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HOROCYCLIC BOUNDARY PROPERTIES OF MEROMORPHIC FUNCTIONS

BY

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1. Introduction

Myrberg's approximation theorem [11] asserts that certain automorphic functions in the unit disk D come arbitrarily close to every complex value on every curve that intersects the unit circle, at any point of a certain set of measure 2π, at a nonzero angle. I have obtained [2] analogous results for some normal functions. Hedlund [8] has established an approximation theorem for various automorphic functions in which the curves involved are arcs of circles tangent to the unit circle instead of curves that intersect the unit circle at an angle different from zero. This motivated me to attempt to find some horocyclic versions of my results for normal functions, and this in turn led to the more fundamental investigation of horocyclic boundary behavior of meromorphic functions presented in this paper.

In Section 2 we gather together for convenient reference the definitions, terminology, and notation that will be required.

Section 3 contains our results. We first prove a fundamental lemma (Lemma 1) which asserts that a particular kind of region has a rectifiable boundary. We then show that for a function meromorphic in D, almost every Plessner point is a right horocyclic Plessner point. This leads to an extension (Corollary 1) of Plessner's theorem (see [14, p. 217]). We also show (Theorem 2) that almost every right horocyclic Fatou point is a Fatou point, and this leads to a horocyclic version (Corollary 3) of Priwalow's uniqueness theorem [14, p. 210]. We obtain a generalization (Corollary 5) of Meier's category-theoretical analogue [10, p. 330, Theorem 5] of Plessner's theorem. We also prove a horocyclic version (Theorem 8) of Noshiro's generalization [13, p. 74, Remark] of Meier's two-chord theorem for holomorphic functions. We conclude with two tangential approximation theorems (Theorems 9 and 10) for normal functions.

2. Terminology and notation

We denote the unit circle in the complex plane by Γ and the open unit disk by D. If $0 < r < 1$, then D_r stands for the open disk $|z| < r$. The extended complex plane will be called Ω. We take $\omega_1, \omega_2, \ldots, \omega_n, \ldots$

to be those points of Ω with both real and imaginary parts rational, enumerated in a specific sequence. When we write $A \subset B$, we mean that A is a proper subset of B.

Suppose that S is a subset of Γ. When we speak of »almost every point of S», we mean every point of S with the exception of a set of linear Lebesgue measure zero; and when we speak of »nearly every point of S», we mean every point of S with the exception of a linear set of first Baire category.

By an arc at a point $\zeta \in \Gamma$ we mean a continous curve $\Lambda : z = z(t)$ $(0 \leqq t < 1)$ such that $|z(t)| < 1$ for $0 \leqq t < 1$ and $\lim_{t \to 1} z(t) = \zeta$. A terminal subarc of Λ means a subarc of Λ of the form $z = z(t)$ $(t_0 \leqq t < 1)$, where $0 \leqq t_0 < 1$. By an admissible arc at ζ we mean an arc at ζ having a tangent at ζ different from the tangent to Γ at ζ.

A circle internally tangent to Γ at a point $\zeta \in \Gamma$ is called a horocycle at ζ, and will be denoted by $h_r(\zeta)$, where r $(0 < r < 1)$ is the radius of the horocycle. The point ζ itself is not reckoned as belonging to $h_r(\zeta)$. The right half of $h_r(\zeta)$ (the terms »right» and »left» throughout this paper are relative to an observer at the origin looking out toward Γ) is denoted by $h_r^+(\zeta)$, and is called the right horocycle at ζ with radius r. Note that $h_r^+(\zeta)$ is an arc at ζ, and includes its initial point in D but not its terminal point ζ. The left horocycle $h_r^-(\zeta)$ is defined analogously. We shal often, in what follows, define formally only the right one of a pair of entities when the definition of the left one is completely analogous. Similarly, we shall state theorems only for right entities when the corresponding theorems for left entities are obviously also valid.

Suppose that $0 < r_1 < r_2 < 1$, $0 < r_3 < 1$, and that r_3 is so large that the circle $|z| = r_3$ intersects both of the horocycles $h_{r_1}(\zeta)$ and $h_{r_2}(\zeta)$. Then r_1, r_2, r_3 taken in that order will be called an admissible triple of numbers. The symbols $\bigcup_{r_1, r_2, r_3}$, $\bigcap_{r_1, r_2, r_3}$ mean that the union or intersection is to be taken over all admissible triples of numbers. Sometimes, however, only admissible triples of rational numbers are involved in a discussion, and then such a union or intersection is to be interpreted accordingly as being taken over all admissible triples of rational numbers.

Given an admissible triple r_1, r_2, r_3, we define the right horocyclic angle $H^+_{r_1, r_2, r_3}(\zeta)$ at ζ with radii r_1, r_2, r_3 as the set of points of intersection of the circles $|z| = r$ $(r_3 \leqq r < 1)$ with the right horocycles $h^+_{r'}(\zeta)$ $(r_1 < r' < r_2)$. The corresponding left horocyclic angle is $H^-_{r_1, r_2, r_3}(\zeta)$. If we wish to refer to a horocyclic angle at ζ, but do not care to specify whether it is a right or a left one, we write it as $H_{r_1, r_2, r_3}(\zeta)$.

