ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

372

A MINIMUM PRINCIPLE FOR POSITIVE HARMONIC FUNCTIONS

BY

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To ROLF NEVANLINNA on his 70th birthday

HELSINKI 1965
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1966.372

KESKUSKIRJAPAINO
HELSINKI 1965

A minimum principle for positive harmonic functions

Let D be a simply-connected region in the plane $z = x + iy$, ζ_0 a given boundary point of D and $S = \{z_n\}_1^\infty \subset D$ a sequence of points tending to ζ_0 as $n \to \infty$. Let φ denote a Martin harmonic function corresponding to ζ_0, i.e. φ is positive in D and vanishes at each boundary point $\neq \zeta_0$. The object of this article is to characterize the sequences S which possess the property that foı each positive harmonic u in D, the inequalities

(1) $$u(z_n) \geq \lambda\, \varphi(z_n)\ ,\ n = 1, 2, \ldots, \lambda > 0\ ,$$

imply

(2) $$u(z) \geq \lambda\, \varphi(z)\ ,\ z \in D\ .$$

If the implication (2) is true we shall call S an equivalence sequence for ζ_0, therewith and henceforth allowing this paper some freedom from orthodox notions and terminologies.

Theorem 1. *S is an equivalence sequence for ζ_0 if and only if it contains a subset $\{z_{n_\nu}\}_{\nu=1}^\infty$ with the properties*

(3) $$\sup_{\mu \neq \nu} g(z_{n_\mu}, z_{n_\nu}) < \infty$$

(4) $$\sum_{\nu=1}^{\infty} g(z, z_{n_\nu})\, \varphi(z_{n_\nu}) = \infty\ ,\ z \in D\ ,$$

where g is the Green function for D.

It is convenient to restate and to prove the theorem for the upper half-plane Ω, $z = x + iy$, $y > 0$, letting ζ_0 be the infinite boundary point and $\varphi = y$. When $z = re^{i\Theta}$ tends to ∞ in Ω we have

$$g(i, z) \sim \frac{2 \sin \Theta}{r}\ ,$$

$$g(i, z)\, \varphi(z) \sim 2 \sin^2 \Theta\ .$$

By virtue of these relations the theorem can be reformulated as follows: »The points

$$z_n = x_n + i\, y_n = r_n\, e^{i\Theta_n},\ n = 1, 2, \ldots$$

form an equivalence sequence for the infinite boundary point of the upper half-plane if and only if they contain a subset $\{z_{n_\nu}\}$ satisfying the separation condition

$$\inf_{\mu \neq \nu} \left| \frac{z_{n_\mu} - z_{n_\nu}}{z_{n_\mu} - \bar{z}_{n_\nu}} \right| > 0 , \tag{5}$$

and such that

$$\sum_{\nu=1}^{\infty} \sin^2 \Theta_{n_\nu} = \infty . » \tag{6}$$

The necessity of the conditions is easily established. To each $z_0 \in \Omega$ and to each $\varepsilon , 0 < \varepsilon < 1$, we assign the circular disc

$$\Delta (z_0, \varepsilon) = \left\{ z : \left| \frac{z - z_0}{z - \bar{z}_0} \right| < \varepsilon \right\} , \tag{7}$$

and we recall that Harnack's inequalities for positive harmonic functions in Ω can be written

$$\frac{1 - \varepsilon}{1 + \varepsilon} \leq \frac{u (z)}{u (z_0)} \leq \frac{1 + \varepsilon}{1 - \varepsilon} , z \in \Delta (z_0 , \varepsilon) . \tag{8}$$

If (5) and (6) were not necessary conditions there would exist an equivalence sequence S such that each of its subsets satisfying the separation condition would make the series (6) convergent. However, from any given S it is always possible to select a subsequence $\{z_{n_\nu}\}$ such that, ε being given, the union $\bigcup_{\nu=1}^{\infty} \Delta (z_{n_\nu} , \varepsilon)$ covers S, whereas each z_{n_ν} is contained in the sole disc $\Delta(z_{n_\nu} , \varepsilon)$. The separation condition is therefore satisfied. If (6) were convergent the same would be true of the series

$$u (z) = \sum_{\nu=1}^{\infty} \frac{y\, y_{n_\nu}^2}{(x - x_{n_\nu})^2 + y^2} , \tag{9}$$

and u would represent a positive harmonic function in Ω with the properties

$$u (z_{n_\nu}) > y_{n_\nu} , \quad \nu = 1, 2, \ldots \tag{10}$$

$$u (i\, y) = o (y) , \quad y \to + \infty . \tag{11}$$

On applying (8) both to u and to $\varphi = y$ we find that in each $\Delta(z_{n_\nu} , \varepsilon)$,

$$u (z) > \left(\frac{1 - \varepsilon}{1 + \varepsilon}\right)^2 y = \lambda\, y . \tag{12}$$

This inequality would therefore remain valid on S, but violated at other points of Ω, in view of (11). This proves the necessity of the stated conditions. The sufficiency will be derived from this more precise result:

Lemma I. *Let u be positive and harmonic in Ω and let $E(\lambda)$ denote the set*

$$E(\lambda) = \{z = x + i\,y \,:\, y > 0\,,\quad u(z) \geq \lambda\, y\}\,. \tag{13}$$

Then the divergence of the integral

$$\int_{E(\lambda)} \frac{dx\, dy}{1 + |z|^2} \tag{14}$$

implies $E(\lambda) = \Omega$.

