ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

336/11

THE CARATHÉODORY CONVERGENCE THEOREM FOR QUASICONFORMAL MAPPINGS IN SPACE

BY

F. W. GEHRING

HELSINKI 1963
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1964.336-11

Communicated 10 May 1963 by P. J. MYRBERG and O. LEHTO

Corrigendum (25 January 1964)

The second sentence in the first paragraph of section 9 should read as follows: We say that $\mathcal{F}$ is a *normal* family if each sequence of homeomorphisms in $\mathcal{F}$, which are bounded at a pair of points in D, contains a subsequence which converges uniformly on each compact subset of D.

Ann. Acad. Scient. Fennicæ A. I. 336/11 F. W. Gehring

KESKUSKIRJAPAINO
HELSINKI 1963

The Carathéodory convergence theorem for quasiconformal mappings in space *

1. *Introduction.* The main purpose of this paper is to give an analogue of Carathéodory's theorem, on the convergence of conformal mappings of variable domains, for quasiconformal mappings in space. The results we obtain are what one would expect to be true for quasiconformal mappings. However, the proof differs in several respects from the usual arguments for Carathéodory's theorem. Moreover the space form of this theorem has several important applications. For example, one of the more interesting problems in space is to determine whether or not a given domain D is quasiconformally equivalent to the unit sphere. Theorem 3 shows that if D has a finite boundary point and is the kernel of a convergent sequence of domains, each of which can be mapped K-quasiconformally onto the unit sphere, then D can also be mapped K-quasiconformally onto the unit sphere. In a quite different direction, one can use Theorem 3 to show that the space of domains D, which are quasiconformally equivalent to the unit sphere, is a complete metric space[1]. Here the distance between two domains D and D' is defined as

$$d(D, D') = \inf(\log K),$$

where the infimum is taken over all K for which there exists a K-quasiconformal mapping of D onto D'. Finally in a recent paper [1], P. P. Belinskiĭ has announced an interesting strengthened form of Liouville's theorem on the conformal mappings in space, the proof of which is apparently based on a result very similar to Theorem 3.

In the final section of this paper, we give a space analogue of a theorem due to Lindelöf on the boundary behaviour of conformal mappings.

---

* This paper supplies proofs of some hitherto unpublished results which were first announced in a survey lecture, given at the Colloquium on Mathematical Analysis in Helsinki, 24 August 1962. This lecture has already been published under the title »Quasiconformal mappings in space». See [6].

This research was supported by a grant from the National Science Foundation, Contract NSF-G-18913.

[1]) A proof will appear shortly in a joint paper by J. Väisälä and the present author.

2. *Quasiconformal mappings.* Suppose that D and D' are finite domains in Euclidean 3-space and that $y(x)$ is a homeomorphism of D onto D'. We say that $y(x)$ is a *K-quasiconformal mapping,* $1 \leqq K < \infty$, if the inequality

$$\frac{1}{K} \operatorname{mod} R \leqq \operatorname{mod} R' \leqq K \operatorname{mod} R \tag{1}$$

holds for all bounded rings R with closure $\bar{R} \subset D$, where R' is the image of R under $y(x)$. A quasiconformal mapping is one which is K-quasiconformal for some K. Here a finite domain R is said to be a *ring* if its complement with respect to the extended or Möbius space consists of two components C_0 and C_1, the *conformal capacity* of R is defined as

$$\Gamma(R) = \inf_u \int_R |\nabla u|^3 \, d\omega ,$$

where the infimum is taken over all functions $u = u(x)$ which are continuously differentiable in R with boundary values 0 on C_0 and 1 on C_1, and the *modulus* of R is given by

$$\operatorname{mod} R = \left(\frac{4\pi}{\Gamma(R)} \right)^{1/2} .$$

See [4], [5], and [8]. An essentially equivalent class of mappings has also been studied by J. Väisälä in [11] and [12].

If $y(x)$ is a continuously differentiable homeomorphism of D onto D' and if $y(x)$ maps each infinitesimal sphere in D onto an infinitesimal ellipsoid so that the ratio of the major to minor axes never exceeds K, then $y(x)$ is a K-quasiconformal mapping by the above definition. Hence the class of K-quasiconformal mappings considered here includes those mappings which are K-quasiconformal according to the classical definition.

