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DISCRETE OPEN MAPPINGS ON MANIFOLDS

BY

JUSSI VÄISÄLÄ

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Discrete open mappings on manifolds

1. *Introduction.* The purpose of this paper is to give a new proof for the following recent result of Černavskii [2], [3]: If f is an open mapping of an n-manifold into an n-manifold such that each point-inverse consists of isolated points, then f is a local homeomorphism except for a set whose topological dimension is at most $n-2$. Our proof is more direct and elementary than that of Černavskii, who obtained this result as a corollary of a more general result concerning mappings whose range space was not required to be a manifold. In particular, we avoid completely the use of the rather deep theory of Smith on periodic homeomorphisms of manifolds. Instead, we will make use of the topological index (= degree) of a mapping. We will also give an application concerning light open mappings.

2. *Terminology, notation and preliminary results.* All topological spaces considered in this paper are assumed to be Hausdorff. All manifolds are assumed to be connected and to have a countable base. All mappings are assumed to be continuous. If X is a space and if $E \subset A \subset X$, we let $\partial_A E$ denote the boundary of E with respect to A, and we abbreviate $\partial_X E$ to ∂E. Let f be a mapping of a space X into a space Y. It is *open* (*closed*) if the image of every open (closed) subset of X is open (closed) in Y. It is *light* if for each $y \in Y$, the inverse-image $f^{-1}(y)$ is totally disconnected. It is *discrete* if each point-inverse is discrete, i.e. consists of isolated points. It is *proper* if the inverse-image of each compact subset of Y is compact. If X and Y are locally compact, then a mapping $f: X \to Y$ is proper if and only if f is closed and each point-inverse is compact. The *branch set* B_f of f is the set of all points in X at which f fails to be a local homeomorphism. The *multiplicity* $N(x, f)$ of f at a point $x \in X$ is the number of points in $f^{-1}f(x)$, and we set $N(f) = \sup N(x, f)$ over all $x \in X$. The set of all points $x \in X$ for which $N(x, f) \leq i$ is denoted by $K_i(f)$. It is well known that if f is open, then $N(x, f)$ is a lower semicontinuous function of x. In other words, for each $a \in X$ and for each integer $k \leq N(a, f)$ there exists a neighborhood U of a such that $N(x, f) \geq k$ for all $x \in U$. In particular, if $N(a, f) < \infty$, then $N(x, f) \geq N(a, f)$ in some neighborhood of a. From this it follows that the sets $K_i(f)$ are closed.

By a *domain* in a space we mean an open connected non-empty subset. A subset A of a space X is said to *separate* X *locally* at a point x if there is a neighborhood U of x such that for each neighborhood V of x such that $V \subset U$, $V - A$ is not connected.

Lemma 2.1. *Let* X *be a locally connected space, and let* A *be a closed subset of* X *such that* $\operatorname{int} A = \emptyset$ *and* $X - A$ *is not connected. If* F *is the closure of the set of all points at which* A *separates* X *locally, then* $X - F$ *is not connected.*

Proof. Since $X - A$ is not connected, we can express it as the union of two disjoint non-empty open sets, U_1 and U_2. Denote $V_i = (\operatorname{int} \bar{U}_i) - F$. Then it is easy to see that $X - F = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and $U_i \subset V_i$. Hence $X - F$ is not connected.

Lemma 2.2. *If* X *is an n-manifold and if* A *is a closed subset of* X *such that* $\dim A = n - 1$, *then* $\operatorname{int} A = \emptyset$, *and* A *separates* X *locally at some point.*

Proof. This is almost the same as Proposition I.4.9.b in Borel [1]. However, Borel's definition for local separation is slightly different from ours. Anyway, there is a domain D such that $D - A$ is not connected. The assertion follows then from Lemma 2.1.

3. *Discrete open mappings on locally compact spaces.* In order to prove that $\dim B_f \leq n - 2$ for discrete open mappings of n-manifolds, we first show $\dim B_f < n$, that is, $\operatorname{int} B_f = \emptyset$. In fact, this assertion is true for all locally compact spaces.

Lemma 3.1. *Let* $f: X \to Y$ *be open and let* $N(f) = k < \infty$. *Then* $N(x, f) < k$ *for every* $x \in B_f$.

