto an annulus whose module equals $\log(k/h)$. The map g is $p(z_0)$-quasiconformal, so that

$$\frac{1}{p(z_0)} \log(k/h) \le M(C_h) .$$

Since $M(C_h)\to 0$ as $h\to 0$, it follows that $k/h\to 1$ as $h\to 0$.

The preimage of the circle Y_h with $f(z_0)$ as centre and radius equal to $\max|f(z)-f(z_0)|$, $z\in E_h$, is contained in $\bar{C}_h$ and has at least one point in common with the boundary curves E_h and E_k. Hence

$$\max\left[\frac{ph}{k}, \frac{k}{ph}\right] \le \frac{\max\limits_{f(z)\in Y_h} |f^{-1}(f(z)) - f^{-1}(f(z_0))|}{\min\limits_{f(z)\in Y_h} |f^{-1}(f(z)) - f^{-1}(f(z_0))|} \le \frac{pk}{h},$$

where $p=p(z_0)$. Since $k/h\to 1$ as $h\to 0$, this yields $H_{f^{-1}}(f(z_0)) = p(z_0)$.

In Lemma 5 we need the following result on conformal maps:

Lemma 4. *Let $\emptyset$ be any conformal map from the unit disc D, $\emptyset(0)=0$. For every $\varepsilon>0$ there exists a natural number n, the same for all maps $\emptyset$, such that*

$$\left|\frac{\emptyset(z_1)}{\emptyset(z_2)} : \frac{z_1}{z_2} - 1\right| < \varepsilon \tag{3}$$

for any pair of points z_1, z_2, $0<|z_i|\le 1/n$, $i=1,2$.

Proof. It clearly suffices to establish (3) in the case $\emptyset'(0) = 1$. We combine the relation

$$|\emptyset(z)| = |\emptyset(z) - \emptyset(0)| = \left| \int \emptyset'(z)\, dz \right| \leq |z| \max |\emptyset'(z)| , \quad |z| \leq r < 1 ,$$

with Koebe's inequalities

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |\emptyset'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3} .$$

It follows that

$$\left| \frac{\emptyset(z)}{z} \right| \leq \frac{1 + r}{(1 - r)^3} .$$

Hence, the functions $\tilde{\emptyset}$,

$$\tilde{\emptyset}(z) = \emptyset(z)/z , \quad \tilde{\emptyset}(0) = 1 ,$$

are locally uniformly bounded. Therefore, by a well-known theorem, they constitute a family which is equicontinuous at $z = 0$. From this (3) follows immediately, since $\tilde{\emptyset}(z) \to 1$ as $z \to 0$.

In the following we use the notation $E_{h_1, h_2}(p , \Theta ; z)$ for the ring with boundary curves $E_{h_1}(p , \Theta ; z)$ and $E_{h_2}(p , \Theta ; z)$, $h_1 < h_2$. If the direction Θ has no significance, it is omitted.

Lemma 5. *Let* $p \geq 1$ *and* f *be an* ε_1*-Lavrentjev map of* D *onto itself such that* $f(0) = 0$. *The inequalities*

$$\left| \left| \frac{f(z_1) - f(z_0)}{f(z_2) - f(z_0)} \right| - \left| \frac{z_1 - z_0}{z_2 - z_0} \right| \right| < \eta_p(\varepsilon_1 , \varepsilon)$$

$$(4) \qquad \left| \arg \frac{f(z_1) - f(z_0)}{f(z_2) - f(z_0)} - \arg \frac{z_1 - z_0}{z_2 - z_0} \right| < \eta_p (\varepsilon_1 , \varepsilon)$$

are then satisfied for every $z_0 \in D$ *and any pair of points* z_1 , z_2 *in* $E_{(1-\varepsilon)h, h}(p ; z_0)$ *with* $0 < hp \leq 1 - |z_0|$, $0 \leq \varepsilon < 1$, *where the function* η_p *depends only on* ε_1 *and* ε *and* $\lim \eta_p(\varepsilon_1 , \varepsilon) = 0$ *as* $\varepsilon_1 , \varepsilon \to 0$.