Now suppose that $y(x)$ is a K-quasiconformal mapping of D onto D'. Then we see from (1) that the inverse mapping $x(y)$ is a K-quasiconformal mapping of D' onto D. It is also obvious that the restriction of $y(x)$ to any subdomain Δ of D is again a K-quasiconformal mapping. We require two additional properties of quasiconformal mappings. The first of these is the following closure property [5].

Lemma 1. *Suppose that* $\{ y_n(x) \}$ *is a sequence of K-quasiconformal mappings of* D, *that*

$$\lim_{n \to \infty} y_n(x) = y(x)$$

uniformly on each compact subset of D, *and that* $y(x)$ *is a homeomorphism. Then* $y(x)$ *is a K-quasiconformal mapping.*

The second of these properties is given by the following distortion theorem [5].

Lemma 2. *For each* K, $1 \leqq K < \infty$, *there exists a distortion function* $\Theta(t) = \Theta_K(t)$ *which is continuous and increasing in* $0 \leqq t < 1$ *with* $\Theta(0) = 0$ *and which has the following property. If* $y(x)$ *is a* K*-quasiconformal mapping of* D *onto* D' *and if* $P \in D$, *then*

$$\frac{|y(Q) - y(P)|}{\varrho(y(P), \partial D')} \leqq \Theta\left(\frac{|Q-P|}{\varrho(P, \partial D)}\right), \tag{2}$$

for all Q *with* $|Q-P| < \varrho(P, \partial D)$.

Here ∂D and $\partial D'$ denote the boundaries of D and D' taken with respect to the Möbius space, and $\varrho(P, \partial D)$ and $\varrho(y(P), \partial D')$ denote the distances from P to ∂D and from $y(P)$ to $\partial D'$, respectively. Hence, for example, $\varrho(P, \partial D) = \infty$ whenever D is the finite space and (2) then implies that D' is also the finite space. The distortion function $\Theta(t)$ depends only on K and t, not on D or the mapping $y(x)$. It can be expressed in terms of the moduli of the space analogues of the Grötzsch and Teichmüller rings.

3. *Equicontinuity.* We begin by establishing some equicontinuity properties for sequences of quasiconformal mappings.

Lemma 3. *Suppose that* $\{ y_n(x) \}$ *is a sequence of* K*-quasiconformal mappings of* D *which are uniformly bounded on each compact subset of* D. *Then the* $y_n(x)$ *are equicontinuous on each compact subset of* D.

Proof. Let E be a compact subset of D, choose $P_0 \in D - E$, and let Δ be the domain consisting of D minus the point P_0. By hypothesis there exists a finite constant $A = A(E, P_0)$ such that

$$|y_n(P) - y_n(P_0)| \leqq |y_n(P)| + |y_n(P_0)| \leqq A$$

for $P \in E$ and all n. Hence

$$\varrho(y_n(P), \partial \Delta_n') \leqq |y_n(P) - y_n(P_0)| \leqq A \tag{3}$$

for $P \in E$ and all n, where Δ_n' is the image of Δ under $y_n(x)$. Since E is compact

$$\varrho(P, \partial \Delta) \geqq a > 0 \tag{4}$$

for $P \in E$. Now fix $P \in E$ and choose Q so that $|Q-P| < a$. Then Lemma 2 applied to the restriction of $y_n(x)$ to Δ yields

$$\frac{|y_n(Q) - y_n(P)|}{(\varrho(y_n(P), \partial \Delta_n')} \leqq \Theta\left(\frac{|Q-P|}{\varrho(P, \partial \Delta)}\right), \tag{5}$$

and combining (3), (4) and (5) we obtain

$$|y_n(Q) - y_n(P)| \leqq A\,\Theta\left(\frac{|Q-P|}{a}\right).$$

Since $\lim_{t\to 0+} \Theta(t) = 0$, this implies the desired equicontinuity of E .