Proof. Suppose $N(x_1, f) = k$. Let $f^{-1}f(x_1) = \{x_1, \ldots, x_k\}$ and choose disjoint neighborhoods U_i of x_i. Then the set $V = U_1 \cap f^{-1}(fU_1 \cap \ldots \cap fU_k)$ is a neighborhood of x_1 such that $f \mid V$ is injective. Since f is open, it is a local homeomorphism at x_1.

Theorem 3.2. *If* X *is locally compact and if* $f: X \to Y$ *is discrete and open, then* $\operatorname{int} B_f = \emptyset$.

Proof. Suppose that $\operatorname{int} B_f \neq \emptyset$. Then B_f contains an open set U with compact closure. The restriction $g = f \mid U$ is an open mapping for which $N(x, g) < \infty$ for all $x \in U$. Hence U is the union of the sets $K_i(g)$. From the Baire theorem [6, p. 200] it follows that $\operatorname{int} K_i(g) = V \neq \emptyset$ for some i. Since $N(g \mid V) \leq i$, Lemma 3.1 implies that there is a point x in V at which $g \mid V$, and hence f, is a local homeomorphism. But $x \in U \subset B_f$, and we have reached a contradiction.

4. *The topological index.* We recall the definition and the basic properties of the topological index (= local degree) of a mapping. For any locally

compact space X, we let $H^p(X)$ be the p-dimensional Alexander-Spanier cohomology group of X, with coefficients in the group Z of the integers, and with compact supports. If X is an n-manifold, then $H^n(X)$ is isomorphic to Z or $Z/2Z$, according as X is orientable or not [1, I.4.3 and I.4.8]. An orientable manifold X together with a preferred generator g_X of $H^n(X)$ is called an *oriented manifold.* If X is an oriented manifold, and if D is a domain in X, then the standard homomorphism $j : H^n(D) \to H^n(X)$ is an isomorphism. Setting $g_D = j^{-1}(g_X)$, we obtain a simultaneous orientation of all domains in X.

Let X and Y be oriented n-manifolds, and let $f : X \to Y$ be a mapping. Given a domain D in X, a point $y \in Y$ is called *(f, D)-admissible* if there is a connected neighborhood U of y such that f defines a proper mapping $f_1 : D \cap f^{-1}U \to U$. For example, if $\bar{D}$ is compact, each point in $Y - f\partial D$ is (f, D)-admissible. If f defines a proper mapping $D \to fD$, each point in $Y - \partial fD$ is (f, D)-admissible.

For each (f, D)-admissible point y, we can define the *topological index* $\mu(y, f, D)$ as follows: Take any neighborhood U of y as above. Then there is an integer k such that $jf_1^*(g_U) = kg_D$, where $j : H^n(D \cap f^{-1}U) \to H^n(D)$ is the standard homomorphism. This integer is independent of the choice of U, and is defined to be $\mu(y, f, D)$. If f is proper, then $\mu(y, f, X)$ is constant for all $y \in Y$, and is denoted by $\mu(f)$. If f defines a homeomorphism $D \to fD$, then $\mu(y, f, D) = \pm 1$ for all $y \in fD$. If f is a local homeomorphism at x, there is a connected neighborhood D of x such that f defines a homeomorphism $D \to fD$. The topological index $\mu(f(x), f, D)$ is then independent of the choice of D, and is denoted by $i(x, f)$. Thus $i(x, f)$ is $+1$ or -1, according as f is sense-preserving or sense-reversing at x. It is constant in each component of $X - B_f$.

The following well-known property of the topological index is needed later:

Lemma 4.1. *Let y be an (f, D)-admissible point which does not belong to $f(D \cap B_f)$. Then the set $A = D \cap f^{-1}(y)$ is finite, and*

$$\mu(y, f, D) = \sum_{x \in A} i(x, f) .$$

5. *Discrete open mappings on manifolds.* In this section we prove the theorem of Černavskii, mentioned in the introduction. The proof is preceded by three lemmas.

Lemma 5.1. *Let X be locally compact and locally connected, and let $f : X \to Y$ be light. Then each point in X has arbitrarily small connected neighborhoods U such that f defines a closed mapping $U \to fU$.*

Proof. Let $x \in X$ and let V be a neighborhood of x. Since f is light, there exists a neighborhood W of x such that $W \subset V$, $\bar{W}$ is compact and $\partial W \cap f^{-1}f(x) = \emptyset$. Choose a neighborhood D of $f(x)$ such that $D \cap f\partial W = \emptyset$, and let U be the x-component of $f^{-1}D$. Then $U \subset W$, and the mapping $U \to fU$ is closed.

