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SETS OF ASYMPTOTIC VALUES OF POSITIVE LINEAR MEASURE

BY

F. BAGEMIHL

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SUOMALAINEN TIEDEAKATEMIA

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Sets of asymptotic values of positive linear measure

It will be shown that certain sets of asymptotic values contain sets of positive linear measure, whereas heretofore they were either known merely to contain sets of positive harmonic measure or not discussed at all. This is accomplished by generalizing a result given by Collingwood and Cartwright [3, p. 103, Lemma 4].

Let C be the unit circle, D be the open unit disk, and Ω be the Riemann sphere. Consider a meromorphic function $f(z)$ in D. The set of asymptotic values of f will be denoted by $A(f)$. If Γ is a closed subarc of C, then $A(f, \Gamma)$ represents the set of asymptotic values of f approached along paths whose ends are contained in Γ, $A_{\mathfrak{p}}(f, \Gamma)$ is the set of asymptotic values of f approached along paths terminating in single points of Γ, and $A_{\mathfrak{a}}(f, \Gamma)$ denotes the set of asymptotic values of f approached along paths whose ends are subarcs of Γ, so that we have

$$A(f, \Gamma) = A_{\mathfrak{a}}(f, \Gamma) \cup A_{\mathfrak{p}}(f, \Gamma) .$$

If $\zeta \in C$, then $A(f, \zeta)$ means the set of asymptotic values of f approached along paths terminating in ζ. The set of Fatou points of f on Γ is denoted by $F(f, \Gamma)$, and $A_F(f, \Gamma)$ represents the set of angular limits that f has at points of Γ.

By an arc at a point $\zeta \in C$ we mean a simple continuous curve $\Lambda : z = z(t)$ $(0 \leqq t < 1)$ such that $|z(t)| < 1$ for $0 \leqq t < 1$ and $\lim_{t \to 1} z(t) = \zeta$. If $0 \leqq t_0 < 1$ and $z(t_0) = \zeta_0$, then the arc $\Lambda' : z = z(t)$ $(t_0 \leqq t < 1)$ is called the terminal subarc of Λ determined by ζ_0.

We shall be concerned with the following

Hypothesis $\mathfrak{H}$. (i) *$f(z)$ is a meromorphic function in D;*

(ii) *Γ is a closed subarc of C with distinct end points ζ_1 and ζ_2;*

(iii) *J is a Jordan arc lying, except for its end points ζ_1 and ζ_2, in D;*

(iv) *G is the subregion of D whose frontier is $J \cup \Gamma$;*

(v) *there exist distinct finite values ω_1 and ω_2 such that*

$$\lim_{\substack{z \to \zeta_1 \\ z \in J}} f(z) = \omega_1 , \qquad \lim_{\substack{z \to \zeta_2 \\ z \in J}} f(z) = \omega_2 ;$$

(vi) *f has no asymptotic path in G on which f tends to ∞.*

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Our basic result is

Theorem 1. *Assume* $\mathfrak{H}$ *and that* f *is holomorphic in* G. *Then* $A_{\mathrm{p}}(f, \Gamma)$ *contains a set of positive linear measure.*

Proof: The arc J minus its end points will be called J_0. Take a point z_0 on J_0; z_0 divides J_0 into an arc Λ_1 at ζ_1 and an arc Λ_2 at ζ_2. Denote by L the open rectilinear segment joining ω_1 and ω_2. Choose an arbitrary point $\mu \in L$ and consider the straight line M perpendicular to L and containing μ. Let δ be a positive number smaller than the distance between M and the set $\{\omega_1, \omega_2\}$. Then there exist terminal subarcs Λ_1', Λ_2' of Λ_1, Λ_2, respectively, such that

$$|f(z) - \omega_1| \leqq \delta \quad (z \in \Lambda_1'), \quad |f(z) - \omega_2| \leqq \delta \quad (z \in \Lambda_2').$$

Since the set of zeros of $f'(z)$ in G is isolated, we can find a circular arc K with center at the origin and lying in G, with one end point λ_1 on Λ_1' and the other end point λ_2 on Λ_2', such that $f'(z) \neq 0$ on K. Since f is also holomorphic on K, K is mapped by $w = f(z)$ onto a bounded simple analytic arc T, and T intersects M at least once, but only a finite number of times. Thus either (a) $T \cap M$ consists of a single point p, or else (b) there are distinct points p_1, p_2 in $T \cap M$ such that any other point in $T \cap M$ lies on M between p_1 and p_2. Let G_0 be the subregion of G whose frontier consists of Γ, K, the terminal subarc Λ_1'' of Λ_1' determined by λ_1, and the terminal subarc Λ_2'' of Λ_2' determined by λ_2.

