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ON HOMOLOGY THEORIES IN LOCALLY CONNECTED SPACES

BY

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Introduction

In this paper we use the method of normal refinements to derive the results of Mardešić [6] simultaneously for many homology theories. The notations and results of § 2 in [3] are assumed.

Section 1 introduces the homology theories used here and the second section presents a uniqueness theorem for them. In the third section similar results are outlined for relative homology. The last section is devoted to some applications to paracompact spaces. The appendix contains some corrections to my previous paper [3].

§1. Fine Complexes

1. Let Z denote the ring of integers. R shall be a fixed ring, M_R the category of R-modules, and C_R the category of augmented chain complexes in M_R.

Let X be a set. The set $P(X)$ of subsets of X together with the inclusions $i^{P,Q} : P \to Q$ $(P \subset Q \subset X)$ is a category (cf. [2], p. 16). If $X' \subset X$, then all functors of $P(X)$ are identified on $P(X')$ with their restrictions to $P(X')$.

Definition 1. A complex C on X is a covariant functor $P(X) \to C_R$, i.e. the value of C on $P \subset X$ is an augmented chain complex $C(P) = (C_n(P))_{n \in Z}$. The n-dimensional homologyfunctor H_n of C is the covariant functor $P(X) \to M_R$, whose value $H_n(P)$ on $P \subset X$ is the module $H_n(C(P))$, the functorial homomorphisms

$$H_n(i^{P,Q}) : H_n(P) \to H_n(Q) \qquad (P \subset Q \subset X)$$

being induced by the functorial homomorphisms

$$C(i^{P,Q}) : C(P) \to C(Q) .$$

2. A covering $\alpha = (A_i)_{i \in I}$ of X is a family of subsets of X, whose union is X. The covering α is said to be proper, if $A_i = A_j$ implies $i = j$ $(i, j \in I)$. If $s \subset \alpha$, $s \neq \emptyset$, we shall denote by $\widehat{s}$ the intersection of the members of s. If $A \subset X$ then $\alpha \cap A$ is the

covering $(A_i \cap A)_{i \in I}$ of A. If $\varepsilon = (E_j)_{j \in J}$ is another covering of X, then $\alpha \cap \varepsilon$ stands for the covering $(A_i \cap E_j)_{i \in I, j \in J}$ of X.

If α is a covering of X and C a complex on X, we shall denote by $\overset{\alpha}{C}_{p,q}$ $(p, q \in Z)$ (i) the zero-module, if $p < -1$, (ii) $C_q(X)$, if $p = -1$ and (iii) the direct sum $\sum_{i_0, \ldots, i_p \in I} C_q(A_{i_0} \cap \ldots \cap A_{i_p})$, if $p > -1$. Let φ_k $(k = 0, \ldots, p)$ denote the functorial homomorphism

$$C_q(A_{i_0} \cap \ldots \cap A_{i_p}) \to C_q(A_{i_0} \cap \ldots \cap \hat{A}_{i_k} \cap \ldots \cap A_{i_p})$$

for $p > 0$, and

$$C_q(A_{i_0}) \to C_q(X)$$

for $p = 0$. If $p \geq 0$, we define a homomorphism

$$\partial' : \overset{\alpha}{C}_{p,q} \to \overset{\alpha}{C}_{p-1,q} \qquad (q \in Z)$$

by

(1) $$\partial'(c) = \sum_{k=0}^{p} (-1)^k \varphi_k(c)$$

for $c \in C_q(A_{i_0} \cap \ldots \cap A_{i_p})$. It is easy to see that $\partial'\partial' = 0$. The family $'\overset{\alpha}{C}_{*q} = (\overset{\alpha}{C}_{p,q})_{p \in Z}$ becomes thus an augmented chain complex. We shall denote by $H_p(\alpha)$ the module $H_p('\overset{\alpha}{C}_{*-1})$ for each $p \in Z$.

Let β be another covering of X. The covering α is said to be a refinement of β, if there exists a mapping $\pi^{\alpha\beta} : \alpha \to \beta$, called projection, such that $A_i \subset \pi^{\alpha\beta}(A_i)$ for each $A_i \in \alpha$. We shall then write $\alpha > \beta$. The projection $\pi^{\alpha\beta}$ induces for each $q \in Z$ a homomorphism

$$\pi^{\alpha\beta}_{*q} : '\overset{\alpha}{C}_{*q} \to '\overset{\beta}{C}_{*q}$$

with components

$$\pi^{\alpha\beta}_{p,q} : \overset{\alpha}{C}_{p,q} \to \overset{\beta}{C}_{p,q} \qquad (p \in Z)$$

defined by

(2) $$\pi^{\alpha\beta}_{p,q}(c) = C_q(\widehat{i^{s}, \pi^{\alpha\beta}(s)})(c)$$

and linearity $(c \in C_q(A_{i_0} \cap \ldots \cap A_{i_p}),\ s = (A_{i_0}, \ldots, A_{i_p}))$. We shall denote by $\pi^{\alpha\beta}_{(p)q}$ the functorial homomorphism

$$H_p(\pi^{\alpha\beta}_{*q}) : H_p('\overset{\alpha}{C}_{*q}) \to H_p('\overset{\beta}{C}_{*q})$$

and by $\pi^{\alpha\beta}_{(p)}$ the homomorphism $\pi^{\alpha\beta}_{(p)-1} : H_p(\alpha) \to H_p(\beta)$ $(p, q \in Z)$.