Lemma 4. *Suppose that* $\{ y_n(x) \}$ *is a sequence of* K*-quasiconformal mappings of* D *onto* D'_n , *that*

$$\sup_n |y_n(P_0)| < \infty$$

for some fixed point $P_0 \in D$, *and that*

$$\sup_n \varrho(O\,, \partial D'_n) < \infty\,,$$

where O *denotes the origin. Then the* $y_n(x)$ *are uniformly bounded and equicontinuous on each compact subset of* D .

Proof. Fix a so that $0 < a < 1$. Then if we choose $P \in D$ and Q so that $|Q-P| < a\,\varrho(P\,, \partial D)$, Lemma 2 implies that

$$|y_n(Q) - y_n(P)| \leqq \Theta(a)\,\varrho(y_n(P)\,, \partial D'_n)$$

for all n . Since

$$\varrho(y_n(P)\,, \partial D'_n) \leqq |y_n(P)| + \varrho(O\,, \partial D'_n)\,,$$

we thus obtain

$$|y_n(Q)| \leqq A\,|y_n(P)| + B\,, \tag{6}$$

where A and B are constants,

$$A = 1 + \Theta(a)\,, \qquad B = \Theta(a) \sup_n \varrho(O\,, \partial D'_n) < \infty\,. \tag{7}$$

In particular we conclude that each point $P \in D$ has a neighborhood $U = U(P) \subset D$ such that (6) holds for all $Q \in U$.

Next if we choose $Q \in D$ and P so that $|P-Q| < \frac{1}{2}\,a\,\varrho(Q\,, \partial D)$, then it is easy to show that $|Q-P| < a\,\varrho(P\,, \partial D)$. Hence we see that each point $Q \in D$ has a neighborhood $V = V(Q) \subset D$ such that (6) holds for all $P \in V$.

Now let G denote the set of points $P \in D$ for which

$$\sup_n |y_n(P)| = C(P) < \infty\,. \tag{8}$$

If $P \in G$ and if U is the neighborhood described above, then (6) implies that

$$\sup_n |y_n(Q)| \leqq A \sup_n |y_n(P)| + B < \infty$$

for all $Q \in U$. Hence $U \subset G$ and G is open. Similarly if $Q \in D - G$ and if V is the neighborhood described above, the same argument shows that $V \subset D - G$ and hence that $D - G$ is open. Since D is connected and $P_0 \in G$, we conclude that (8) holds for all $P \in D$.

Finally suppose that E is a compact subset of D. Then the neighborhoods $U(P)$ described above cover E as P ranges through E, and we can choose $P_1, \ldots, P_m$ so that

$$E \subset \bigcup_{i=1}^{m} U(P_i) .$$

It then follows from (6) that

$$|y_n(Q)| \leqq A \max (C(P_1), \ldots, C(P_m)) + B < \infty$$

for $Q \in E$, and hence the $y_n(x)$ are uniformly bounded on E. The equicontinuity is now a consequence of Lemma 3.

Lemma 5. *Suppose that* $\{ y_n(x) \}$ *is a sequence of K-quasiconformal mappings of* D *and that*

$$\sup_n |y_n(P_0)| < \infty , \qquad \sup_n |y_n(P_1)| < \infty$$

for a pair of distinct fixed points $P_0, P_1 \in D$. *Then the* $y_n(x)$ *are uniformly bounded and equicontinuous on each compact subset of* D.

Proof. Let Δ denote D minus the point P_1, and let Δ_n' denote the image of Δ under $y_n(x)$. Then $y_n(P_1) \in \partial \Delta_n'$ and hence

$$\sup_n \varrho(O, \partial \Delta_n') \leqq \sup_n |y_n(P_1)| < \infty .$$

Lemma 4 now implies the desired conclusions on each compact subset of Δ. Interchanging the roles of P_0 and P_1 then yields these results on each compact subset of D.

4. *Hurwitz property.* We next apply Lemmas 2 and 5 to obtain space analogues for Hurwitz's theorem on the limit functions of normal families of analytic functions.