Lemma 5.2. *Let X be an n-manifold, and let U_1, U_2 be disjoint domains in X such that $\partial U_1 = \partial U_2$ and $\bar{U}_1 \cup \bar{U}_2 \neq X$. Then there is no homeomorphism of $\bar{U}_1$ onto $\bar{U}_2$ which keeps the points of ∂U_1 fixed.*

Proof. Assume that there is a homeomorphism $f : \bar{U}_1 \to \bar{U}_2$ which satisfies the condition of the lemma. Then f defines a homeomorphism $g : U_1 \to U_2$, and we obtain a commutative diagram

$$\begin{array}{ccc} H^{n-1}(\partial U_1) & \xrightarrow{\delta_1} & H^n(U_1) \\ \uparrow \text{id} & & \uparrow g^* \\ H^{n-1}(\partial U_2) & \xrightarrow{\delta_2} & H^n(U_2) \end{array}$$

On the other hand, $H^n(U_1 \cup U_2)$ can be written as $H^n(U_1) \oplus H^n(U_2)$, where the projection mappings $r_i^* : H^n(U_1 \cup U_2) \to H^n(U_i)$ are the homomorphisms induced by the inclusions $r_i : U_i \to U_1 \cup U_2$. Moreover, since $\partial U_1 = \partial(U_1 \cup U_2)$, there is a coboundary homomorphism $\delta : H^{n-1}(\partial U_1) \to H^n(U_1 \cup U_2)$ such that the diagrams

$$\begin{array}{ccc} H^{n-1}(\partial U_1) & \xrightarrow{\delta} & H^n(U_1 \cup U_2) \\ & \searrow^{\delta_i} & \downarrow r_i^* \\ & & H^n(U_i) \end{array}$$

are commutative for $i = 1, 2$. Thus δ is given by $\delta(a) = (\delta_1(a), \delta_2(a))$. Since $\delta_1 = g^*\delta_2$, Im δ does not contain elements of the form $(b, 0)$ where $b \neq 0$. On the other hand, we have the exact cohomology sequence

$$H^{n-1}(\partial U) \xrightarrow{\delta} H^n(U_1 \cup U_2) \longrightarrow H^n(\bar{U}_1 \cup \bar{U}_2)$$

Here $H^n(\bar{U}_1 \cup \bar{U}_2) = 0$ by [1, I.4.3]. Hence δ is surjective, and we obtain a contradiction.

Lemma 5.3. *Let X and Y be oriented n-manifolds, and let $f : X \to Y$ be closed and open satisfying $N(x, f) < \infty$ for all $x \in X$. Let A be a closed subset of X such that $X - A$ is not connected and $N(x, f) = 1$ for $x \in A$. Then each component U of $X - A$ is mapped onto a component V of*

$Y - fA$. Moreover, the induced mapping $U \to V$ is closed, and $\mu(y, f, U) = \pm 1$ for all $y \in V$.

Proof. Clearly fU is contained in a component V of $Y - fA$. To show that the mapping $g : U \to V$ defined by f is closed, let E be closed in U, that is, $E = \bar{E} \cap U$. Then fE is contained in $f\bar{E} \cap V$, which is closed in V. But since $V \cap fA = \emptyset$, $f\bar{E} \cap V = fE$. Hence g is closed. Since it is also open, $fU = V$. Moreover, since point-inverses are finite, g is proper.

We next show that f defines a homeomorphism $h : \partial U \to \partial V$. Since $f \mid A$ is injective and $\partial U \subset A$, it suffices to show that $f\partial U = \partial V$. Since $V \cap f\partial U \subset V \cap fA = \emptyset$, $f\partial U \subset \partial V$. On the other hand, since f is closed, $f\bar{U} = \bar{V}$, whence $\partial V = f\bar{U} - fU \subset f\partial U$.