3. The differentials

$$\partial : C_q(A_{i_0} \cap \ldots \cap A_{i_p}) \to C_{q-1}(A_{i_0} \cap \ldots \cap A_{i_p}) \tag{3}$$

of $C(A_{i_0} \cap \ldots \cap A_{i_p})$ commute with the differentials (1). If we denote by $(-1)^{p+1} \partial''$ the homomorphism of $\overset{\alpha}{C}_{p,q}$ into $\overset{\alpha}{C}_{p,q-1}$ induced by the homomorphisms (3), it follows that the homomorphisms ∂' and ∂'' make the family $\overset{\alpha}{C}_* = (\overset{\alpha}{C}_{p,q})_{p,q \in Z}$ an augmented double chain complex (see [3], p. 21). It follows immediately from the definitions (1), (2) and (3), that the homomorphisms $\pi^{\alpha\beta}_{p,q}$, ∂' and $(-1)^{p+1} \partial''$ commute with each other. Hence they induce functorially a homomorphism $\pi^{\alpha\beta} : \overset{\alpha}{C}_* \to \overset{\beta}{C}_*$ of augmented double chain complexes, which coincides with $\pi^{\alpha\beta}_{*-1}$ on $'\overset{\alpha}{C}_{*-1}$ and is the identity map on $''\overset{\alpha}{C}_{-1*} = ''\overset{\beta}{C}_{-1*}$.

Definition 2. The complex C is said to be fine with respect to α, if $'\overset{\alpha}{C}_{*q}$ is acyclic for $q > -1$; i.e., if $H_p('\overset{\alpha}{C}_{*q}) = 0$ for each $p \in Z$ and $q < -1$.

Henceforth we consider only those coverings with respect to which C is fine; this is equivalent to condition (2.2) of [3] on page 22. We obtain thus the canonical homomorphisms

$$h^\alpha_n : H_n(X) = H_n(''\overset{\alpha}{Z}_{-1*}) \xrightarrow{\overset{\alpha}{D}_n} H_{-1}(''\overset{\alpha}{Z}_{n*}) \xrightarrow{j} H_n('\overset{\alpha}{C}_{*-1}) = H_n(\alpha)\,, \tag{4}$$

where j is a canonical epimorphism (see [3], pp. 22, 29). The equation

$$h^\beta_n = \pi^{\alpha\beta}_{(n)}\, h^\alpha_n \tag{5}$$

follows immediately from the definitions.

Definition 3. The coverings α and β are said to determine $H_n(X)$, if h^β_n is an isomorphism onto $\pi^{\alpha\beta}_{(n)}(H_n(\alpha))$ (see [1], p. 320).

Definition 4. (cf. [5], p. 214) A net of X is a set N of coverings of X, directed by the relation $<$. For each $n \in Z$ we shall denote by $H_n(N)$ the inverse limit of the modules $H_n(\alpha)$ $(\alpha \in N)$ under the homomorphisms $\pi^{\alpha\beta}_{(n)}$ and by $\pi^\beta_{(n)} : H_n(N) \to H_n(\beta)$ the inverse limit of the homomorphisms $\pi^{\alpha\beta}_{(n)}$ $(\alpha \in N)$. The equations (5) give rise to limit homomorphisms

$$t_n : H_n(X) \to H_n(N) \quad (n \in Z) \tag{6}$$

of the homomorphisms h^α_n $(\alpha \in N)$ satisfying the equations

$$h^\alpha_n = \pi^\alpha_{(n)}\, t_n \qquad (\alpha \in N\,,\ n \in Z)\,. \tag{7}$$