Lemma 6. *Suppose that* $\{ y_n(x) \}$ *is a sequence of K-quasiconformal mappings of* D, *that*

$$\lim_{n \to \infty} y_n(x) = y(x) , \qquad |y(x)| < \infty ,$$

in D, *that* $y_n(x) \neq Q_n'$ *in* D, *and that*

$$\lim_{n \to \infty} Q_n' = Q' .$$

Then either $y(x) \neq Q'$ *in* D *or* $y(x) \equiv Q'$ *in* D.

Proof. Let G be the set of points $P \in D$ for which $y(P) = Q'$. Lemma 5 implies that the $y_n(x)$ are equicontinuous on each compact subset of D. Hence $y(x)$ is continous and G is closed in D. Now suppose $P \in G$ and let U be the set of points Q for which $|Q-P| < a\,\varrho(P, \partial D)$, where a is some fixed constant, $0 < a < 1$. Then Lemma 2 implies that

$$|y_n(Q) - y_n(P)| \leqq \Theta(a)\, \varrho(y_n(P), \partial D'_n)$$

for all $Q \in U$, where D'_n is the image of D under $y_n(x)$. Since $Q'_n \notin D'_n$, we see that

$$\varrho(y_n(P), \partial D'_n) \leqq |y_n(P) - Q'_n|,$$

and hence that

$$|y(Q) - y(P)| = \lim_{n \to \infty} |y_n(Q) - y_n(P)| \leqq \Theta(a) \lim_{n \to \infty} |y_n(P) - Q'_n| = 0$$

for all $Q \in U$. Hence $U \subset G$ and G is open. Since D is connected, we conclude that either $G = \emptyset$ or that $G = D$. Thus either $y(x) \neq Q'$ in D or else $y(x) \equiv Q'$ in D as desired.

Lemma 7. *Suppose that* $\{ y_n(x) \}$ *is a sequence of* K*-quasiconformal mappings of* D *and that*

$$\lim_{n \to \infty} y_n(x) = y(x), \qquad |y(x)| < \infty,$$

in D. *Then* $y(x)$ *is either a homeomorphism or a constant.*

Proof. Lemma 5 implies that the $y_n(x)$ are equicontinuous on each compact subset of D. Hence $y(x)$ is continuous in D. If $y(x)$ is not one to one, we can find a pair of distinct points $P, Q \in D$ such that $y(P) = y(Q) = Q'$. Let Δ be the domain D minus the point Q and let $y_n(Q) = Q'_n$. Then $y_n(x) \neq Q'_n$ in Δ and

$$\lim_{n \to \infty} Q'_n = Q'.$$

Since $P \in \Delta$ and $y(P) = Q'$, Lemma 6 implies that $y(x) \equiv Q'$ in Δ and hence that $y(x)$ is constant in D. The desired conclusion now follows from a well known theorem in topology. (See, for example, p. 137 in [10].)

5. *Uniformly convergent sequences of homeomorphisms.* We consider next the following result.

Theorem 1. *Suppose that* $\{ y_n(x) \}$ *is a sequence of homeomorphisms of* D_n *onto* D'_n, *that each compact subset of a domain* D *is contained in all but a finite number of* D_n, *that*

$$\lim_{n \to \infty} y_n(x) = y(x)$$

uniformly on each compact subset of D, *and that* $y(x)$ *is a homeomorphism of* D *onto* D'. *Then each compact subset of* D' *is contained in all but a finite number of* D'_n *and*

$$\lim_{n \to \infty} x_n(y) = x(y) \tag{9}$$

uniformly on each compact subset of D', *where* $x_n(y)$ *and* $x(y)$ *are the inverses of* $y_n(x)$ *and* $y(x)$.

Proof. We see that the first assertion in the conclusion of Theorem 1 is contained in the following result.

Lemma 8. *Under the hypotheses of Theorem* 1, *for each compact set* $E' \subset D'$ *we can find a compact set* F *and an integer* n_1 *such that* $F \subset D_n$ *and* $E' \subset F'_n$ *for* $n \geqq n_1$, *where* F'_n *is the image of* F *under* $y_n(x)$.