We define an admissible tangential arc at a point $\zeta \in \Gamma$ to be an arc Λ at ζ for which there exists a sequence $\{H_{r^{(n)}, r_2^{(n)}, r_3^{(n)}}(\zeta)\}$ of nested right or of nested left horocycles at ζ with $\lim_{n\to\infty} (r_2^{(n)} - r_1^{(n)}) = 0$, each term of which contains some terminal subarc of Λ.

Given a perfect nowhere dense subset P of Γ and an admissible triple of numbers r_1, r_2, r_3, we define the right region associated with P, r_1, r_2, r_3 to be

$$G^+(P\,; r_1, r_2, r_3) = (\bigcup_{\zeta \in P} H^+_{r_1, r_2, r_3}(\zeta)) \cup D_{r_3}\,.$$

The corresponding left region is denoted by $G^-(P\,; r_1, r_2, r_3)$, and $G(P\,; r_1, r_2, r_3)$ means either one of the two regions thus defined.

If z and z' are points of D, then $\varrho(z, z')$ represents the non-Euclidean hyperbolic distance between z and z' (see [5, p. 343]).

In what follows we shall be concerned with a function $f(z)$ that is meromorphic or holomorphic in D. The reader should know what is meant by a normal function [13, p. 86]. It is also assumed that he knows the rudiments cf cluster set theory (see [13]) and the customary notation for cluster sets; in particular, he should know what is meant by a Plessner point, as well as by a Fatou point of f and the corresponding Fatou value [13, p. 61]. By a Meier point of f is meant a point $\zeta \in \Gamma$ such that, for every chord χ at ζ,

$$C_\chi(f, \zeta) = C(f, \zeta) \subset \Omega\,.$$

We use the customary notation $A(f)$ for the set of asymptotic values of f.

Define the right outer horocyclic angular cluster set of f at a point $\zeta \in \Gamma$ tc be

$$C_{\mathfrak{A}^+}(f, \zeta) = \bigcup_{H^+} C_{H^+}(f, \zeta)\,,$$

and the right inner horocyclic angular cluster set of f at ζ to be

$$C_{\mathfrak{B}^+}(f, \zeta) = \bigcap_{H^+} C_{H^+}(f, \zeta)\,,$$

where in each case H^+ ranges over all right horocyclic angles at ζ. Then we define the outer horocyclic angular cluster set of f at ζ to be

$$C_{\mathfrak{A}}(f, \zeta) = C_{\mathfrak{A}^+}(f, \zeta) \cup C_{\mathfrak{A}^-}(f, \zeta)\,,$$

and the inner horocyclic angular cluster set of f at ζ to be

$$C_{\mathfrak{B}}(f, \zeta) = C_{\mathfrak{B}^+}(f, \zeta) \cap C_{\mathfrak{B}^-}(f, \zeta)\,.$$

Define the right principal horocyclic cluster set of f at ζ to be

$$\Pi^+_\omega(f, \zeta) = \bigcap_{0<r<1} C_{h_r^+(\zeta)}(f, \zeta)\,,$$

and the principal horocyclic cluster set of f at ζ to be

$$\Pi_\omega(f, \zeta) = \Pi_\omega^+(f, \zeta) \cap \Pi_\omega^-(f, \zeta) .$$

We call a point $\zeta \in \Gamma$ a right horocyclic Fatou point of f provided that

$$C_{\mathfrak{A}^+}(f, \zeta) = \{\alpha\}$$

for some $\alpha \in \Omega$; α is then called the corresponding right horocyclic Fatou value of f at ζ. We call ζ a horocyclic Fatou point of f if ζ is either a right horocyclic Fatou point of f or a left horocyclic Fatou point of f or both; we then speak of a horocyclic Fatou value of f at ζ (so that there may be either one or two such values at a horocyclic Fatou point). The sets of right horocyclic, left horocyclic, horocyclic Fatou points of f will be denoted by $F_\omega^+(f)$, $F_\omega^-(f)$, $F_\omega(f)$, respectively.

We call a point $\zeta \in \Gamma$ a right horocyclic Plessner point of f provided that

$$C_{\mathfrak{B}^+}(f, \zeta) = \Omega .$$

We say that $\zeta \in \Gamma$ is a horocyclic Plessner point of f provided that

$$C_{\mathfrak{B}}(f, \zeta) = \Omega .$$

The sets of right horocyclic, left horocyclic, horocyclic Plessner points of f will be denoted by $I_\omega^+(f)$, $I_\omega^-(f)$, $I_\omega(f)$, respectively. A point $\zeta \in \Gamma$ that is both a Plessner point and a horocyclic Plessner point of f will be termed a generalized Plessner point of f.

We call a point $\zeta \in \Gamma$ a right horocyclic Meier point of f provided that

$$\Pi_\omega^+(f, \zeta) = C(f, \zeta) \subset \Omega .$$

We say that $\zeta \in \Gamma$ is a horocyclic Meier point of f provided that

$$\Pi_\omega(f, \zeta) = C(f, \zeta) \subset \Omega .$$

A point $\zeta \in \Gamma$ that is both a Meier point and a horocyclic Meier point of f will be termed a generalized Meier point of f.

We define $K_\omega(f)$ to be the set of points $\zeta \in \Gamma$ such that

$$C_H(f, \zeta) = C_{H'}(f, \zeta)$$

for every pair of horocyclic angles H, H' at ζ.