Examples: 1° Let X be a topological space, A a subspace of X, M a module and α an open covering of X. If $P \subset X$, let $C^A(P)$ be the augmented complex $S(P, P \cap A) \otimes M$ of singular chains of $(P, P \cap A)$ with coefficients in M and $C^{\alpha A}(P)$ the subcomplex $S_\alpha(P, P \cap A) \otimes M$ of $C^A(P)$ generated by singular simplices s, for which one can find an element of α which contains the carrier $||s||$ of s (see [6], p. 152). Let T_n^P be the set of those generators of $C_n^{\alpha A}(P)$ whose carriers are not in A. Then $C_n^{\alpha A}(P)$ is canonically isomorphic with a direct sum $\sum_{s \in T_n^P} M_s$ of modules M_s isomorphic with M. If $P \subset Q \subset X$, then the functorial homomorphism $C_n^{\alpha A}(P) \to C_n^{\alpha A}(Q)$ is induced by the inclusion $T_n^P \subset T_n^Q$. Thus $'\overset{\alpha}{C}{}_{*n}^{\alpha A}$ $(n > -1)$ is canonically isomorphic with a direct sum $\sum_{s \in T_n^X} C(\alpha^s, M_s)$, where α^s is the closed subcomplex of the nerve of α composed of those members of α which contain $||s||$. Because the complexes α^s are simplices, it follows that $C^{\alpha A}$ is fine with respect to α (cf. [4], p. 706).

We have by definition $C_{-1}^{\alpha A}(P) = M$ for $P \neq \emptyset$ and $P \cap A = \emptyset$ and $C_{-1}^{\alpha A}(P) = 0$ otherwise. Thus $'\overset{\alpha}{C}{}_{-1}^{\alpha A}$ is isomorphic with the augmented complex $C(X_\alpha, A_\alpha, M)$ of the M-valued chains of the nerve of α modulo A. On the other hand it is well known that the inclusion $C^{\alpha A}(P) \subset C^A(P)$ induces a homotopy equivalence of complexes for each $P \subset X$ and for each open covering α of X. These facts will be used in the last paragraph.

2° As in [3], pp. 10, 14, let M be a covariant functor $P(X) \to M_R$ such that $M(\emptyset) = 0$; let S be a spectrum of X and α a finite covering of X belonging to S. Then S and M define a complex C^S on X, whose components C_n^S $(n \in Z)$ are the functors $C_n(S, \overset{\circ}{M})$ (see [3], p. 17). We have shown in [3], pp. 28—29, that C^S is fine with respect to α (on page 29, φ should be added in front of c_m^β, $-c_m^\beta$ and $-c_{mk}^\beta$ in the equations (3.4) and (3.5)).

If β is a finite covering of X, we may use S to construct a new spectrum which contains β, as follows: Let $S \cap \beta$ be the set $\{\alpha \cap \beta ; \alpha \in S\}$. If $\alpha > \alpha'$ $(\alpha, \alpha' \in S)$ and $\pi^{\alpha\alpha'}$ is the corresponding projection belonging to S, we define a projection π of $\alpha \cap \beta$ in $\alpha' \cap \beta$ by $\pi(A_i \cap B_j) = \pi^{\alpha\alpha'}(A_i) \cap B_j$ $(A_i \in \alpha, B_j \in \beta)$. Then $S \cap \beta$ with these projections is a spectrum which contains β.

If β has a refinement γ in S, then C^S is fine with respect to β. In fact, without changing the complex C^S on X, we may substitute for S the spectrum $S(\beta) = \{\alpha > \gamma ; \alpha \in S\} \cup \beta$, where $\pi^{\gamma\beta} : \gamma \to \beta$ is fixed arbitrarily and $\pi^{\alpha\beta} : \alpha \to \beta$ is the composed projection $\pi^{\gamma\beta}\pi^{\alpha\gamma}$, $\pi^{\alpha\gamma}$belonging to S.

Remark: If X is a topological space, then the spectra in the examples $A - D$ of [3], p. 15, are rather uninteresting from the homological viewpoint, because $C_n(U(X), M|P) = C_n(F(X), M|P) = 0$ for $n > 0$ and for each $P \subset X$. Instead the grating spectrum Σ of X yields non-trivial homology theories (cf. [5], p. 279).

We take this opportunity to note that remark 2° in [3], p. 32, can be corrected by transferring it into cohomological form. The theorem of Floyd then follows by Pontryagin duality.

§2. n-refinements

4. Let α and β be two coverings of X, and N a net of X.

Definition 5. (cf. [1], p. 320) The covering α is said to be a (strong) n-refinement of β if there exists a projection $\pi^{\alpha\beta} : \alpha \to \beta$ such that

$$\operatorname{Im} (H_n (i^{A_i, \pi^{\alpha\beta}(A_i)}) : H_n (A_i) \to H_n (\pi^{\alpha\beta} (A_i))) = 0$$

$$(\operatorname{Im} (H_k (i^{\widehat{s}, \widehat{\pi^{\alpha\beta}(s)}}) : H_k (\widehat{s}) \to H_k (\widehat{\pi^{\alpha\beta} (s)})) = 0)$$

for each $A_i \in \alpha$ (for each $s \subset \alpha$ and $k < n + 1$). We shall then write $\alpha \mid n > \beta$ $(\alpha \mid n \gg \beta)$.