To show that $\mu(y, f, U) = \pm 1$, it suffices to prove that $g^* : H^n(V) \to H^n(U)$ is surjective. Consider the diagram

$$\begin{array}{ccccc} H^{n-1}(\partial U) & \xrightarrow{\ \delta\ } & H^n(U) & \longrightarrow & H^n(\bar{U}) \\ \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle g^*} & & \\ H^{n-1}(\partial V) & \longrightarrow & H^n(V) & & \end{array}$$

Here $H^n(\bar{U}) = 0$ by [1, I.4.3], whence δ is surjective. Since h^* is an isomorphism, g^* is surjective, and the lemma is proved.

Theorem 5.4. *If X and Y are n-manifolds and if $f : X \to Y$ is discrete and open, then* $\dim B_f \leq n - 2$.

Proof. It suffices to prove that $\dim (B_f \cap D) \leq n - 2$ for each domain D in X such that D and fD are orientable and $\bar{D}$ is compact. Hence we may assume that X and Y are oriented and that $N(x, f) < \infty$ for all $x \in X$. From Theorem 3.2 it follows that $\dim B_f < n$. Suppose that $\dim B_f = n - 1$. Let A be the closure of the set of all points in B_f at which B_f locally separates X. Then $A \neq \emptyset$ by Lemma 2.2. Since A is the union of the closed sets $A \cap K_i(f)$, it follows from the Baire theorem that there is an integer i such that the interior G of $A \cap K_i(f)$ with respect to A is not empty. Let U be an open set in X such that $G = U \cap A$. Since $G \subset K_i(f)$, $N(x, f \mid U) \leq i$ for $x \in G$. Let x_1 be a point in G at which $N(x, f \mid U)$ attains its maximum, and let $x_2, \ldots, x_k$ be the other points of $U \cap f^{-1}f(x_1)$. Choose disjoint neighborhoods $U_j \subset U$ of x_j, and set $V = U_1 \cap f^{-1}(fU_1 \cap \ldots \cap fU_k)$. Then $N(x, f \mid V) = 1$ for $x \in V \cap A$.

Since $x_1 \in A$, there is a point p in $V \cap B_f$ at which B_f separates X locally. Choose a connected neighborhood D of p such that $D \subset V$,

$D - B_f$ is not connected, and f defines a closed mapping $D \to fD = D'$. This is possible by Lemma 5.1. By Lemma 2.1, $D - A$ is not connected. If W is a component of $D - A$, it follows from Lemma 5.3 that f defines a closed mapping g of W onto a component W' of $D' - f(A \cap D)$ and that $\mu(y, f, W) = \pm 1$ for $y \in W'$.

We show that g is a homeomorphism. If this were not the case, there would exist two distinct points z_1, z_2 in W such that $f(z_1) = f(z_2)$. Choose disjoint neighborhoods $Q_i \subset W$ of z_i. Since $\operatorname{int} fB_f = \emptyset$ by [4, 2.1], there is a point $y \in fQ_1 \cap fQ_2 - fB_f$. Then $g^{-1}(y)$ contains at least two points, and

$$\mu(y, f, W) = \sum_{x \in g^{-1}(y)} i(x, f)$$

by Lemma 4.1. But $W - B_f$ is connected, since otherwise B_f would separate X locally at some point of W (Lemma 2.1). Hence $i(x, f)$ is constant for $x \in W - B_f$. This implies $|\mu(y, f, W)| \geq 2$, which contradicts the previous result. Thus g is a homeomorphism.

Since $D \cap B_f \neq \emptyset$, there must exist two components, say W_1 and W_2, of $D - A$ which are mapped onto the same component W' of $D' - f(A \cap D)$. Since f defines the homeomorphisms $f_i : W_i \cup \partial_D W_i \to W' \cup \partial_{D'} W'$, we obtain a homeomorphism $f_2^{-1} f_1 : W_1 \cup \partial_D W_1 \to W_2 \cup \partial_D W_2$, which keeps the points of $\partial_D W_1$ fixed. From Lemma 5.2 it follows that $D - A$ has only the components W_1 and W_2. But this means that f defines a homeomorphism $W_1 \cup \partial_D W_1 \to D'$. This leads to a contradiction, because $W_1 \cup \partial_D W_1$ would be both open and closed in D. The theorem is proved.