In [2] we have defined the set $\Pi_T(f, \zeta)$ as $\bigcap_\Lambda C_\Lambda(f, \zeta)$, where the intersection is taken over all admissible arcs Λ at ζ. We now define the set $\Pi_{T_\omega}(f, \zeta)$ as $\bigcap_\Lambda C_\Lambda(f, \zeta)$, where the intersection is taken over all admissible tangential arcs Λ at ζ, and we put

$$\Pi_{T^*}(f, \zeta) = \Pi_T(f, \zeta) \cap \Pi_{T_\omega}(f, \zeta) .$$

3. Results and proofs

Lemma 1. *The boundary of every region* $G(P\,;r_1, r_2, r_3)$ *is a rectifiable Jordan curve.*

Proof. We shall assume that the region in question is a right region.

The boundary of the region consists of the set P, of arcs of the circle $|z| = r_3$, and of arcs of all right horocycles $h^+_{r_1}(\zeta)$ and $h^+_{r_2}(\zeta')$ such that ζ is the right endpoint and ζ' is the left endpoint of an arc Θ of Γ complementary to the set P; it is obviously a Jordan curve. The length of the boundary is not greater than

$$\text{meas}\,(P) + 2\,\pi\,r_3 + \sum_{\Theta} l\,(\Theta)\,,$$

where meas (P) denotes the (linear) Lebesgue measure of P, Θ ranges over the open subarcs of Γ complementary to the set P, and $l(\Theta)$ represents either the sum of the lengths of $h^+_{r_1}(\zeta)$ and $h^+_{r_2}(\zeta')$ if these right horocycles do not intersect, or if they do intersect, the sum of the lengths of those subarcs of these horocycles that extend from Γ to their point of intersection. Only a finite number of these pairs of right horocycles do not intersect, because only a finite number of the arcs Θ exceed any given positive number in length (we shall denote the length of an arc Θ by Θ). We are going to show that there exists a positive constant K, depending only on r_1 and r_2, such that, if $h^+_{r_1}(\zeta)$ and $h^+_{r_2}(\zeta')$ intersect, and if s_1, s_2 denote their respective lengths, then $s_1 + s_2 < K \cdot \Theta$. This will establish the convergence of $\sum_{\Theta} l\,(\Theta)$ and hence the rectifiability of the boundary of $G(P\,; r_1, r_2, r_3)$.

Referring to Fig. 1, we take

$$AT = r_1\,,\ BT = r_2\,,$$

so that

$$s_1 = r_1\,\varphi_1\,,\ s_2 = r_2\,\varphi_2\,,\ l\,(\Theta) = s_1 + s_2\,. \tag{1}$$

It is evident that

$$\varphi_1 = \alpha + \beta = \beta + \gamma + \delta\,,\ \varphi_2 = \alpha + \Theta = \Theta + \gamma + \delta\,. \tag{2}$$

Let d be the length of AB. Applying the law of cosines to triangle ATB, we obtain

$$d^2 = r_1^2 + r_2^2 - 2\,r_1\,r_2 \cos\beta\,. \tag{3}$$

Now

$$OA = 1 - r_1\,,\ OB = 1 - r_2\,.$$

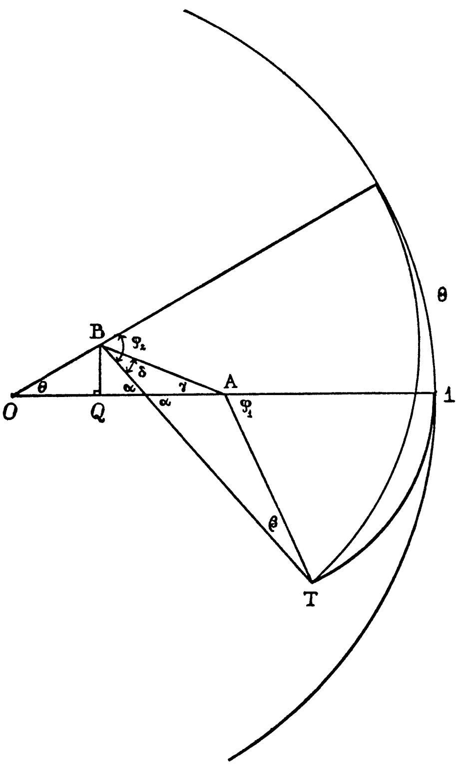

Fig. 1

Applying the law of cosines to triangle AOB, we obtain

(4) $$d^2 = (1 - r_1)^2 + (1 - r_2)^2 - 2\,(1 - r_1)\,(1 - r_2)\cos\Theta .$$

Elimination of d^2 from (3) and (4) yields

$$\beta = \operatorname{arc\,cos} \frac{(r_1 + r_2 - 1) + (1 - r_1)\,(1 - r_2)\cos\Theta}{r_1\, r_2}$$

and hence

$$\beta = \operatorname{arc\,sin} \frac{\sqrt{r_1^2 r_2^2 - [(r_1 + r_2 - 1) + (1 - r_1)\,(1 - r_2)\cos\Theta]^2}}{r_1\, r_2} .$$

Observing that

$$QB = (1 - r_2)\sin\Theta ,$$

and

$$OQ = (1 - r_2)\cos\Theta$$

so that

$$QA = 1 - r_1 - (1 - r_2)\cos\Theta ,$$

we obtain

$$\gamma = \operatorname{arc\,tan} \frac{(1-r_2)\sin\Theta}{1-[r_1+(1-r_2)\cos\Theta]} \; .$$