Definition 6. The net N is said to be semi-lc_n if there exists an element of N which has an n-refinement in N; N is said to be lc_n if every element of N has a strong n-refinement in N.

Lemma 1. If $\overset{n}{\alpha} \mid n - 2 \gg \ldots \mid n - 2 \gg \overset{0}{\alpha} \mid n - 1 > \overset{-1}{\alpha}$ $(\overset{i}{\alpha} \in N ; i = - 1, 0, \ldots, n)$, then $\overset{n}{h}{}^{\alpha}_{n-1}$ is injective and $\operatorname{Im} (\overset{-1}{h}{}^{\alpha}_{n}) = \operatorname{Im} (\pi^{\overset{n}{\alpha}, \overset{-1}{\alpha}}_{(n)})$.

Proof. Let $\overset{i}{\pi} : \overset{i}{\alpha} \to \overset{i-1}{\alpha} (i = 0, \ldots, n)$ be the projections defining the relations $\mid n - 2 \gg$ and $\mid n - 1 >$ above and $\overset{n,-1}{\pi}$ the composite projection $\overset{0}{\pi}\overset{1}{\pi} \ldots \overset{n}{\pi}$. For each $k \in Z$ we have the commutative diagram

$$\begin{array}{ccc} & H_k(X) & \\ \swarrow \overset{k}{h}{}^{\alpha}_{n} & & \overset{-\alpha}{h}{}^{1}_{k} \searrow \\ H_k(\overset{n}{\alpha}) & \xrightarrow{\overset{n,-1}{\pi}_{(k)}} & H_k(\overset{-1}{\alpha}) \, , \end{array}$$

which is identical with the commutative diagram

$$\begin{array}{ccc} H_k(''\overset{n}{Z}_{-1*}) & = & H_k(''\overset{-1}{Z}_{-1*}) \\ \swarrow & & \searrow \\ H_k('\overset{n}{C}_{*-1}) & \longrightarrow & H_k('\overset{-1}{C}_{*-1}) \end{array}$$

of the corresponding double complexes $\overset{i}{C}_* = \overset{\overset{i}{\alpha}}{C}_* \; (-1 \leq i \leq n)$. In virtue of the assumption the image of the canonical homomorphism

$$H_{n-k-1}(''\overset{k}{C}_{k*}) \to H_{n-k-1}(''\overset{k-1}{C}_{k*}) \,, \tag{8}$$

induced by the homomorphism $\overset{k}{\pi}_*$, is zero for each $k = 0, \ldots, n$. Setting $k = n$ in (8) we see as in [3] p. 31, that

$$\begin{aligned} &\operatorname{Ker}(H_{-1}(''\overset{n}{Z}_{n-1*}) \xrightarrow{\;j\;} H_{n-1}('\overset{n}{C}_{*-1})) \subset \\ &\operatorname{Ker}(H_{-1}(''\overset{n}{Z}_{n-1*}) \longrightarrow H_{-1}(''\overset{n-1}{Z}_{n-1*})) \,. \end{aligned} \tag{9}$$

Theorem 1 and Corollary 1 of [3], pp. 25, 26, then imply the assertion.

Corollary 1. If $\alpha^{2n+2} | n-1 \gg \ldots . | n-1 \gg \alpha^0 | n > \alpha^{-1}$, then α^{2n+2} and α^{n+1} determine $H_p(X)$ for each $p < n+1$.

Theorem 1. If a net N of X is lc_{n-1} and semi-lc_n, then the homomorphism $t_p : H_p(X) \to H_p(N)$ is bijective for each $p < n+1$.

Proof: In virtue of Corollary 1, h_p^α is an isomorphism onto $\pi_{(p)}^\alpha(H_p(N))$ for each sufficiently fine element α of N. If $\beta > \alpha$ $(\beta \in N)$, the relations (5) imply that

$$\pi_{(p)}^{\beta\alpha} : \pi_{(p)}^\beta(H_p(N)) \to \pi_{(p)}^\alpha(H_p(N))$$

is bijective. Because $H_p(N)$ is the inverse limit of the modules $\pi_{(p)}^\alpha(H_p(N))$ under the homomorphisms $\pi_{(p)}^{\beta\alpha}$, the theorem follows from the equations (7).