We only prove the former inequality because the latter can be verified in a similar way.

If the lemma is false, there exist real numbers $p \geq 1$ and $\eta_0 > 0$ such that to arbitrarily small $\varepsilon_1 > 0$ and $\varepsilon > 0$, there corresponds a map f and a ring $E_{(1-\varepsilon)h, h}(p ; z_0)$ both satisfying the conditions in the lemma, and the ring containing a pair of points z_1 and z_2 for which

$$(5) \qquad \left| \left| \frac{f(z_1) - f(z_0)}{f(z_2) - f(z_0)} \right| - \left| \frac{z_1 - z_0}{z_2 - z_0} \right| \right| > \eta_0 .$$

Here two possibilities can occur: Either $\limsup h > 0$, as $\varepsilon_1, \varepsilon \to 0$ or $\lim h = 0$. In the former case we apply the following theorem of Lavrentjev [5]:

Let f be an ε_1-Lavrentjev map of D onto itself such that $f(0) = 0$, $f(1) = 1$. Then

$$|f(z) - z| < \lambda(\varepsilon_1) ,$$

where λ depends only on ε_1 and $\lim \lambda = 0$ as $\varepsilon_1 \to 0$.
From this result it follows immediately that (5) cannot hold for all positive values of ε_1.

In the latter case $\lim h = 0$. We note first that there is no loss of generality of we assume $z_0 = f(z_0) = 0$. In order to see this we continue f by setting

$$f(z) = [\overline{f(1/\bar{z})}]^{-1} .$$

Let Z map the disc $|z| < 2$ conformally onto D such that z_0 is carried to the origin. Then

$$\left| 1 - \frac{Z(z_1)}{Z(z_2)} : \frac{z_1 - z_0}{z_2 - z_0} \right| = \left| \frac{\bar{z}_0(z_2 - z_1)}{4 - \bar{z}_0 z_1} \right| < \frac{|z_2 - z_1|}{3} .$$

Since $h \to 0$ as $\varepsilon_1, \varepsilon \to 0$, the ratio

$$\frac{Z(z_1)}{Z(z_2)} : \frac{z_1 - z_0}{z_2 - z_0}$$

is then arbitrarily near to unity for ε_1 and ε small enough. The same argument can be used to show that $f(z_0) = 0$ is no essential restriction.

With the assumptions $z_0 = f(z_0) = 0$ the ineqaality (5) assumes the form

$$\left| \left| \frac{f(z_1)}{f(z_2)} \right| - \left| \frac{z_1}{z_2} \right| \right| > \eta_0 \tag{6}$$

where $z_i \in E_{(1-\varepsilon)h, h}(p ; 0)$, i = 1, 2.

Let $\emptyset$ be any conformal map from D, $\emptyset(0) = 0$. From Lemma 4 we conclude: For any $\eta > 0$ there exists a natural number m such that for all points in $E_{(1-\varepsilon)\frac{1}{n}, \frac{1}{n}}(p ; 0)$

$$\left| \left| \frac{\emptyset(z_1)}{\emptyset(z_2)} \right| - \left| \frac{z_1}{z_2} \right| \right| \le \left| \frac{\emptyset(z_1)}{\emptyset(z_2)} - \frac{z_1}{z_2} \right| < \frac{\eta}{2} \tag{7}$$

for $n \geq m$.

Let m_0 be the number corresponding to η_0. Since $\lim h = 0$, we can choose h such that $m_0 h < 1$. Let G be the f-image of $\{z \mid |z| < m_0 h\}$ and r the distance of its boundary and the origin. The function F,

$$F(z) = \frac{1}{r} f(m_0 hz) ,$$

is an ε_1-Lavrentjev map, which maps D onto a domain T, similar to G and containing the unit disc. Since f satisfies (6), $E_{(1-\varepsilon)\frac{1}{m_0}, \frac{1}{m_0}}(p\,;0)$ contains a pair of points $a_i = z_i/(m_0 h)$, $i = 1, 2$, such that

$$\left| \left| \frac{F(a_1)}{F(a_2)} \right| - \left| \frac{a_1}{a_2} \right| \right| > \eta_0 . \tag{8}$$