Finally, applying the law of sines to triangle ABT, we have

$$\delta = \operatorname{arc\,sin} \frac{r_1 \sin\beta}{d} \; .$$

It is clear that, as $\Theta \to 0$,

$$\frac{\sqrt{r_1^2 r_2^2 - [(r_1+r_2-1)+(1-r_1)(1-r_2)\cos\Theta]^2}}{r_1 r_2} \to 0 \; ,$$

so that $\beta \to 0$ and consequently $\dfrac{r_1 \sin\beta}{d} \to 0$. Furthermore,

$$\frac{(1-r_2)\sin\Theta}{1-[r_1+(1-r_2)\cos\Theta]} \to 0 \; .$$

We are now in a position to estimate β, γ, and δ for small Θ. First, since $\operatorname{arc\,sin} x = O(x)$ as $x \to 0$, we have

$$\beta = O\left(\frac{\sqrt{r_1^2 r_2^2 - [(r_1+r_2-1)+(1-r_1)(1-r_2)\cos\Theta]^2}}{r_1 r_2}\right) .$$

But $\cos\Theta = 1 + O(\Theta^2)$, and hence

$$\begin{aligned}
&\sqrt{r_1^2 r_2^2 - [(r_1+r_2-1)+(1-r_1)(1-r_2)\cos\Theta]^2} \\
&\quad = \sqrt{r_1^2 r_2^2 - [(r_1+r_2-1)+(1-r_1)(1-r_2)\{1+O(\Theta^2)\}]^2} \\
&\quad = \sqrt{r_1^2 r_2^2 - [r_1 r_2 + (1-r_1)(1-r_2)\,O(\Theta^2)]^2} \\
&\quad = \sqrt{-2r_1 r_2 (1-r_1)(1-r_2)\,O(\Theta^2) - (1-r_1)^2 (1-r_2)^2\,O(\Theta^4)} \\
&\quad = O(\Theta) \; .
\end{aligned}$$

Consequently,

$$\beta = O(\Theta) \; . \tag{5}$$

Next, since $\operatorname{arc\,tan} x < x$ for small positive x, we have

$$\gamma < \frac{(1-r_2)\sin\Theta}{1-[r_1+(1-r_2)\cos\Theta]} = O(\Theta) \; . \tag{6}$$

Finally,

$$\delta = O\left(\frac{r_1 \sin \beta}{d}\right) ,$$

$$\frac{r_1 \sin \beta}{d} = O(\beta) ,$$

$$\beta = O(\Theta) ,$$

and hence

$$\delta = O(\Theta) . \tag{7}$$

If we combine (1) and (2) with (5), (6), and (7), we see that the asserted constant K exists, and the proof of Lemma 1 is complete.

Lemma 2. *Let the function* $f(z)$ *be meromorphic in* D , *and let* M *be a measurable subset of* Γ *of positive measure. Suppose that, for some admissible triple of numbers* r_1, r_2, r_3 , *we have*

$$C_{H^+_{r_1, r_2, r_3}(\zeta)}(f , \zeta) \neq \Omega \tag{8}$$

for every $\zeta \in M$. *Then* M *contains a subset* M_1 *of positive measure such that every point of* M_1 *is a Fatou point of* f .

Proof. Let $\{Q_1 , Q_2 , \ldots , Q_n, \ldots\}$ be the enumerable set of closed squares (interior and boundary points) in the plane, the coordinates of whose vertices are rational. Suppose that $\zeta \in M$. Because of (8) and the fact that $C_{H^+_{r_1, r_2, r_3}(\zeta)}(f , \zeta)$ is a closed set, there exists an n such that

$$C_{H^+_{r_1, r_2, r_3}(\zeta)}(f , \zeta) \cap Q_n = \emptyset .$$

Consequently, there exists a natural number k so large, that if $r_3' = 1 - \dfrac{1}{k}$, then $r_3 \leqq r_3'$, and in $H^+_{r_1, r_2, r_3'}(\zeta)$, $f(z)$ omits all values belonging to Q_n . Let the set of points $\zeta \in M$ with which the natural numbers n , k can thus be associated be denoted by $E_{n,k}$. Then the set of such sets is enumerable, and $M = \bigcup\limits_{n,k} E_{n,k}$. Denote by $E^*_{n,k}$ the set of all points $\zeta \in \Gamma$ such that in $H^+_{r_1, r_2, r_3'}(\zeta)$, where $r_3' = 1 - \dfrac{1}{k}$, $f(z)$ omits all values belonging to Q_n . Then evidently $E^*_{n,k}$ is a closed set, and hence it is measurable. Since M is also measurable, and $E_{n,k} = E^*_{n,k} \cap M$, the set $E_{n,k}$ is measurable too. And since M is of positive measure, at least one of the sets $E_{n,k}$, call it E_{n_0, k_0} , is of positive measure.

