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REMARKS ON THE INTEGRABILITY OF THE DERIVATIVES OF QUASICONFORMAL MAPPINGS

BY

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To Rolf Nevanlinna on his 70th birthday

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1. *Introduction.* Let w be a K-quasiconformal (not necessarily homeomorphic) mapping of a domain D of the euclidean plane R^2 into R^2. In D, w is then a weak L^2-solution of a Beltrami equation

$$w_{\bar{z}} = \varkappa w_z ,$$

where $\varkappa$, the complex dilatation of w, is a Borel-measurable function which is defined for almost all $z \in D$ and satisfies the inequality $|\varkappa(z)| \leq (K - 1)/(K + 1)$. (For the general theory of quasiconformal mappings, we refer to [3].)

Let Sf,

$$Sf(z) = -\frac{1}{\pi} \iint_{R^2} \frac{f(\zeta)}{(\zeta - z)^2} \, d\xi d\eta , \quad (\zeta = \xi + i\eta) ,$$

denote the Hilbert-transformation of the function f. With a proper interpretation of the singular integral, S is a bounded linear mapping of L^p, $p > 1$, into itself (Calderón—Zygmund [2]). Its L^p-norm $\|S\|_p$ is continuous in p, has the value 1 for $p = 2$, and grows to ∞ as $p \to \infty$.

Bojarski [1] proved that the partial derivatives of a K-quasiconformal mapping are locally in L^p for every value of p for which

$$\|S\|_p < (K + 1)/(K - 1) . \tag{1}$$

In conjunction with the above mentioned properties of $\|S\|_p$, this gave him the following result: If $p(K)$ denotes the least upper bound of the positive numbers p such that every K-quasiconformal mapping has locally L^p-integrable partial derivatives, then $p(K) > 2$ for all values of K, $1 \leq K < \infty$.

Calderón and Zygmund proved that

$$C_0 = \limsup_{p \to \infty} \frac{\|S\|_p}{p} < \infty . \tag{2}$$

On the other hand, the example

$$w(z) = z|z|^{\frac{1}{K}-1} \tag{3}$$

shows that there exist K-quasiconformal mappings whose derivatives are not locally in L^p for $p = 2K/(K-1)$. Hence, in view of the condition (1), $\|S\|_{2K/(K-1)} \geqq (K+1)/(K-1)$, which is equivalent to

$$\|S\|_p \geqq p - 1 .$$

The value of $p(K)$ is not known for any K, $1 < K < \infty$. From the above example (3) it follows that

$$p(K) \leqq \frac{2K}{K-1} . \tag{4}$$

For $K = 1$ we obviously have $p(K) = \infty$. Additional information about the behaviour of $p(K)$ for K near to 1 is obtained if the Bojarski condition (1) is combined with (2). With the notation

$$C = \liminf_{K \to 1} (K-1)p(K)$$

it follows that

$$C \geqq 2/C_0 ,$$

i.e. C is positive. By (4), $C \leqq 2$, and if (4) actually holds as an equality, as it seems reasonable to conjecture, then we would have $C = 2$.

In this paper we shall establish the inequality

$$p(K) \geqq \frac{2K^{2/C}}{K^{2/C}-1} \tag{5}$$

for all values of $K > 1$. In view of (4) it is likely that the problem of determining $p(K)$ for an arbitrary $K > 1$ has thus been reduced to studying the asymptotic behaviour of $p(K)$ as $K \to 1$.

Since

$$\frac{2K^{2/C}}{K^{2/C}-1} \geqq \frac{CK}{K-1}$$

for $0 < C \leqq 2$, we also have the lower estimate

$$p(K) \geqq \frac{CK}{K-1}$$

for all values of K.

2. *Decomposition formula for a quasiconformal mapping.* A K-quasiconformal mapping is an analytic function of a K-quasiconformal homeomorphism. Hence, there is no loss of generality from the standpoint of our problem, if we restrict ourselves in the following to homeomorphic mappings.