Proof of Lemma 8.[1]) Let U' be any open sphere with closure $\bar{U}' \subset D'$, choose a second open sphere V' such that $\bar{U}' \subset V'$ and $\bar{V}' \subset D'$, and let $\bar{U}$ and $\bar{V}$ be the preimages of $\bar{U}'$ and $\bar{V}'$ under $y(x)$. Then $\bar{V}$ is a compact subset of D and there exists an integer n_0 such that $\bar{V} \subset D_n$ for $n \geqq n_0$. We shall show that there exists an $n_1 \geqq n_0$ such that $\bar{U}' \subset \bar{V}'_n$ for $n \geqq n_1$, where $\bar{V}'_n$ is the image of $\bar{V}$ under $y_n(x)$.

If this were not the case, we could find a subsequence $\{n_j\}$, $n_j \geqq n_0$, such that $\bar{U}' - \bar{V}'_{n_j} \neq \emptyset$ for all j. Let P be the point which $y(x)$ maps onto the center of U' and let $r > 0$ be the radius of U'. Then there exists a j_0 such that

$$|y_{n_j}(P) - y(P)| < r$$

for $j \geqq j_0$, and hence $\bar{U}' \cap V'_{n_j} \neq \emptyset$ for $j \geqq j_0$. Since $\bar{U}'$ is connected, we can find a sequence of points $\{P'_j\}$ such that

$$P'_j \in \bar{U}' \cap \partial V'_{n_j} \tag{10}$$

for $j \geqq j_0$. Because $x_{n_j}(P'_j) \in \partial V$ and ∂V is compact, we may assume, by choosing a second subsequence and then relabeling, that

$$\lim_{j \to \infty} x_{n_j}(P'_j) = P \in \partial V. \tag{11}$$

Since the $y_{n_j}(x)$ converge uniformly on ∂V, it is easy to see that

$$P' = y(P) = \lim_{j \to \infty} y(x_{n_j}(P'_j)) = \lim_{j \to \infty} y_{n_j}(x_{n_j}(P'_j)) = \lim_{j \to \infty} P'_j. \tag{12}$$

Now (10) implies that $P' \in \bar{U}'$ while (11) and (12) imply that $P' \in \partial V'$. Thus $\bar{U}' \cap \partial V' \neq \emptyset$, and this contradicts the way in which V' was chosen.

---

[1]) The argument given here is essentially due to Carathéodory [3].

Now suppose that E' is a compact subset of D'. Then we can cover E' by a finite number of open spheres $U'_1, \ldots, U'_m$ whose closures lie in D'. Choose open spheres $V'_1, \ldots, V'_m$ so that $\bar{U}'_i \subset V'_i$ and $\bar{V}'_i \subset D'$ for $i = 1, \ldots, m$, and let F be the preimage of

$$F' = \bigcup_{i=1}^{m} \bar{V}'_i$$

under $y(x)$. Then F is a compact subset of D, and if we apply what was proved above to each U'_i, it follows we can find an integer n_1 such that $F \subset D_n$ and $E' \subset F'_n$ for $n \geqq n_1$. This completes the proof of Lemma 8.

Now to complete the proof of Theorem 1, let E' be a compact subset of D' and choose F and n_1 as in Lemma 8. Then $F \subset D_n$ and $E' \subset F'_n$ for $n \geqq n_1$. We want to show that (9) holds uniformly on E'. If this were not the case, we could find an $\varepsilon > 0$, a subsequence $\{n_j\}$ with $n_j \geqq n_1$, and a sequence of points $\{P'_j\}$ in E' such that

$$|x_{n_j}(P'_j) - x(P'_j)| \geqq \varepsilon \tag{13}$$

for all j. Since $x_{n_j}(P'_j) \in F$ and F is compact, we may assume as in the proof of Lemma 8 that

$$\lim_{j \to \infty} x_{n_j}(P'_j) = P \in F, \tag{14}$$

and arguing as in (12), we get

$$P' = y(P) = \lim_{j \to \infty} P'_j .$$

But $x(y)$ is continuous at $P' \in E'$,

$$\lim_{j \to \infty} x(P'_j) = x(P') = P, \tag{15}$$

and we see that (14) and (15) contradict (13), thus completing the proof of Theorem 1.