Let $\emptyset$ be a conformal map from D onto T such that $\emptyset(0) = 0$, $\emptyset(1) = F(1)$. We apply the result (7) to $\emptyset$ with $z_i = a_i$:

$$\left| \left| \frac{\emptyset(a_1)}{\emptyset(a_2)} \right| - \left| \frac{a_1}{a_2} \right| \right| < \frac{\eta_0}{2} . \tag{9}$$

Combined with the triangle inequality, (8) and (9) yield

$$\left| \left| \frac{\emptyset(a_1)}{\emptyset(a_2)} \right| - \left| \frac{F(a_1)}{F(a_2)} \right| \right| > \frac{\eta_0}{2} . \tag{10}$$

On the other hand,

$$|\emptyset(a_i)| \geq (1 - \varepsilon)/m_0 , \quad i = 1 , 2 , \tag{11}$$

because the domain T contains the unit disc.

Let $\varepsilon < 1/2$ and choose ε_1 so small that, for every $z \in E_{(1-\varepsilon)\frac{1}{m_0}, \frac{1}{m_0}}(p\,;0)$,

$$|\emptyset(z) - F(z)| < \eta_0 / [4 m_0 (1 + 2p + \eta_0)] . \tag{12}$$

This possible because of Theorem 1′ [5]:

Let F be an ε_1-Lavrentjev map and $\emptyset$ a conformal map, both from the unit disc onto a Jordan domain G such that $F(0) = \emptyset(0) = 0$, $F(1) = \emptyset(1)$. Then there exists a function $\lambda(\varepsilon_1, r, d)$, $\varepsilon_1 > 0$, $0 \leq r < 1$, $d > 0$ denoting the distance between $\emptyset(0)$ and the boundary of G, which tends decreasingly to zero with ε_1, such that

$$|\emptyset(z) - F(z)| < \lambda(\varepsilon_1 , r , d)$$

for $|z| \leq r$.

By (9), (11), and (12),

$$\left| \left| \frac{F(a_1)}{F(a_2)} \right| - \left| \frac{\emptyset(a_1)}{\emptyset(a_2)} \right| \right| < \max_{i=1,2} |F(a_i) - \emptyset(a_i)| \frac{1 + |\emptyset(a_1)| / |\emptyset(a_2)|}{|\emptyset(a_2)| - |F(a_2) - \emptyset(a_2)|} < \frac{\eta_0}{2} .$$

This is in contradiction with (10).

An estimate derived from Lemma 5. Let f be an ε_1-Lavrentjev map satisfying the conditions in Lemma 5. We use the inequalities (4) to examine the distortion under f of a ring E bounded by two similar ellipses. Since the inequalities (4) are invariant under similarity transformations of the z- and $f(z)$-plane, we may assume that E has the form $E = E_{h_1, h_2}(p , \pi/2 ; 0)$ and $f(0) = 0$, $f(iph_2) = iph_2$. By (4), any point $z \in E$ then satisfies

$$\text{(i)} \qquad \begin{aligned} &||f(z)| - |z|| < \eta\, p h_2 \\ &|\arg f(z) - \arg z| < \eta \qquad \eta = \eta_p(\varepsilon_1 , 1 - h_1/h_2) . \end{aligned}$$

Because of Lemma 5, we can choose ε_1 and $1 - h_1/h_2$ so small that

$$\text{(i)}' \qquad \eta_p(\varepsilon_1 , 1 - h_1/h_2) < [2(2p + p^2(p^2 - 1))]^{-1} .$$

We also require

$$\text{(i)}'' \qquad h_2/h_1 \leq 2 .$$

We want to show that there exists a ring $\tilde{E} = E_{H_1, H_2}(p , \pi/2 ; 0)$, $\tilde{E} \supset f(E)$, such that

$$H_1 = h_1(1 - C_1\eta)$$

$$H_2 = h_2(1 + C_2\eta) ,$$

where C_1 and C_2 are positive constants.