The proof of (5) is based on the possibility of representing a plane quasiconformal homeomorphism as a composition of mappings with lower maximal dilatations:

L e m m a. *Let w be a K-quasiconformal homeomorphism and K_1, K_2 two numbers $\geqq 1$ such that $K_1 K_2 = K$. Then w admits a representation*

$$w = w_1 \circ w_2 \,, \tag{6}$$

where w_i is a K_i-quasiconformal homeomorphism, $i = 1, 2$.

Proof: We write w in the form (6), where w_1 and w_2 are quasiconformal homeomorphisms. In order to show that w_1 and w_2 can be chosen to be K_1- and K_2-quasiconformal, respectively, we first express the complex dilatation $\varkappa_{w_2}$ of w_2 in terms of the complex dilatations of w and w_1^{-1}. From $w_2 = w_1^{-1} \circ w$ it follows after formal computation that

$$|\varkappa_{w_2}(z)| = \left| \frac{\varkappa_w(z) + \varkappa_{w_1^{-1}}(w(z)) e^{-2i\arg w_z(z)}}{1 + \varkappa_w(z) \varkappa_{w_1^{-1}}(w(z)) e^{-2i\arg w_{\bar{z}}(z)}} \right| = \left| \frac{\varkappa^*(z) + \varkappa_{w_1^{-1}}(w(z))}{1 + \overline{\varkappa^*(z)} \varkappa_{w_1^{-1}}(w(z))} \right| , \tag{7}$$

where

$$\varkappa^*(z) = \varkappa_w(z) e^{2i\arg w_z(z)} \,.$$

Next we make use of the fundamental Existence Theorem, which says that the complex dilatation of a K-quasiconformal mapping can be prescribed almost everywhere as a measurable function whose modulus does not exceed $(K - 1)/(K + 1)$. Therefore, w_1^{-1} can be constructed such that

$$\varkappa_{w_1^{-1}}(w(z)) = - \frac{K_1 - 1}{K_1 + 1} e^{i\arg \varkappa^*(z)} \qquad \text{a.e.}$$

The first conclusion is that w_1^{-1}, and thus w_1, is K_1-quasiconformal. From (7) it further follows that

$$|\varkappa_{w_2}(z)| \leqq \frac{\left| |\varkappa_w(z)| - |\varkappa_{w_1^{-1}}(w(z))| \right|}{1 - |\varkappa_w(z) \varkappa_{w_1^{-1}}(w(z))|} \leqq \frac{\dfrac{K-1}{K+1} - \dfrac{K_1 - 1}{K_1 + 1}}{1 - \dfrac{K-1}{K+1} \cdot \dfrac{K_1 - 1}{K_1 + 1}}$$

$$= \frac{K - K_1}{K + K_1} = \frac{K_2 - 1}{K_2 + 1} \qquad \text{a.e.}$$

Hence, w_2 is K_2-quasiconformal, and the Lemma is proved.

3. *Application of the decomposition formula.* Let w, w_1, and w_2 be as in the above Lemma. For the Jacobians of these mappings we then have

$$J_w(z) = J_{w_1}(w_{w_2}(z))J_{w_2}(z) \tag{8}$$

a.e. in the considered domain D.

Let $F \subset D$ be a compact set. From (8) it follows, by Hölder's inequality, that

$$\iint_F J_w(z)^p dxdy \leqq \left(\iint_F J_{w_1}(w_2(z))^{pq} J_{w_2}(z) dxdy\right)^{\frac{1}{q}} \left(\iint_F J_{w_2}(z)^{\left(p-\frac{1}{q}\right)q'} dxdy\right)^{\frac{1}{q'}}$$

$$= \left(\iint_{w_2(F)} J_{w_1}(\zeta)^{pq} d\xi d\eta\right)^{\frac{1}{q}} \left(\iint_F J_{w_2}(z)^{\left(p-\frac{1}{q}\right)q'} dxdy\right)^{\frac{1}{q'}} \tag{9}$$

for all values of p and q for which the above integrals exist, with $1/q + 1/q' = 1$.