By means of a suitable linear transformation $W = \varphi(w)$, we map the complement of Q_n onto a region in the disk $|W| < 1$. Then the function $W = f^*(z) \equiv \varphi(f(z))$ is meromorphic in D , and for every $\zeta \in E_{n_0, k_0}$, the set $C_{H^+_{r_1,r_2,r_3'}(\zeta)}(f, \zeta)$, where $r_3' = 1 - \frac{1}{k}$, is a subset of the disk $|W| < 1$. Let $\{\varrho_m\}$ be a monotonically increasing sequence of numbers greater than r_3' and converging to 1. For every m, let

$$A_m = \{\zeta \in E_{n_0,k_0} : |f^*(z)| < 1 \ (z \in H^+_{r_1,r_2,r_3'}(\zeta),\ |z| \geqq \varrho_m)\}.$$

Then $A_1 \subseteqq A_2 \subseteqq \ldots \subseteqq A_m \subseteqq \ldots$ and $E_{n_0,k_0} = \bigcup_m A_m$. Hence, there exists an m_0 such that A_{m_0} is of positive measure, and there is a perfect nowhere dense subset P of A_{m_0} having positive measure. Set $\varrho_{m_0} = \varrho$. Then for every $\zeta \in P$, we have $|f^*(z)| < 1$ for $z \in H^+_{r_1,r_2,\varrho}(\zeta)$.

Consider the region $G(P; r_1, r_2, \varrho)$. According to Lemma 1, the boundary $\mathfrak{J}$ of this region is a rectifiable Jordan curve. Let $a_1, a_2, \ldots, a_p$ be all the poles of $f^*(z)$ in $|z| \leqq \varrho$, and form the function

$$g(z) = (z - a_1)(z - a_2) \ldots (z - a_p) f^*(z);$$

it is evidently holomorphic and bounded in $G(P; r_1, r_2, \varrho)$. By an extension of Fatou's theorem [14, p. 129], $g(z)$, and consequently also $f(z)$, has angular limits at almost all points of $\mathfrak{J}$. Now $\mathfrak{J} \cap \Gamma = P$, and at almost every point $\zeta \in P$, $\mathfrak{J}$ has a tangent that coincides with the tangent of Γ at ζ, since P is perfect. Hence, the angular limits of $f(z)$ in D and in $G(P; r_1, r_2, \varrho)$ coincide at almost every point $\zeta \in P$. This implies the existence of a set M_1 with the properties described in Lemma 2.

Theorem 1. *Let the function $f(z)$ be meromorphic in D. Then almost every Plessner point of f is a right horocyclic Plessner point of f.*

Proof. According to Plessner's theorem [14, p. 217], $\Gamma = E_1 \cup E_2 \cup E_3$, where E_1 is the set of Fatou points of f, every point of E_2 is a Plessner point of f, and E_3 is of measure zero. The sets E_1 and E_2 are measurable, and if E_2 is of measure zero, then there is nothing to be provėd.

Assume then that E_2 is of positive measure. Let r_1, r_2, r_3 be an admissible triple of rational numbers (there are only enumerably many such triples), and define $M^{(r_1,r_2,r_3)}$ as the set of all points $\zeta \in \Gamma$ for which $C_{H^+_{r_1,r_2,r_3}(\zeta)}(f, \zeta) = \Omega$. It is readily seen that $M^{(r_1,r_2,r_3)}$ is a Borel set, and hence it is measurable. Therefore, if we set

$$E_2^{(r_1,r_2,r_3)} = E_2 \cap M^{(r_1,r_2,r_3)}, \quad R_2^{(r_1,r_2,r_3)} = E_2 - E_2^{(r_1,r_2,r_3)},$$

then $E_2^{(r_1,r_2,r_3)}$ is measurable and $E_2 = E_2^{(r_1,r_2,r_3)} \cup R_2^{(r_1,r_2,r_3)}$. According to Lemma 2, $R_2^{(r_1,r_2,r_3)}$ is of measure zero.

Now let

$$E = \bigcap_{r_1,r_2,r_3} E_2^{(r_1,r_2,r_3)}, \quad R = \bigcup_{r_1,r_2,r_3} R_2^{(r_1,r_2,r_3)}.$$

Then

$$E_2 = E \cup R ,$$

R is of measure zero, and every point of E is a right horocyclic Plessner point of f.

Corollary 1. *Let the function $f(z)$ be meromorphic in D. Then $\Gamma = M_1 \cup M_2 \cup M_3$, where M_1 is the set of Fatou points of f, every point of M_2 is a generalized Plessner point of f, and M_3 is of measure zero.*

Proof. According to Plessner's theorem, $\Gamma = E_1 \cup E_2 \cup E_3$, where E_1 is the set of Fatou points of f, every point of E_2 is a Plessner point of f, and E_3 is of measure zero. By Theorem 1, almost every point of E_2 is a right horocyclic Plessner point, and it can be shown analogously that almost every point of E_2 is a left horocyclic Plessner point. Hence, $E_2 = M_2 \cup R_2$, where every point of M_2 is a horocyclic Plessner point and R_2 is of measure zero. If we set $M_1 = E_1$ and $M_3 = R_2 \cup E_3$, we obtain Corollary 1.

Remark 1. In connection with Theorem 1, it is natural to ask the following question: Suppose that $f(z)$ is meromorphic in D. Is it true that almost every right horocyclic Plessner point of f is a Plessner point of f? I do not know the answer, but I would guess that it is in the negative.

It is false that if $f(z)$ is meromorphic in D then almost every Fatou point is a right horocyclic Fatou point; this follows from a well-known result due to Littlewood [9]. However, we do have the following theorem:

Theorem 2. *Let the function $f(z)$ be meromorphic in D. Then almost every right horocyclic Fatou point of f is a Fatou point of f.*

Proof. Corollary 1 implies that almost every point of Γ is either a Fatou point of f or a right horocyclic Plessner point of f. The conclusion of Theorem 2 now follows from the fact that no right horocyclic Fatou point of f is a right horocyclic Plessner point of f.