Denote

$$R(x) = p \sqrt{\frac{1 + \tan^2 x}{p^2 + \tan^2 x}} .$$

The point with polar co-ordinates $r = h_1 R(\alpha)$, $\varphi = \alpha$, lies on the ellipse $E_{h_1}(p , \pi/2 ; 0)$ and may be transformed under f to a point with $r = h_1 R(\alpha) - \eta p h_2$, $\alpha - \eta \leq \varphi_1 \leq \alpha + \eta$. Similarly, the point on $E_{h_2}(p , \pi/2 ; 0)$ in the direction α has polar co-ordinates $r = h_2 R(\alpha)$, $\varphi = \alpha$. Hence, $\tilde{E}$ must cover the points with $r = h_2 R(\alpha) + \eta p h_2$, $\alpha - \eta \leq \varphi_2 \leq \alpha + \eta$. Because of symmetry, we may assume $\alpha \in [-\eta , \pi/2 + \eta]$. In fact, it suffices to consider the cases $\varphi_1 = \alpha + \eta$, $\alpha \in [0 , \pi/2 - \eta]$, $\varphi_2 = \alpha - \eta$, $\alpha \in [\eta , \pi/2]$. This yields the inequalities

$$\text{(ii)} \qquad \begin{aligned} &h_1(1 - C_1\eta)\, R(\alpha + \eta) \leq h_1 R(\alpha) - \eta p h_2 \\ &h_2(1 + C_2\eta)\, R(\alpha - \eta) \geq h_2 R(\alpha) + \eta p h_2 . \end{aligned}$$

The mean value theorem applied to the function R yields

$$|R(\alpha + \eta) - R(\alpha)| \leq \eta p^2(p^2 - 1) .$$

The inequalities (ii) are thus satisfied for

$$C_1 = 2p + p^2(p^2 - 1)$$

$$C_2 = p + p^2(p^2 - 1) .$$

Hence, by (i)′ and (i)″, there exists a ring $\tilde{E} \supset f(E)$ such that

$$H_2/H_1 \leq h_2/h_1 + [12p + 8p^2(p^2 - 1)]\, \eta_p\, (\varepsilon_1 , 1 - h_1/h_2) . \tag{14}$$

Remark 3. Let a sequence of ε_1-Lavrentjev maps g_n converge uniformly towards a Lavrentjev map g. If the conditions (i)′ and (i)″ are fulfilled for $g_n , n \geq n_0$, and a ring $E = E_{h_1 , h_2}(p , \Theta ; z)$, then (14) is applicable to the limit function g and E. This follows from the fact that g satisfies the inequalities (4) with $\eta = \eta_p(\varepsilon_1 , 1 - h_1/h_2)$.

We are now prepared to establish the main result of this section.

Theorem 5. *For a Lavrentjev map $p(z) = H(z)$ at all points.*

Proof. H and p are local properties and conformal invariants. Hence, we can assume that f maps the unit disc D onto itself such that $f(0) = 0$. By the proof for the existence theorem, there exists a sequence of Lavrentjev maps $f_n , f_n \in C^1$, $f_n(0) = 0$, such that $f_n \to f$ uniformly in D. Moreover, the characteristics p_n , Θ_n of f_n converge uniformly to the characteristics p , Θ of f in D. By this we mean that $p_n \to p$ uniformly in D and $\Theta_n \to \Theta$ uniformly in every closed subset of D not containing points at which p has the value 1.

Let $z_0 \in D$, and denote $\Upsilon_h(z_0) = \{z \mid |z - z_0| = h\}$. We want to show that for all sufficiently small values of h, the f-image of $\Upsilon_h(z_0)$ is covered by a ring $E_{a , b}(p(z_0) ; f(z_0))$, where b/a is arbitrarily near to unity.

The first step is to show that to every $\varepsilon_1 > 0$, there corresponds a natural number m such that every map f_n, $n \geq m$, admits a representation

$$f_n = (f_n \circ f_m^{-1}) \circ f_m , \tag{15}$$

where $f_n \circ f_m^{-1}$ is an ε_1-Lavrentjev map.