The particular value of λ is immaterial and we may therefore assume $\lambda = 1$ and set $E(1) = E$. As a consequence of Harnack's inequalities we have

$$\frac{\partial u}{\partial y} \leq \frac{u}{y}\,,\; y > 0\,,$$

where the sign of equality is excluded unless $u = ay$, in which case the lemma is trivially true. We may therefore assume that the upper sign holds throughout Ω. This implies that $u(x + iy)/y$ for fixed x is strictly decreasing with increasing y. If not void the open set $\Omega - E$ has thus a boundary which meets vertical lines in at most one finite point. Each component of $\Omega - E$ is therefore an unbounded simply-connected region. Let D be a component and Γ its boundary. Without loss of generality we assume that D contains a point $z = iy_0$ on the imaginary axis. The function $v(z) = y - u(z)$ is by assumption harmonic and strictly positive in D, vanishes at all finite boundary points and is thus a Martin function for D. We shall prove that this implies that (14) converges.

In the sequel we shall denote by C_r, $r > 1 + y_0$, the region

$$C_r = \{z = x + iy \,:\, y > 0\,,\; |z + i| < r\}$$

and by γ_r the largest open arc of the circle $|z + i| = r$ contained in D and containing the point $i\,(r - 1)$. Together with Γ the arc γ_r forms the boundary of a well defined simply-connected subregion D_r of C_r.

In the continuation of the proof we shall use the fruitful notion of harmonic measure which plays such a prominent role in the work of Rolf Nevanlinna. The harmonic measure, $h(z_0\,, \gamma_r)$, of γ_r is by definition the value at z_0 of the bounded harmonic function in D_r which equals 1 on γ_r and vanishes elsewhere on the boundary. By the minimum principle for harmonic functions,

(15) $$v(z_0) \leq h(z_0, \gamma_r) \max_{z \in \gamma_r} v(z) < h(z_0, \gamma_r) \cdot r .$$

In order to estimate h we recall this result ([1], p. 10).

Lemma II. *Let D be simply-connected, z_0 a point in D and γ a boundary continuum. Let ψ be harmonic in D and have the properties:* $\psi(z_0) = 0$, $\psi(z) \geq L > 0$ *on* γ,

$$A = \int_D |\operatorname{grad} \psi|^2 \, dxdy < \infty$$

Then

(16) $$h(z_0, \gamma) < e^{-\frac{\pi L^2}{A}} .$$

For the region D_r the choice

$$\psi(z) = \log \left| \frac{z + i}{z_0 + i} \right|$$

yield

$$L = \log r - \log (1 + y_0) .$$

Define $E_r = E \cap C_r$, let $m(r)$ be determined by the relation

$$\pi \, m(r) = \int_{E_r} |\operatorname{grad} \psi|^2 \, dx \, dy = \int_{E_r} \frac{dx \, dy}{|z + i|^2} ,$$

and observe that

$$\int_{C_r} \frac{dx \, dy}{|z + i|^2} < \pi \log r .$$

Hence,

$$A < \pi (\log r - m(r))$$

and

$$\frac{\pi L^2}{A} > \frac{(\log r - \log (1 + y_0))^2}{\log r - m(r)} \geq$$

$$\geq \log r - 2 \log (1 + y_0) + m(r) \left\{ 1 + O\left(\frac{1}{\log r}\right) \right\}$$

If (14) diverges, then $m(r)$ will increase to ∞ with r and we would have $h = o\left(\frac{1}{r}\right)$, and consequently $v(z_0) = 0$, contradictory to the assumption $v(z_0) > 0$. This proves Lemma I.

We can now continue the proof of the sufficiency of the conditions in Theorem I. In order to simplify the notations we let $\{z_n\}$ denote the subsequence of S satisfying (5) and (6). By virtue of the separation condition (5) the discs $\Delta(z_n, \varepsilon)$ are disjoint if ε is small enough, and they are contained, according to (12), in the set $E(\lambda')$ if

$$\lambda' = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 \lambda .$$

The divergence of

$$\sum_1^\infty \sin^2 \Theta_n$$

therefore implies that the integral (14) for $E(\lambda')$ diverges, the radius of $\Delta(z_n, \varepsilon)$ being $> 2\varepsilon y_n$. Lemma I asserts that everywhere in Ω, $u(z) \geq \lambda' y$, and this concludes the proof since λ' can be taken arbitrary close to λ.

We want to point out that Lemma I remains true also for positive superharmonic functions. The proof is the same except for one important difference. The region replacing D_r will be multiply-connected and Lemma II not valid. The proof can however be carried through by means of the following more general but still unpublished result.

Let D be limited by a finite number of Jordan curves $\{\Gamma_\nu\}^n$, and let γ be a closed boundary set carried by *one and the same* boundary component, say Γ_1. Let α be an arc joining the given point z_0 with some point belonging to the set $\Gamma_1 - \gamma$. Then

$$\max_{z \in \alpha} h(z, \gamma) < 5\, e^{-\pi\lambda}$$

where λ stands for the extremal length of the family of curves joining α and γ within D.

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Printed October 1965