Corollary 2. *Let $f(z)$ be a nonconstant meromorphic function in D, and E_z be a subset of Γ of positive measure. Suppose that every point of E_z is a horocyclic Fatou point of f, and let E_w denote the set of corresponding horocyclic Fatou values. Then E_w is of positive inner harmonic measure.*

Proof. It clearly suffices to consider the case that $E_w \neq \Omega$, and then there is no loss of generality in assuming that $\infty \notin E_w$. The set $F_\omega^+(f)$ is readily seen to be a Borel set, and hence it is measurable; so, likewise, is the set $F_\omega^-(f)$. Consequently the sets

$$E_z^+ = E_z \cap F_\omega^+(f) , \; E_z^- = E_z \cap F_\omega^-(f)$$

are measurable too. Since $E_z = E_z^+ \cup E_z^-$, and E_z is of positive measure, at least one of the sets E_z^+, E_z^- is also of positive measure; suppose that E_z^+ is.

It follows from Egoroff's theorem that E_z^+ contains a perfect subset P_z of positive measure on which f approaches the corresponding right horocyclic Fatou values uniformly; the set P_w of these values is therefore a closed and bounded subset of E_w.

We infer now from Theorem 2 that P_z contains a subset Q_z of equal measure such that every point of Q_z is a Fatou point of f. Consequently [14, p. 210] the set Q_w of Fatou values of f at the points of Q_z contains a closed subset S of positive harmonic measure. My ambiguous-point theorem [1, p. 382, Corollary 1] implies that every point of S, with at most enumerably many exceptions, belongs to P_w. Hence

$$S = (P_w \cap S) \cup R,$$

where R is at most enumerable. Since $P_w \cap S$ is closed and bounded, it follows [12, p. 127] that $P_w \cap S$ is a subset of E_w of positive harmonic measure, and Corollary 2 is proved.

Corollary 3. *Let $f(z)$ be a meromorphic function in D, E_z be a subset of Γ of positive measure, and $\alpha \in \Omega$. Suppose that every point of E_z is a horocyclic Fatou point of f with a corresponding horocyclic Fatou value α. Then $f(z) \equiv \alpha$.*

Theorem 3. *Let the function $f(z)$ be meromorphic in D. Then almost every Meier point of f is a right horocyclic Meier point of f.*

Proof. By Plessner's theorem, almost every point of Γ is either a Fatou point or a Plessner point of f. Since no Meier point of f is a Plessner point of f, it follows that almost every Meier point of f is a Fatou point of f. But a point $\zeta \in \Gamma$ that is both a Meier point and a Fatou point of f is a point at which

$$\lim_{\substack{z \to \zeta \\ z \in D}} f(z)$$

exists, and this implies that ζ is also a right horocyclic Meier point of f. Hence almost every Meier point of f is a right horocyclic Meier point of f.

Remark 2. If $f(z)$ is meromorphic in D, is almost every right horocyclic Meier point of f a Meier point of f? I suspect that this is not so.

Theorem 4. *Let the function $f(z)$ be meromorphic in D. Then nearly every Plessner point of f is a right horocyclic Plessner point of f, and nearly every right horocyclic Plessner point of f is a Plessner point of f.*

Proof. If a point $\zeta \in \Gamma$ is a Plessner point or a right horocyclic Plessner point of f, then $C(f, \zeta) = \Omega$. By a theorem of Collingwood [7, p. 1241, Theorem 4], for nearly every point $\zeta \in \Gamma$ we have

$$C_\Delta(f, \zeta) = C(f, \zeta), \tag{9}$$

where Δ is an arbitrary Stolz angle at ζ. An appropriate modification of the proof of Collingwood's theorem shows that for nearly every point $\zeta \in \Gamma$ we have

$$C_{H^+}(f, \zeta) = C(f, \zeta), \tag{10}$$

where H^+ is an arbitrary right horocyclic angle at ζ. Relations (9) and (10) obviously yield our theorem.

Lemma 3. *Let $f(z)$ be a normal meromorphic function in D. If there exists a right horocycle $h_r^+(\zeta)$ at a point $\zeta \in \Gamma$ such that $C_{h_r^+(\zeta)}(f, \zeta) \subset C(f, \zeta)$, then there exists a right horocyclic angle $H^+_{r_1, r_2, r_3}(\zeta)$ at ζ such that $C_{H^+_{r_1, r_2, r_3}(\zeta)}(f, \zeta) \subset C(f, \zeta)$.*

Proof. Let

$$\eta \in C(f, \zeta) - C_{h_r^+(\zeta)}(f, \zeta). \tag{11}$$

Suppose that $C_{H^+_{r_1, r_2, r_3}(\zeta)}(f, \zeta) = C(f, \zeta)$ for every right horocyclic angle $H^+_{r_1, r_2, r_3}(\zeta)$ at ζ. Then, in particular, $\eta \in C_{H^+_{r_1, r_2, r_3}(\zeta)}(f, \zeta)$ for every right horocyclic angle $H^+_{r_1, r_2, r_3}(\zeta)$ at ζ. Consequently there exists a sequence of points $\{z_n'\}$ in D, where $\lim_{n\to\infty} z_n' = \zeta$ and $\lim_{n\to\infty} f(z_n') = \eta$, such that, for an appropriate sequence of points $\{z_n\}$ on $h_r^+(\zeta)$ with $\lim_{n\to\infty} z_n = \zeta$, we have $\lim_{n\to\infty} \varrho(z_n, z_n') = 0$. Since $f(z)$ is a normal meromorphic function in D, we infer [4, p. 10, Lemma 1] that $\lim_{n\to\infty} f(z_n) = \eta$, which contradicts (11). This proves Lemma 3.