In other words, it must be proved that the characteristic p^*_{nm} of $f_n \circ f_m^{-1}$ is at most $1 + \varepsilon_1$. We show this by direct computation. For the composed map we have

$$|\varkappa_{f_n \circ f_m^{-1}}(f_m(z))| = \left| \frac{\varkappa_{f_n}(z) - \varkappa_{f_m}(z)}{1 - \overline{\varkappa_{f_n}(z)}\varkappa_{f_m}(z)} \right| \leq \frac{|\varkappa_{f_n}(z) - \varkappa_{f_m}(z)|}{1 - |\varkappa_{f_n}(z)\varkappa_{f_m}(z)|} .$$

Let $K = \sup\limits_n \sup\limits_z p_n(z)$. It follows that

$$|\varkappa_{f_n \circ f_m^{-1}}(f_m(z))| \leq \frac{||\varkappa_{f_n}(z)| - |\varkappa_{f_m}(z)|\, e^{2i(\Theta_m - \Theta_n)}|}{1 - \left(\dfrac{K-1}{K+1}\right)^2}$$

$$\leq \frac{\dfrac{1}{2}\,|p_m(z) - p_n(z)| + 2\,|\Theta_m(z) - \Theta_n(z)|\,\dfrac{K-1}{K+1}}{1 - \left(\dfrac{K-1}{K+1}\right)^2}\,.$$

For every $\varepsilon_1 > 0$ there exists a natural number m such that for all $n \geq m$ and every $z \in D$ either

$$(16)\qquad \begin{aligned} |p_n(z) - p(z)| &\leq \frac{\varepsilon_1}{2(2+\varepsilon_1)}\left[1 - \left(\frac{K-1}{K+1}\right)^2\right] \\ |\Theta_n(z) - \Theta(z)| &\leq \left(\frac{K+1}{K-1}\right)\frac{\varepsilon_1}{8(2+\varepsilon_1)}\left[1 - \left(\frac{K-1}{K+1}\right)^2\right] \end{aligned}$$

or

$$(17)\qquad |p_n(z) - 1| \leq \frac{\varepsilon_1}{3}\,.$$

Hence, if (16) holds

$$|\varkappa_{f_n \circ f_m^{-1}}(f_m(z))| \leq \varepsilon_1/(2+\varepsilon_1)$$

and so

$$p^*_{nm}(f_m(z)) \leq 1 + \varepsilon_1\,.$$

In the case of (17), we assume $\varepsilon_1 < 1$ and use the inequality

$$p^*_{nm}(f_m(z)) \leq p_n(z)\, p_{f_m^{-1}}(f_m(z)) = p_n(z)\, p_m(z) \leq 1 + \varepsilon_1\,,$$

cf. 1.4. The representation (15) with the desired properties has thus been established.

Fix a point $z_0 \in D$. Given an ε, $1 > \varepsilon > 0$, we show that for $n \geq n_0$, $0 < h \leq h_0$, n_0 and h_0 depending only on ε, the f_n-image of $Y_h(z_0)$ is covered by a ring $E_{H_1,H_2}(p(z_0); f_n(z_0))$, where $H_2/H_1 \leq 1 + \varepsilon$.

Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ so small that

$$(18)\qquad \begin{aligned} &(1+\varepsilon_2)^2/(1-\varepsilon_2) \leq 1 + \varepsilon/2\,, \\ &\eta_p(\varepsilon_1, d) \leq \varepsilon/[24p + 16p^2(p^2-1)] \quad \text{for} \quad 0 \leq d \leq 1 - \frac{1-\varepsilon_2}{(1+\varepsilon_2)^2}\,. \end{aligned}$$

Together with (16) or (17) we finally require

$$(19)\qquad |p_n(z) - p(z)| \leq \varepsilon_2\,,$$

$z \in D$, for all sufficiently large values of n.