6. *Carathéodory kernels.* Suppose that $\{D_n\}$ is a sequence of domains which contain the origin O. We define the *kernel* D of the sequence $\{D_n\}$ as follows.

(i) If there exists no fixed neighborhood U of the origin which is contained in all of the D_n, then D consists only of the origin.

(ii) If there exists a fixed neighborhood U of the origin which is contained in all of the D_n, then D is the domain with the following three properties.

$$(16)\quad \begin{cases} \text{(a)} & O \in D . \\ \text{(b)} & \text{Each compact set } E \subset D \text{ lies in all but a finite number of } D_n . \\ \text{(c)} & \text{If } \Delta \text{ is a domain satisfying (a) and (b), then } \Delta \subset D . \end{cases}$$

In the first case, the kernel D is said to be degenerate. In the second case, it is not *a priori* obvious that any such domain D exists. However we may, for example, set

$$D = \bigcup_{\alpha} \Delta_{\alpha} ,$$

where the union is taken over the collection of all domains $\{\Delta_\alpha\}$ which satisfy (a) and (b). Since the neighborhood U satisfies (a) and (b) this collection is not empty. Then D is clearly a domain which satisfies (a) and (c), and it is easy to verify that D also has the property (b). For an interesting alternative characterization of D in the second case, see [9].

Finally the D_n are said to *converge to their kernel* D if every subsequence of domains $\{D_{n_j}\}$ also has D as its kernel.

7. *Convergence theorems.* We now apply the results of §§ 3—5 to prove a pair of convergence theorems for quasiconformal mappings in space.

Theorem 2. *Suppose that* $\{ y_n(x) \}$ *is a sequence of* K*-quasiconformal mappings of* D_n *onto* D_n' *, that each compact subset of a domain* D *is contained in all but a finite number of* D_n *, and that*

$$(17)\qquad \lim_{n \to \infty} y_n(x) = y(x) , \qquad |y(x)| < \infty ,$$

in D *. Then the convergence is uniform on each compact subset of* D *and* $y(x)$ *is either a constant or a* K*-quasiconformal mapping of* D *onto* D' *. In this last case, each compact subset of* D' *is contained in all but a finite number of* D_n' *and*

$$\lim_{n \to \infty} x_n(y) = x(y)$$

uniformly on each compact subset of D' *, where* $x_n(y)$ *and* $x(y)$ *are the inverses of* $y_n(x)$ *and* $y(x)$ *, respectively.*

Proof. Let Δ be any domain with compact closure in D . Then $\Delta \subset D_n$ for $n \geqq n_0$, and we can apply Lemma 5 to conclude that the $y_n(x)$ are equicontinuous, and hence converge uniformly, by a familiar argument, on each compact subset of Δ . Lemma 7 further implies that $y(x)$ is either constant in Δ or a homeomorphism of Δ . In this last case we then see from Lemma 1 that $y(x)$ is a K-quasiconformal mapping of Δ .

Now Δ may be chosen arbitrarily. Hence the convergence in (17) is uniform on each compact subset of D and $y(x)$ is either constant in D

or a K-quasiconformal mapping of D onto a domain D'. Finally in this last case, the remaining conclusions follow from Theorem 1.

We next have the following space form of the Carathéodory convergence theorem [2].

Theorem 3. *Suppose that $\{D_n\}$ is a sequence of domains which contain the origin, that the D_n converge to their kernel D, and that D is a domain with a finite boundary point. Suppose further that $\{y_n(x)\}$ is a sequence of K-quasiconformal mappings of D_n onto D'_n and that $y_n(O) = O$. If*

$$\lim_{n\to\infty} y_n(x) = y(x), \qquad |y(x)| < \infty, \tag{18}$$

in D, then the D'_n converge to their kernel D' and D' has a finite boundary point. Conversely if the D'_n converge to their kernel D' and if D' has a finite boundary point, then there exists a subsequence $\{n_j\}$ such that

$$\lim_{j\to\infty} y_{n_j}(x) = y(x), \qquad |y(x)| < \infty, \tag{19}$$

in D. In each case, D' is the image of D under $y(x)$ and $y(x)$ is either a constant or a K-quasiconformal mapping, depending on whether or not D' is degenerate.