There is an ordinary element of the inverse $f^{-1}(w)$ of the function $f(z)$ in case (a) at p and in case (b) at p_1 and at p_2. In case (a) this element can be continued towards ∞ in either direction along M without encountering another point of T, and in case (b) the elements at p_1 and p_2 can be continued towards ∞ in suitable directions so as not to encounter another point of T. In case (a), a direction of continuation can be chosen, and in case (b) one of the two points p_1, p_2 can be chosen, so that the curve m on which the appropriate path of continuation along M is mapped lies in G_0. The continuation in question cannot be made to the point ∞ because ∞ is neither an assumed value nor an asymptotic value of $f(z)$ in G_0. Consequently there is a finite point q on M such that either the continuation has a boundary element (see [3, p. 100]) at q or the continuation is terminated by a transcendental singularity at q. The curve m is consequently an asymptotic path of $f(z)$ which either terminates in a point of Γ or has its end on Γ, and on which $f(z) \to q$ as $|z| \to 1$. Hence $q \in A(f, \Gamma)$.

Now consider $f(z)$ restricted to G_0. According to Kierst [8], $A(f, \Gamma \cup \Lambda_1'' \cup K \cup \Lambda_2'')$ is an analytic subset of Ω. Evidently $A(f, \Lambda_1'' \cup K \cup \Lambda_2'')$ $= A_{\mathrm{p}}(f, \Lambda_1'' \cup K \cup \Lambda_2'')$ is an F_σ. Now $A_{\mathrm{a}}(f, \Gamma)$ contains at most enumerably many finite values, because otherwise, as was pointed out to me by J. E. McMillan, there would exist two such values with corresponding asymptotic paths which meet on Γ, and this is readily seen to contradict (vi); let us assume that μ was chosen so that $q \notin A_a(f, \Gamma)$. Then $A_p(f, \Gamma \cup \Lambda_1'' \cup K \cup \Lambda_2'')$ is also an analytic set. Consequently [5, p. 370, 42 · 1 · 3] the set

$$E = A_p(f, \Gamma \cup \Lambda_1'' \cup K \cup \Lambda_2'') - A_p(f, \Lambda_1'' \cup K \cup \Lambda_2'')$$

is an analytic set; it is clearly a subset of $A_p(f, \Gamma)$ and contains q. The set E is Carathéodory linearly measurable [6, p. 83, 7. 1. 211 and p. 84, Bibliography; p. 105, 8. 5. 1]. Since μ was an arbitrary point of L with at most enumerably many exceptions, the orthogonal projection of E onto L contains almost all of L, and hence [13, p. 533] E is of positive linear measure. This completes the proof.

In Theorem 1 we may not drop the hypothesis that f be holomorphic in G. This is evident from

Theorem 2. *Under the assumption* $\mathfrak{H}$, *it is possible to have* $A(f) = \{\omega_1, \omega_2\}$.

Proof: We take $\zeta_1 = +1, \zeta_2 = -1$; Γ to be the semicircle $|z| = 1$, $\Im(z) \geqq 0$; J to be the segment $-1 \leqq z \leqq +1$; and finally $\omega_1 = +1, \omega_2 = -1$. Define B_0 to be the arc $\frac{1}{2} \leqq z < 1$ and B_1 to be the arc $-1 < z \leqq -\frac{1}{2}$; and for $n = 1, 2, 3, \ldots,$ let

$$J_n = \left\{z : |z| = \frac{n+1}{n+2}\right\}.$$

We put

$$S = B_0 \cup B_1 \cup \left(\bigcup_{n=1}^{\infty} J_n\right)$$

and call S the skeleton.