Theorem 5.5. (Cf. Cernavskii [2, Theorems 1 and 2.]) *Let X and Y be n-manifolds, and let $f : X \to Y$ be discrete, open and closed. Then $N(f) < \infty$, and $N(x, f) = N(f)$ for all $x \in X - f^{-1}fB_f$, where the exceptional set $f^{-1}fB_f$ has dimension at most $n - 2$. If X and Y are oriented, then $N(f) = |\mu(f)|$.*

Proof. The inequality $\dim f^{-1}fB_f \leq n - 2$ follows directly from the preceding theorem and from [4, 2.1]. We next show that $N(x, f) < \infty$ for each $x \in X$. Suppose that $f^{-1}(y)$ is infinite for some $y \in Y$. Choose metrics d and d_1 on X and Y, respectively. Arrange the points of $f^{-1}(y)$ into a sequence $x_1, x_2, \ldots$ and choose points $z_i \in X$ such that $d(z_i, x_i) < 1/i$, $d_1(f(z_i), y) < 1/i$ and $f(z_i) \neq y$. Then the points z_i form a closed set whose image is not closed. This contradicts the closedness of f and proves that the point-inverses are finite.

We next show that $N(x, f)$ is constant for $x \in U = X - f^{-1}fB_f$. Since U is connected, it suffices to prove that $N(\, , f)$ is continuous in U. It is lower semicontinuous, since f is open. Let $x_1 \in U$, let $N(x_1, f) = k$, and let $f^{-1}f(x_1) = \{x_1, \ldots, x_k\}$. Choose disjoint neighborhoods U_i of x_i such that the restrictions $f|U_i$ are injective. Since f is closed, the set $V = Y - f(X - (U_1 \cup \ldots \cup U_k))$ is a neighborhood of $f(x_1)$. Then $N(x, f) \leq k$ for $x \in U_1 \cap f^{-1}V$. Thus $N(\ , f)$ is also upper semicontinuous in U.

To prove that $N(f) = N(x, f)$ for $x \in U$, it suffices to show that for each $x \in f^{-1}fB_f$ there is $x_1 \in U$ such that $N(x, f) \leq N(x_1, f)$. This follows directly from the lower semicontinuity of $N(\ , f)$ and from $\operatorname{int} f^{-1}fB_f = \emptyset$.

Assume now that X and Y are oriented. Choose a point $z \in U$. Then Lemma 4.1 implies

$$\mu(f) = \sum_{x \in f^{-1}f(z)} i(x, f) .$$

Since U is connected, $i(x, f)$ is constant for $x \in U$, and we obtain $|\mu(f)| = N(z, f) = N(f)$. The theorem is proved.

Remark. The mapping f in Theorem 5.4 is a pseudo-covering map in the sense of Church and Hemmingsen [4].

6. *An application.* Let f be a mapping of an n-manifold X into an n-manifold Y. We let E_f denote the set of all points $x \in X$ which are not isolated in $f^{-1}f(x)$. For $n = 2$, Stoïlow [7, p. 113] has proved that if f is light and open, then $E_f = \emptyset$, i.e., f is discrete. It is not known whether this is true for $n > 2$.

Theorem 6.1. *Let X and Y be n-manifolds, and let $f: X \to Y$ be light and open. If $\dim \bar{E}_f \leq n - 2$ and if $\dim f\bar{E}_f < n$, then $E_f = \emptyset$.*

Proof. The restriction of f to $X - \bar{E}_f$ is discrete and open. Hence, by Theorem 5.4, $\dim (B_f - \bar{E}_f) \leq n - 2$. Since $\dim \bar{E}_f \leq n - 2$, this implies $\dim B_f \leq n - 2$. Moreover, $\dim f(B_f - \bar{E}_f) \leq n - 2$ by [4, 2.1]. Hence $\dim fB_f < n$. The theorem follows now directly from [4, 2.2] or [8, Corollary of 5.2].

Corollary. *If $f: X \to Y$ is light and open, then either $E_f = \emptyset$ or $\dim f\bar{E}_f \geq n - 1$.*

Proof. Suppose that $\dim f\bar{E}_f \leq n - 2$. By [5, VI 7, p. 91], $\dim \bar{E}_f \leq n - 2$. Hence $E_f = \emptyset$ by the above theorem.

7. *Remark.* All results of this paper remain true if the word »manifold» is replaced by »cohomology manifold over Z», in the sense of Borel [1].

University of Helsinki
Helsinki, Finland

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