§3. Relative homology

5. In this section we shall indicate briefly how the ideas above can be transferred to relative homology. We shall suppose that $C(i^{P,Q}) : C(P) \to C(Q)$ defines $C(P)$ as a subcomplex of $C(Q)$ for all $P \subset Q \subset X$. Let A be a fixed subset of X. Setting $H_n(X, A) = H_n(C(X)/C(A))$ for each $n \in Z$, we obtain the exact homology sequence S_C of C:

$$S_C : \ldots H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \ldots$$

A covering $\alpha = (\alpha_1, \alpha_2)$ of (X, A) is a covering α_1 of X and a subfamily α_2 of α_1, whose members cover A (cf. [6], p. 151). If $\beta = (\beta_1, \beta_2)$ is another covering of (X, A), then a projection $\pi^{\alpha\beta}$: $\alpha \to \beta$ is a projection $\alpha_1 \to \beta_1$ such that $\pi^{\alpha\beta}(\alpha_2) \subset \beta_2$. The covering α is then said to be a refinement of β (in symbols $\alpha > \beta$).

Henceforth we shall write $\overset{\alpha_2}{C}_*$ and $H_n(\alpha_2)$ for $\overset{\alpha_2 \cap A}{C}_*$ and $H_n(\alpha_2 \cap A)$ respectively. Then $\overset{\alpha_2}{C}_*$ may be identified with a subcomplex of $\overset{\alpha_1}{C}_*$. Writing $H_n(\alpha) = H_n('\overset{\alpha_1}{C}_{*-1}/'\overset{\alpha_2}{C}_{*-1})$ for each $n \in Z$ we obtain the exact homology sequences S_α and S_β below. The projection $\pi^{\alpha\beta}$ induces a homomorphism $\pi_S^{\alpha\beta}$ of S_α into S_β, which is represented by the commutative diagram

$$\begin{array}{lccccccccc}
S_\alpha : & \ldots \to & H_n(\alpha_2) & \to & H_n(\alpha_1) & \to & H_n(\alpha) & \to & H_{n-1}(\alpha_2) & \to \ldots \\
\pi_S^{\alpha\beta} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
S_\beta : & \ldots \to & H_n(\beta_2) & \to & H_n(\beta_1) & \to & H_n(\beta) & \to & H_{n-1}(\beta_2) & \to \ldots
\end{array}$$

The complexes $'\overset{\alpha_1}{C}_{*q}$ and $'\overset{\alpha_2}{C}_{*q}$ are by assumption acyclic for $q > -1$. Denoting by $\overset{\alpha}{C}_*$ the quotient complex $\overset{\alpha_1}{C}_*/\overset{\alpha_2}{C}_*$, it follows that $'\overset{\alpha}{C}_{*q} = '\overset{\alpha_1}{C}_{*q}/'\overset{\alpha_2}{C}_{*q}$ is acyclic for $q > -1$. We obtain in this way the homomorphisms $h_S^\alpha : S_C \to S_\alpha$ such that

$$h_S^\beta = \pi_S^{\alpha\beta} \, h_S^\alpha . \tag{10}$$

A net of (X, A) is a set N of coverings of (X, A) directed by the relation $<$. The coverings α_1 and $\alpha_2 \cap A$ $(\alpha \in N)$ are the elements of the nets N_1 and N_2 of X and A respectively. Let $H_n(N)$ be the inverse limit of the modules $H_n(\alpha)$ $(\alpha \in N)$ induced by the projections. In general the limit sequence

$$S_N : \ldots \to H_n(N_2) \to H_n(N_1) \to H_n(N) \to H_{n-1}(N_2) \to \ldots$$

of the sequences S_α $(\alpha \in N)$ is not exact. If $\pi_S^\beta : S_N \to S_\beta$ is the limit homomorphism of the homomorphisms $\pi_S^{\alpha\beta}$ $(\alpha \in N , \alpha > \beta)$, the equations (10) give rise the a homomorphism $t_S : S_C \to S_N$ such that

$$h_S^\alpha = \pi_S^\alpha \, t_S \tag{11}$$

for each $\alpha \in N$.

6. If N_1 and N_2 are lc_{n-1} and semi-lc_n, there exists for each sufficiently fine $\delta \in N$ a sequence $\alpha > \beta > \gamma > \delta$ of elements of N, where $\alpha_2 \cap A$ and $\beta_2 \cap A$, $\beta_2 \cap A$ and $\gamma_2 \cap A$ as well as $\gamma_2 \cap A$ and $\delta_2 \cap A$ determine $H_p(A)$ for each $p < n + 1$ and α_1 and β_1, β_1 and γ_1 as well as γ_1 and δ_1 determine $H_p(X)$ for each $p < n + 1$. Consider the following part of the composite homomorphism $\pi_S^{\gamma\delta} \, \pi_S^{\beta\gamma} \, \pi_S^{\alpha\beta} \, h_S^\alpha$:

$$\begin{array}{cccccccccc}
\longrightarrow & H_p(A) & \xrightarrow{a} & H_p(X) & \xrightarrow{b} & H_p(X , A) & \xrightarrow{c} & H_{p-1}(A) & \xrightarrow{d} & \ldots \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \\
\longrightarrow & H_p(\alpha_2) & \longrightarrow & H_p(\alpha_1) & \longrightarrow & H_p(\alpha) & \xrightarrow{e} & H_{p-1}(\alpha_2) & \longrightarrow & \ldots \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} & & \\
\longrightarrow & H_p(\beta_2) & \longrightarrow & H_p(\beta_1) & \xrightarrow{f} & H_p(\beta) & \xrightarrow{c} & H_{p-1}(\beta_2) & \xrightarrow{d} & \ldots \\
& \downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow & & \\
\longrightarrow & H_p(\gamma_2) & \xrightarrow{a} & H_p(\gamma_1) & \xrightarrow{m} & H_p(\gamma) & \longrightarrow & H_{p-1}(\gamma_2) & \longrightarrow & \ldots \\
& \downarrow{\scriptstyle l} & & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & H_p(\delta_2) & \longrightarrow & H_p(\delta_1) & \longrightarrow & H_p(\delta) & \longrightarrow & H_{p-1}(\delta_2) & \longrightarrow & \ldots ,
\end{array}$$

where $p < n + 1$ and $H_p(A)$, $H_p(X)$ and $H_{p-1}(A)$ are identified canonically with a common submodule of the modules of the corresponding vertical rows. The letters a to m are symbols of homomorphisms already defined above.

We show first, that khg is injective. If $x \in H_p(X , A)$ and $(khg)(x) = 0$, it follows that $c(x) = 0$. There exists a $y \in H_p(X)$ such that $b(y) = x$. Then $m(y) = 0$. There exists a $z \in H_p(\gamma_2)$ such that $a(z) = y$. Then $l(z) \in H_p(A)$, $(al)(z) = y$ and $(bal)(z) = x = 0$. Thus $H_p(X , A)$ may be identified with a common submodule of $H_p(\alpha)$, $H_p(\beta)$ and $H_p(\gamma)$. We will show that $\pi_{(p)}^{\alpha\gamma}(H_p(\alpha)) = H_p(X , A)$. If $x \in H_p(\alpha)$, then $(ie)(x) \in H_{p-1}(A)$ and $(die)(x) = 0$. There exists a $y \in H_p(X , A)$ such that $c(y) = (ie)(x)$; i.e. $c(h(x) - y) = 0$. There exists a $z \in H_p(\beta_1)$ such that $f(z) = h(x) - y$. Then $j(z) \in H_p(X)$ and $(mj)(z) + y = (kh)(x) \in H_p(X , A)$.

We have thus shown that α and γ determine $H_p(X, A)$ for each $p < n + 1$. Then a similar proof as in Theorem 1 yields

Theorem 2. If N_1 and N_2 are lc_{n-1} and semi-lc_n, then the homomorphism

$$t_p : H_p(X, A) \to H_p(N)$$

of t_S is bijective for each $p < n + 1$.

§4. Local connectedness in paracompact spaces

7. Let X be a paracompact space.

Definition 7. (cf. [6], p. 153) The space X is said to be semi-lc_n if each $x \in X$ has a neighbourhood $U(x)$ such that

$$\text{(12)} \qquad \operatorname{Im}(H_n(i^{U(x), X})) = 0 .$$

We call X lc_n if there exists for each $x \in X$, and for each neighbourhood $U(x)$ of x, a neighbourhood $V(x)$ in $U(x)$ such that

$$\text{(13)} \qquad \operatorname{Im}(H_k(i^{V(x), U(x)})) = 0$$

for each $k < n + 1$.

We denote by $\operatorname{Cov}(X)$ the set of open and proper coverings of X. The module $H_n(\operatorname{Cov}(X))$ is denoted by $\check{H}_n(X)$ $(n \in Z)$. If A is a paracompact subspace of X, then $\operatorname{Cov}(X, A)$ shall be the set of open and proper coverings of (X, A) and $\check{H}_n(X, A)$ $(n \in Z)$ the module $H_n(\operatorname{Cov}(X, A))$.

Lemma 2. If X is (semi-)lc_n, then $\operatorname{Cov}(X)$ is (semi-) lc_n.

Proof. If X is semi-lc_n, we choose for each $x \in X$ an open neighbourhood $U(x)$ satisfying (12). Then every open and proper refinement of $(U(x))_{x \in X}$ is an n-refinement of the covering $\{X\}$ of X.

If X is lc_n and $\alpha \in \operatorname{Cov}(X)$, let $\alpha' \in \operatorname{Cov}(X)$ be a star refinement of α. Each $x \in X$ has an open neighbourhood $U(x)$, which belongs to α', and an open neighbourhood $V(x)$ in $U(x)$ satisfying (13). Then every open and proper refinement of $(V(x))_{x \in X}$ is a strong n-refinement of α (cf. [1], p. 320).