Let n_0 be an integer such that (19), and (16) or (17) are satisfied for $n \geq n_0$, and fix a map f_m, $m \geq n_0$. The reasoning in Theorem 4 is applicable to f_m^{-1}, since f_m^{-1} is a Lavrentjev map. It follows that there exists an $h_0 > 0$ such that the f_m-image of any circle $Y_h(z_0)$, $h \leq h_0$, is covered by a ring

$$E_1 = E_{k_1, k_2}(p_{f_m^{-1}}(f_m(z_0)), \Theta_{f_m^{-1}}(f_m(z_0)); f_m(z_0)),$$

where

$$(19)' \qquad k_2/k_1 \leq (1 + \varepsilon_2)/(1 - \varepsilon_2).$$

Since $p_{f_m^{-1}}(f_m(z_0)) = p_m(z_0)$, the ring E_1 lies in a ring

$$E_2 = E_{h_1, h_2}(p(z_0), \Theta_{f_m^{-1}}(f_m(z_0)); f_m(z_0)),$$

where

$$\frac{h_2}{h_1} \leq \frac{k_2}{k_1} \max \left[\frac{p_m(z_0)}{p(z_0)}, \frac{p(z_0)}{p_m(z_0)}\right].$$

By (19) and (19)′

$$(20) \qquad h_2/h_1 \leq (1 + \varepsilon_2)^2/(1 - \varepsilon_2).$$

The functions $f_n \circ f_m^{-1}$ are ε_1-Lavrentjev maps for $n \geq n_0$. The estimate (14), combined with (20) and (18), yields the result that the $f_n \circ f_m^{-1}$-image of E_2 is covered by a ring

$$E_3 = E_{H_1, H_2}(p(z_0); f_n(z_0)),$$

where

$$H_2/H_1 \leq 1 + \varepsilon.$$

The same is valid for the f_n-images of the circle $Y_h(z_0)$, $h \leq h_0$, $n \geq n_0$, since $f_n(Y_h(z_0)) \subset (f_n \circ f_m^{-1})(E_2)$. As $n \to \infty$, the functions f_n converge uniformly towards f. Remark 3 with $g_n = f_n \circ f_m^{-1}$ implies that also $(f \circ f_m^{-1})(E_2) \subset E_{H_1, H_2}(p(z_0); f(z_0))$, where $H_2/H_1 \leq 1 + \varepsilon$. Since $f(Y_h(z_0)) \subset (f \circ f_m^{-1})(E_2)$, the theorem follows.

The following result is an immediate conclusion of the above theorem.

Corollary. *For a Lavrentjev map* $F(z) = p(z)$ *at all points.*

Proof. By the above theorem, H equals p, so H is continuous. The equality

$$F(z_0) = \limsup_{z \to z_0} \operatorname{ess} H(z)$$

therefore yields $F(z_0) = H(z_0) = p(z_0)$ for every z_0.

5. Counterexamples

We show in 5.1—5.2 that neither the composition of two Lavrentjev maps nor the inverse of a Lavrentjev map need be a Lavrentjev map, and in 5.3 that Lavrentjev maps form a proper subset of generalized Lavrentjev maps. We conclude the paper with some examples of non-differentiable Lavrentjev maps.

5.1. *Composition of Lavrentjev maps.* Let us consider the map f_1,

$$f_1(z) = z\, e^{(i \log |z|)^{1/3}} .$$

From

$$(f_1)_z(z) = \left[1 + \frac{i}{6} (\log |z|)^{-2/3}\right] e^{i (\log |z|)^{1/3}}$$

$$(f_1)_{\bar z}(z) = \frac{i}{6} \frac{z}{\bar z} (\log |z|)^{-2/3} e^{i (\log |z|)^{1/3}}$$

we conclude that $\varkappa_{f_1} = (f_1)_{\bar z}/(f_1)_z$ is continuous also at $z = 0$. Hence, f_1 is a Lavrentjev map.