For a quasiconformal mapping the Jacobian belongs locally to every L^r with $r < \frac{1}{2}p(K)$. Hence, the right-hand integrals are finite if we set

$$pq = \frac{1}{2}\, p(K_1) - \varepsilon\,, \qquad \left(p - \frac{1}{q}\right) q' = \frac{1}{2}\, p(K_2) - \varepsilon\,,$$

$0 < \varepsilon < \frac{1}{2}$. Then

$$p = \frac{(p(K_1) - 2\varepsilon)(p(K_2) - 2\varepsilon)}{2(p(K_1) + p(K_2) - 2 - 4\varepsilon)}\,,$$

and letting $\varepsilon \to 0$ we conclude from (9)

$$p(K) \geqq \frac{p(K_1)p(K_2)}{p(K_1) + p(K_2) - 2} \tag{10}$$

or

$$1 - \frac{2}{p(K)} \geqq \left(1 - \frac{2}{p(K_1)}\right)\left(1 - \frac{2}{p(K_2)}\right). \tag{10'}$$

4. *Continuity of* $p(K)$. As a first application of (10), we set $K = x + \Delta x$, $K_1 = x$, $K_2 = 1 + \Delta x/x$. Then $p(K_2) \to \infty$ as $\Delta x \to 0$, and we obtain from (10)

$$p(x + 0) \geqq p(x)\,.$$

Similarly, if $K = x$, $K_1 = x - \Delta x$, $K_2 = x/(x - \Delta x)$, we get

$$p(x - 0) \leqq p(x)\,.$$

On the other hand, it is obvious from the definition that $p(x)$ does not increase as x increases. We have thus proved: $p(K)$ *depends continuously on* K.

5. *Lower estimate of* $p(K)$. In order to establish (5), we first observe that repeated application of (10′) for $K_1 = K_2$ yields

$$1 - \frac{2}{p(K)} \geqq \left(1 - \frac{2}{p(K^{2^{-n}})}\right)^{2^n} \tag{11}$$

for $n = 1, 2, \ldots$. Given an $\varepsilon > 0$, it follows from the definition of C that

$$p(K^{2^{-n}}) \geqq \frac{(C - \varepsilon)K^{2^{-n}}}{K^{2^{-n}} - 1}$$

for all sufficiently large values of n. Hence, for n large enough,

$$\frac{2}{p(K^{2^{-n}})} \leqq \frac{2}{C - \varepsilon}\left(1 - \frac{1}{K^{2^{-n}}}\right) \leqq \frac{2}{C - \varepsilon} \cdot \frac{\log K}{2^n},$$

and so in view of (11),

$$1 - \frac{2}{p(K)} \geqq \left(1 - \frac{\log K^{\frac{2}{C-\varepsilon}}}{2^n}\right)^{2^n}.$$

Letting $n \to \infty$, we conclude from this

$$1 - \frac{2}{p(K)} \geqq \frac{1}{K^{2/C}}, \tag{12}$$

and (5) follows.

6. *Distortion of the area.* Let again w be a K-quasiconformal homeomorphism of a domain D and F a compact set in D. Since the Jacobian J_w is L^p-integrable over F for any $p < \frac{1}{2}p(K)$, application of Hölder's inequality yields the following result on the distortion of the area under the mapping w ([3], V. 5): To every $\delta < 1 - 2/p(K)$ there corresponds a finite constant A such that

$$m(w(E)) \leqq A\, m(E)^{\delta} \tag{13}$$

(m = two-dimensional Lebesgue measure) for every measurable set $E \subset F$.

Let $\delta(K)$ denote the least upper bound of the numbers δ for which (13) is always valid. The example (3) shows that

$$\delta(K) \leqq \frac{1}{K}\,.$$

On the other hand, since $\delta(K) \geqq 1 - 2/p(K)$, it follows from (12) that

$$\delta(K) \geqq \frac{1}{K^{2/C}}\,.$$

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