We now define a continuous function $g(z)$ on S. Let

$$g(z) \equiv +1 \quad (z \in B_0), \quad g(z) \equiv -1 \quad (z \in B_1).$$

Define C^+ (C^-) to be the subarc of C on which $\Im(z) \geqq 0$ $(\Im(z) \leqq 0)$, and for every n, let J_n^+ (J_n^-) be the subarc of J_n on which $\Im(z) \geqq 0$ $(\Im(z) \leqq 0)$. For $n = 2k-1$ $(k = 1, 2, 3, \ldots)$, define $g(z)$ on J_n as follows: $g(z)$ is a homeomorphism of J_n^- onto C^- and a homeomorphism of J_n^+ onto C^-. For $n = 2k$ $(k = 1, 2, 3, \ldots)$, define $g(z)$ to be a homeomorphism of J_n^- onto C^+ and a homeomorphism of J_n^+ onto C^+. Clearly $g(z)$ is defined and continuous on S.

According to [1], there exists a meromorphic function $f(z)$ in D such that

$$\lim_{\substack{|z| \to 1 \\ z \in S}} |f(z) - g(z)| = 0 .$$

It is evident from this and the definition of $g(z)$ that $\mathfrak{H}$ is satisfied but $A(f) = \{+1, -1\}$.

Theorem 3. *Let $f(z)$ be holomorphic and not constant in D. Suppose that B is a subarc of C, E is a subset of B of positive measure, and f has an angular limit at every point of E, but $\infty \notin A(f, B)$. Then $A_p(f, B)$ contains a set of positive linear measure.*

Proof: A theorem of Priwalow [12, p. 210] implies that there exist points ζ_1, ζ_2 in E and finite values ω_1, ω_2 such that f has the angular limit ω_1, ω_2 at ζ_1, ζ_2, respectively. Let Γ be the subarc of B with end points ζ_1, ζ_2, and let J be the union of the radii at ζ_1 and ζ_2. Then the hypothesis of Theorem 1 is satisfied, and the conclusion of Theorem 3 follows from that of Theorem 1.

Remark. The assumption $\infty \notin A(f, B)$ is necessary in Theorem 3 even if f is of bounded characteristic and the measure of E equals the length of B: see [4, p. 98].

Theorem 4. *Assume $\mathfrak{H}$ and that f is holomorphic and normal in G. Then $A_F(f, \Gamma)$ is of positive linear measure.*

Proof: The set $F(f, \Gamma)$ is a Borel set [7, p. 275], so that $A_F(f, \Gamma)$ is an analytic set [7, p. 269] and hence is linearly measurable. Moreover, by [2, p. 15, Theorem 3], $F(f, \Gamma)$ is of positive measure because of (vi). According to [9, p. 53, Theorem 2], $A_p(f, \Gamma) = A_F(f, \Gamma)$. The conclusion of Theorem 4 now follows from that of Theorem 3.

Corollary. *Let $f(z)$ be a nonconstant normal holomorphic function in D. Suppose that B is a subarc of C for which $\infty \notin A(f, B)$. Then $A_F(f, B)$ is of positive linear measure.*

MacLane [10] has defined $\mathcal{A}$ to be the class of nonconstant holomorphic functions in D having asymptotic values at each point of an everywhere dense set of points on C.

Theorem 5. *Let $f \in \mathcal{A}$. Suppose that B is a subarc of C for which $\infty \notin A_p(f, B)$. Then $A_p(f, B)$ contains a set of positive linear measure.*

Proof: According to [10, p. 28, Corollary], $A(f, B)$ contains a closed set H of positive harmonic measure. If ω_1 and ω_2 are distinct finite values in H, then [10, p. 18, Theorem 4] $\{\omega_1, \omega_2\} \subset A_p(f, B)$. The conclusion now follows from Theorem 1.

Theorem 6. *Let $f(z)$ be a nonconstant holomorphic function in D. Suppose that f has only finitely many asymptotic tracts for ∞, and the ends of the arc tracts of f for ∞ do not cover C. Then $A_p(f)$ is of positive linear measure.*

Proof: The references to [8] and [6] in the proof of Theorem 1 show that the set $A(f)$ is linearly measurable. McMillan has shown [11, Theorem 2] that there exists a subset E of C of positive measure such that f has a finite asymptotic value at each point of E, and [11, Theorem 4] the set of these asymptotic values contains a closed set of positive harmonic measure. It is readily seen that $\mathfrak{H}$ is satisfied for a suitable Γ, and application of Theorem 1 completes the proof.

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University of Wisconsin-Milwaukee

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