Theorem 1 and Lemma 2 imply

Theorem 3. If X is lc_{n-1} and semi-lc_n, then the canonical homomorphism

$$t_p : H_p(X) \to \check{H}_p(X)$$

is bijective for each $p < n + 1$.

If $C(P)$ is a subcomplex of $C(Q)$ for each $P \subset Q \subset X$, then Theorem 2 and Lemma 2 imply

Theorem 4. If X and A are lc_{n-1} and semi-lc_n, then the canonical homomorphism

$$t_p : H_p(X, A) \to \check{H}_p(X, A)$$

is bijective for $p < n + 1$.

8. Singular homology: Let C^A and $C^{\alpha A}$ $(\alpha \in \mathrm{Cov}(X))$ be the singular complexes of the example 1°, pp. 4,5. In virtue of the homotopy equivalence

$$H_n(C^A(P)) \simeq H_n(C^{\alpha A}(P)) \quad (P \subset X, n \in Z), \tag{14}$$

C^A and $C^{\alpha A}$ have the same homology functors H_n $(n \in Z)$. Moreover it is easy to see that

$$\overset{\alpha}{C}{}^{\alpha A}_{p,q} = \overset{\alpha}{C}{}^{A}_{p,q} \tag{15}$$

for $p > -1$. If $\alpha > \beta \in \mathrm{Cov}(X)$, we obtain the commutative diagram

$$\begin{array}{ccc} H_n(C^{\alpha A}(X)) & \xrightarrow{h_n^\alpha} & H_n(\alpha) \\ \| & & \Big\downarrow \pi_{(n)}^{\alpha\beta} \\ H_n(X) & & \\ \| & & \\ H_n(C^{\beta A}(X)) & \xrightarrow{h_n^\beta} & H(\beta) \end{array}$$

and the limit homomorphism

$$t_n : H_n(X) \to \check{H}_n(X)$$

of the homomorphisms h_n^α $(\alpha \in \mathrm{Cov}(X))$ for each $n \in Z$. In virtue of the homotopy equivalences (14) and the equations (15) above Lemma 1 is valid here, and we obtain Theorem 3 for the complex C^A.

If $P \subset Q \subset X$, then $C^{\emptyset}(P)$ and $C^{\alpha\emptyset}(P)$ are subcomplexes of $C^{\emptyset}(Q)$ and $C^{\alpha\emptyset}(Q)$ respectively. In a similar manner Theorem 4 is then obtained for the complex $C^{\emptyset}$.

Definition 8. (see [6], p. 153) Let M be the ring Z and C the complex $C^{\emptyset}$. Then X is semi-q-lc_s if it is semi-lc_q, and lc_s^p if it is lc_p $(p, q \in Z)$.

Lemma 3. If X is semi-n-lc_s and lc_s^{n-1}, $A \subset X$ closed and lc_s^{n-1}, then X is semi-lc_n and lc_{n-1} with respect to C^A for any module M.

Proof: We shall prove first that X is semi-lc_n and lc_{n-1} with respect to $C^{\emptyset}$ for any module M. Let p be an integer, smaller than $n+1$, and U a neighbourhood of $x \in X$. If $p = n$, we assume that $U = X$. Denoting by $H_q(P, M, S)$ the module $H_q(C^{\emptyset}(P))$ $(q \in Z, P \subset X)$ we obtain by assumption neighbourhoods $W \subset V \subset U$ of x such that

(16) $$\mathrm{Im}(H_{p-1}(i^{W,V}) : H_{p-1}(W, Z, S) \to H_{p-1}(V, Z, S)) = 0$$

and

$$\mathrm{Im}(H_p(i^{V,U}) : H_p(V, Z, S) \to H_p(U, Z, S)) = 0 .$$

Setting $H_p(i^{V,U}) \otimes M = a$, $b = H_p(i^{W,V}) : H_p(W, M, S) \to H_p(V, M, S)$
$c = H_p(i^{V,U}) : H_p(V, M, S) \to H_p(U, M, S)$ and
$d = \mathrm{Tor}\,(H_{p-1}(i^{W,V}), M) : \mathrm{Tor}\,(H_{p-1}(W, Z, S), M) \to \mathrm{Tor}\,(H_{p-1}(V, Z, S), M)$,
we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & H_p(W, Z, S) \otimes M & \to & H_p(W, M, S) & \to & \mathrm{Tor}\,(H_{p-1}(W, Z, S), M) & \to 0 \\
& \downarrow & & \downarrow b & & \downarrow d & \\
0 \to & H_p(V, Z, S) \otimes M & \to & H_p(V, M, S) & \to & \mathrm{Tor}\,(H_{p-1}(V, Z, S), M) & \to 0 \\
& \downarrow a & & \downarrow c & & \downarrow & \\
0 \to & H_p(U, Z, S) \otimes M & \to & H_p(U, M, S) & \to & \mathrm{Tor}\,(H_{p-1}(U, Z, S), M) & \to 0 ,
\end{array}$$

with exact rows and $a = d = 0$. Simple diagram chasing yields $cb = 0$, which proves the assertion.