We combine f_1 with the affine map f_2,

$$f_2(w) = u + i\,Kv\,, \quad w = u + iv\,, \quad K > 1\,.$$

From

$$\varkappa_{f_2} \equiv (1 - K)/(1 + K)\,,$$

$$|\varkappa_{f_1}| \to 0 \quad \text{as} \quad z \to 0$$

we conclude that

$$\varkappa_{f_2 \circ f_1}(z) = \frac{1 - K}{1 + K} e^{-2\, i (\log |z|)^{1/3}} + \varepsilon(z)\,,$$

where $\varepsilon(z) \to 0$ as $z \to 0$. Hence, $\varkappa_{f_2 \circ f_1}$ is discontinuous at the origin, and $f_2 \circ f_1$ is not a Lavrentjev map.

5.2. *Inverses of Lavrentjev maps.* For the homeomorphism f,

(1) $$f(z) = [x + i\,Ky]\, e^{i (\log |z|)^{1/3}}\,,$$

we have

$$f_z(z) = \frac{1}{2}\left[1 + K + \frac{i}{3z}\,[x + iKy]\,(\log|z|)^{-2/3}\right] e^{i(\log|z|)^{1/3}}\,,$$

$$\varkappa_f(z) = \frac{1 - K + i(x + iKy)(\log|z|)^{-2/3}(3\bar{z})^{-1}}{1 + K + i(x + iKy)(\log|z|)^{-2/3}(3z)^{-1}}\,.$$

Since $\varkappa_f$ is continuous, f is a Lavrentjev map.

For the complex dilatation of the inverse map f^{-1} we have

$$\varkappa_{f^{-1}}(f(z)) = -\,\varkappa_f(z)\,e^{2i\arg f_z(z)}\,.$$

As $z \to 0$, $\varkappa_f \to (1 - K)/(1 + K)$ and $\arg f_z \to -\infty$. Hence, $\varkappa_{f^{-1}}$ is discontinuous at $z = 0$, and f^{-1} is not a Lavrentjev map.

5.3. *Lavrentjev maps and generalized Lavrentjev maps.* The inverse of a Lavrentjev map f is always a generalized Lavrentjev map. This is seen as follows. Let z_0 and $\xi_0 = f(z_0)$ be regular points for f and f^{-1}, respectively. From

$$\overline{f_{\xi}^{-1}(\xi_0)} = \frac{1}{J_f(z_0)}\,f_z(z_0)\,, \quad f_{\bar{\xi}}^{-1}(\xi_0) = \frac{-1}{J_f(z_0)}\,f_{\bar{z}}(z_0)\,,$$

it follows that f^{-1} is a generalized homeomorphic solution of the equation

$$f_{\bar{\xi}}^{-1}(\xi) + \varkappa_f(f^{-1}(\xi))\,\overline{f_{\xi}^{-1}(\xi)} = 0\,.$$

By 3 (1), f^{-1} is a generalized Lavrentjev map.

The inverse of (1) is thus a generalized Lavrentjev map but not a Lavrentjev map.

5.4. *Non-differentiable Lavrentjev maps.* The map f,

$$f(z) = z/(1 - \log|z|)\,, \tag{2}$$

possesses a continuous complex dilatation

$$\varkappa_f(z) = z/[2\bar{z}(3/2 - \log|z|)]\,.$$

The inverse map satisfies

$$\varkappa_{f^{-1}}(f(z)) = -\,\varkappa_f(z)\,,$$

so that also f^{-1} is a Lavrentjev map. It is not differentiable at $z = 0$, for $\lim|z/f(z)| = \infty$ as $z \to 0$. From (2) it follows that $p_{f^{-1}}(0) = 1$,

Slightly modifying the above example we also get a Lavrentjev map which is non-differentiable at a point z at which $p(z) > 1$. We set

$$f(z) = [Kx + iy]/(1 - \log |z|) ,$$

and conclude that the inverse of f is a Lavrentjev map.

As above, we show that f^{-1} is non-differentiable at the origin. In this case, $p_{f^{-1}}(0) = K$.

A third example is given by the Lavrentjev map f_1 in 5.1. It has no partial derivatives at $z = 0$. However it satisfies the equations

$$\limsup_{z \to 0} |f_1(z)/z| = \liminf_{z \to 0} |f_1(z)/z| = 1 .$$

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