Theorem 5. *Let $f(z)$ be a bounded holomorphic function in D. Then nearly every point of Γ is a horocyclic Meier point of f.*

Proof. Since $f(z)$ is bounded in D, we have $C(f, \zeta) \subset \Omega$ for every $\zeta \in \Gamma$. Now suppose that for some $\zeta \in \Gamma$ we have

$$\Pi_\omega(f, \zeta) \subset C(f, \zeta). \tag{12}$$

Then there is either a right or a left horocycle at ζ, say $h_r^+(\zeta)$, such that

$$C_{h_r^+(\zeta)}(f, \zeta) \subset C(f, \zeta).$$

Hence, according to Lemma 3, there is a right horocyclic angle $H^+_{r_1, r_2, r_3}(\zeta)$ at ζ for which

$$C_{H^+_{r_1, r_2, r_3}(\zeta)}(f, \zeta) \subset C(f, \zeta). \tag{13}$$

But by a ready generalization of a theorem of Collingwood [7, p. 1241, Theorem 4] from Stolz angles to horocyclic angles, the set of $\zeta \in \Gamma$ for which (13) holds is of first category, and therefore the same is true of the set of $\zeta \in \Gamma$ for which (12) is true, which completes the proof.

Corollary 4. *If $f(z)$ is a bounded holomorphic function in D, then nearly every point of Γ is a generalized Meier point of f.*

Proof. This follows immediately from Theorem 5 and a theorem of Meier [10, p. 330, Theorem 6].

Theorem 6. *Let the function $f(z)$ be meromorphic in D. Then nearly every point of Γ is either a right horocyclic Meier point of f or a right horocyclic Plessner point of f.*

Proof. Let E be the set of points of Γ that are not right horocyclic Plessner points of f. We shall show that nearly every point of E is a right horocyclic Meier point of f.

If m, n is a pair of natural numbers and r_1, r_2, r_3 is an admissible triple of rational numbers, define

$$(14)\quad E^+(m, n; r_1, r_2, r_3) = \left\{\zeta \in \Gamma : |f(z) - \omega_m| > \frac{1}{n} \text{ for all } z \in H^+_{r_1, r_2, r_3}(\zeta)\right\}.$$

Then there are only enumerably many of these sets $E^+(m, n; r_1, r_2, r_3)$, and evidently

$$(15)\qquad E = \bigcup_{m, n; r_1, r_2, r_3} E^+(m, n; r_1, r_2, r_3).$$

Now suppose it is false that nearly every point of E is a right horocyclic Meier point of f. Then there exists a subset S of E, where S is of second category, such that no point of S is a right horocyclic Meier point cf f. In view of (15), there exists a pair of natural numbers m', n', and an admissible triple of rational numbers r_1', r_2', r_3' such that the set

$$(16)\qquad S' = S \cap E^+(m', n'; r_1', r_2', r_3')$$

is of second category. Consequently S' is everywhere of second category on some open subarc Γ' of Γ. Let ζ_1 and ζ_2 be the right and left end points of Γ', and denote by R the region bounded by Γ', $h^+_{r_2}(\zeta_1)$, $h^+_{r_1}(\zeta_2)$, and $|z| = r_3$. Then clearly we have

$$R \subset \bigcup_{\zeta \in S'} H^+_{r_1, r_2, r_3}(\zeta).$$

This means, according to (16) and (14), that the function

$$g(z) = \frac{1}{f(z) - \omega_{m'}}$$

is bounded in R. By an obviously valid localization of Theorem 5, nearly every point of Γ' is a right horocyclic Meier point of g. But then nearly every point of Γ', in particular, nearly every point of S', is a right horocyclic Meier point of f, which contradicts the definition of S. This proves Theorem 6.

Theorem 7. *Let the function $f(z)$ be meromorphic in D. Then nearly every Meier point of f is a right horocyclic Meier point of f, and nearly every right horocyclic Meier point of f is a Meier point of f.*

Proof. According to Meier's theorem [10, p. 330, Theorem 5], nearly every point of Γ is a Meier point or a Plessner point of f. But no right horocyclic Meier point of f is a Plessner point of f. Consequently nearly every right horocyclic Meier point of f is a Meier point of f.

From Theorem 6 we know that nearly every point of Γ is a right horocyclic Meier point or a right horocyclic Plessner point of f. Since no Meier point of f is a right horocyclic Plessner point of f, it follows that nearly every Meier point of f is a right horocyclic Meier point of f.

Corollary 5. *Let the function $f(z)$ be meromorphic in D. Then nearly every point of Γ is a generalized Meier point or a generalized Plessner point of f.*

Proof. This follows immediately from Theorem 4, Theorem 7, and the theorem of Meier quoted in the proof of Theorem 7.

Remark 3. We note for the sake of completeness that *nearly every Fatou point is a right horocyclic Fatou point, and nearly every right horocyclic Fatou point is a Fatou point.* This follows readily from horocyclic versions of theorems of Collingwood [7, Theorems 2 and 4].