With U given as above, by assumption we may choose the neighbourhoods $W \subset V \subset U$ of x such that

$$\mathrm{Im}\,(H_p(i^{V,U}) : H_p(V, M, S) \to H_p(U, M, S)) = 0 ,$$

and

$$\mathrm{Im}\,(H_{p-1}(i^{W\cap A, V\cap A}) : H_{p-1}(W \cap A, M, S) \to H_{p-1}(V \cap A, M, S)) = 0 .$$

Writing $a = H_p(i^{V,U})$ and $c = H_{p-1}(i^{W\cap A, V\cap A})$ above, $b = H_p(i^{W,V})$: $H_p(C^A(W)) = H_p(W, W \cap A, M, S) \to H_p(V, V \cap A, M, S) = H_p(C^A(V))$ and $d = H_p(i^{V,U}) : H_p(C^A(V)) = H_p(V, V\cap A, M, S) \to H_p(U, U \cap A, M, S)$ $= H_p(C^A(U))$, we obtain the commutative diagram

$$\begin{array}{ccccccc}
\ldots \to & H_p(W, M, S) & \to & H_p(W, W \cap A, M, S) & \to & H_{p-1}(W \cap A, M, S) & \to \ldots \\
& \downarrow & & \downarrow b & & \downarrow c & \\
\ldots \to & H_p(V, M, S) & \to & H_p(V, V \cap A, M, S) & \to & H_{p-1}(V \cap A, M, S) & \to \ldots \\
& \downarrow a & & \downarrow d & & \downarrow & \\
\ldots \to & H_p(U, M, S) & \to & H_p(U, U \cap A, M, S) & \to & H_{p-1}(U \cap A, M, S) & \to \ldots
\end{array}$$

with exact rows and $a = c = 0$. Diagram chasing yields $db = 0$, which completes the proof.

Theorem 4 then contains a new proof of Theorem 1 of Mardešić in [6]. If A is closed in X, then Theorem 3 and Lemma 3 yield a slight generalization of it.

9. In a compact space X the results of section 7 can be applied to the net $\mathrm{Cov}^f(X)$ of finite open and proper coverings of X and to the complex C^{Σ} of example 2° on page 6 above. If X is lc_{n-1} and semi-lc_n with respect to C^{Σ}, it follows that the canonical homomorphism

$$t_p : H_p(\Sigma\,, M) = H_n(X) \rightarrow \check{H}_p(X) = \check{H}_p(\Sigma\,, M)$$

is bijective for $p < n + 1$. This result may be added to Lemma 10 of [3], p. 18.

Appendix

Corrections to the author's paper [3]:

page	line	for	read
13	25	H^j	H^n
	32	φ	φ_*
	33	φ	φ^*
14	29	§6	§1
16	Example D. see p. 7 above		
19	1	$\pi^{\gamma}_{(n)}$	$\check{\pi}^{\gamma}_{(n)}$
	3	$\check{\pi}^{\gamma}_{(n)}$	$\pi^{\gamma}_{(n)}$
for $\varepsilon_{(n)} = \pi^{\gamma}_{(n)}\,(\check{\pi}^{\gamma}_{(n)})^{-1}$			
read $\varepsilon_{(n)} = (\check{\pi}^{\gamma}_{(n)})^{-1}\pi^{\gamma}_{(n)}$			
20	6	g^{j-1}	g_{j-1}
	19	A^{j-i}_{j-i}	A^{j-i}_{3j-i}
29	1,4,5	see p. 5 above	

page	line	for	read
33	fig. 4	$\bar{\beta}_1$	β_1
35	8	$A\,, B$	$\bar{A}\,, \bar{B}$
36	16	$\bar{A} \cup \bar{B}$	$\bar{A} \cap \bar{B}$
37	16	$\mathring{\delta}$	$\mathring{\bar{\delta}}$
	19	for $C^p(\overset{i}{\alpha}\,, \overset{i}{C}{}^q\,, C^q(\beta\,, \mathring{N}^i))$ read $C^p(\overset{i}{\alpha}\,, C^q(\beta\,, \mathring{N}^i))$	
	19	§6	§1
39	2	$''Z^{-1*}$	$''\overset{i}{Z}{}^{--1*}$
41	12	$\overset{-1}{S}{}^{-1}$	$\overset{-1}{S}$
43	2	A^p	A_p
	29	A	A_0

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