Theorem 8. *Let $f(z)$ be holomorphic in D, and S be a subset of Γ. Suppose that to every $\zeta \in S$ there correspond two arcs at ζ on which f is bounded, such that there exists a horocyclic angle at ζ lying between these two arcs. Then almost every point of S is a Fatou point of f.*

Proof. According to Corollary 1, almost every point of Γ is a Fatou point or a generalized Plessner point of f. Let $\zeta \in S$ and suppose that ζ is a generalized Plessner point of f. Let Λ_1 and Λ_2 be the two arcs at ζ on which f is assumed to be bounded, and let $H_{r_1, r_2, r_3}(\zeta)$ be a horocyclic angle at ζ lying between Λ_1 and Λ_2. Since

$$C_{H_{r_1, r_2, r_3}(\zeta)}(f, \zeta) = \Omega ,$$

it follows from a theorem of Iversen and Gross (see [13, p. 14, Theorem 1] for the special case in which the set E there consists of a single point) that ∞ is an asymptotic value of f at ζ. Consequently ζ is an ambiguous point of f. But f has at most enumerably many ambiguous points [1, p. 380, Theorem 2]. This implies that almost every point of S is a Fatou point of f.

Lemma 4. *Let $f(z)$ be a normal meromorphic function in D, and suppose that $\zeta \in K_\omega(f)$. Then $\Pi_{T_\omega}(f, \zeta) = C_{\mathfrak{A}}(f, \zeta)$.*

Proof. Suppose that $\alpha \in \Pi_{T_\omega}(f, \zeta)$. Then $\alpha \in C_\Lambda(f, \zeta)$ for every admissible tangential arc Λ at ζ. By definition, there exists a horocyclic angle H at ζ containing a terminal subarc of Λ. Clearly $C_\Lambda(f, \zeta) \subseteq C_H(f, \zeta)$, so that $\alpha \in C_{\mathfrak{A}}(f, \zeta)$.

Now suppose that $\alpha \in C_{\mathfrak{A}}(f, \zeta)$. Let Λ be any admissible tangential arc at ζ. Since $\zeta \in K_\omega(f)$, we have $\alpha \in C_H(f)$ for every horocyclic angle H at ζ. Hence, there exists a sequence of points $\{z_n'\}$ in D, where $\lim_{n\to\infty} z_n' = \zeta$ and $\lim_{n\to\infty} f(z_n') = \alpha$, such that, for an appropriate sequence of points $\{z_n\}$ on Λ with $\lim_{n\to\infty} z_n = \zeta$, we have $\lim_{n\to\infty} \varrho(z_n, z_n') = 0$. This implies [4, p. 10, Lemma 1] that $\lim_{n\to\infty} f(z_n) = \alpha$, and hence $\alpha \in C_\Lambda(f, \zeta)$. This holds for an arbitrary admissible tangential arc Λ at ζ, and therefore $\alpha \in \Pi_{T_\omega}(f, \zeta)$.

Theorem 9. *Let $f(z)$ be a nonconstant normal meromorphic function in D, and suppose that $A(f)$ is of harmonic measure zero. Then there exists a residual subset S of Γ of measure 2π such that, for every $\zeta \in S$, $\Pi_{T_\omega}(f, \zeta) = \Omega$.*

Proof. According to Corollary 1, almost every point of Γ is either a Fatou point or a generalized Plessner point of f. Since f is nonconstant and $A(f)$ is of harmonic measure zero, Priwalow's theorem [14, p. 210] implies that the set of Fatou points of f is of measure zero. Hence, the set $I_\omega(f)$ is of measure 2π. An obviously valid horocyclic analogue of [6, p. 382, Theorem 3] implies that $I_\omega(f)$ is also a residual subset of Γ. If $\zeta \in I_\omega(f)$, then $C_{\mathfrak{A}}(f, \zeta) = \Omega$, and since $I_\omega(f) \subseteq K_\omega(f)$, Lemma 4 yields $\Pi_{T_\omega}(f, \zeta) = C_{\mathfrak{A}}(f, \zeta)$. Setting $S = I_\omega(f)$, we obtain Theorem 9.

Remark 4. Theorem 9 is valid if $\Pi_{T_\omega}(f, \zeta)$ is replaced by $\Pi_{T^*}(f, \zeta)$; we have only to apply Theorem 9 in its original form and [2, p. 4, Theorem 1].

Theorem 10. *Let $f(z)$ be a nonconstant normal meromorphic function in D, and suppose that $A(f)$ is of linear measure zero. Then there exists a residual subset R of Γ such that, for every $\zeta \in R$, $\Pi_{T_\omega}(f, \zeta) = \Omega$.*

Proof. Since f is nonconstant and $A(f)$ is of linear measure zero, we have $C(f, \zeta) = \Omega$ for every $\zeta \in \Gamma$ [13, p. 51]. Hence [13, III, §3] $I(f)$ is a residual subset of Γ, and so by Theorem 4 the set $I_\omega(f)$ is also residual. As in the proof of Theorem 9, $\zeta \in I_\omega(f)$ implies that $\Pi_{T_\omega}(f, \zeta) = \Omega$; setting $R = I_\omega(f)$, we obtain Theorem 10.

Remark 5. Theorem 10 is valid if $\Pi_{T_\omega}(f, \zeta)$ is replaced by $\Pi_{T^*}(f, \zeta)$; we have only to apply Theorem 10 in its original form and [2, p. 4, Theorem 2].

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