Proof. Suppose that the $y_n(x)$ converge to a finite limit $y(x)$ in D and let D^* be the image of D under $y(x)$. We prove first that $D^* = D'$. Now Theorem 2 implies that $D^* \subset D'$. For if $y(x)$ is constant, then D^* consists only of the origin. Otherwise $y(x)$ is a K-quasiconformal mapping of D onto D^* and each compact subset of D^* is contained in all but a finite number of D'_n. Since D^* contains the origin, it follows from (c) of (16) that $D^* \subset D'$.

In order to show that $D' \subset D^*$, fix $P' \in D'$ and let Δ' be any bounded domain, containing the origin and P', such that $\overline{\Delta'} \subset D'$. Then $\overline{\Delta'}$ is a compact subset of D' and hence $\Delta' \subset D'_n$ for $n \geqq n_0$. Let Δ_n be the image of Δ' under $x_n(y)$ for $n \geqq n_0$. Then $\Delta_n \subset D_n$ and hence

$$\sup_n \varrho(O, \partial\Delta_n) \leqq \sup_n \varrho(O, \partial D_n). \tag{20}$$

Now the fact that the D_n converge to D, a domain with a finite boundary point, implies that

$$\sup_n \varrho(O, \partial D_n) < \infty. \tag{21}$$

For otherwise we could find a subsequence $\{n_i\}$ such that

$$\lim_{i\to\infty} \varrho(O, \partial D_{n_i}) = \infty,$$

and the kernel for the subsequence of domains $\{D_{n_i}\}$ would be the finite space. Since $x_n(O) = O$, we conclude from (20), (21) and Lemma 4 that the $x_n(y)$ are uniformly bounded and equicontinuous on each compact subset of Δ'. Hence by Ascoli's theorem, we may pick a subsequence $\{n_j\}$ such that

$$\lim_{j \to \infty} x_{n_j}(y) = x_0(y)$$

in Δ'.

Let Δ be the kernel of the sequence of domains $\{\Delta_{n_j}\}$ and let Δ^* be the image of Δ' under $x_0(y)$. Then the argument of the first paragraph, applied to the $x_{n_j}(y)$ on Δ', shows that $\Delta^* \subset \Delta$. Moreover, since D is the kernel of the sequence $\{D_{n_j}\}$, we see that $\Delta \subset D$. Hence $x_0(P') \in D$ and, by virtue of the equicontinuity of the $y_n(x)$, we conclude that

$$y(x_0(P')) = \lim_{j \to \infty} y_{n_j}(x_0(P')) = \lim_{j \to \infty} y_{n_j}(x_{n_j}(P')) = P' .$$

Thus $P' \in D^*$. Since P' was chosen as any point in D', we conclude that $D' \subset D^*$ and hence with the above that $D' = D^*$.

Now let $\{n_k\}$ be any subsequence. By virtue of our hypothesis that (18) holds in D,

$$\lim_{k \to \infty} y_{n_k}(x) = y(x)$$

in D. Then, since the D_{n_k} converge to D, we can apply what was proved above to conclude that $D' = D^*$ is the kernel of the sequence $\{D'_{n_k}\}$. Hence the D'_n converge to D'. It is also clear that D' has a finite boundary point, since D' either consists of the origin or is the image of D under the K-quasiconformal mapping $y(x)$. Thus the proof for the first half of Theorem 3 is complete.

Suppose now that the domains D'_n converge to their kernel D' and that D' has a finite boundary point. Next let Δ be any bounded domain containing the origin such that $\bar{\Delta} \subset D$. Then $\Delta \subset D_n$ for $n \geqq n_0$. Let Δ'_n denote the image of Δ under $y_n(x)$ for $n \geqq n_0$. Then $\Delta'_n \subset D'_n$ and, arguing as in (20) and (21), we have

$$\sup_n \varrho(O, \partial\Delta'_n) \leqq \sup_n \varrho(O, \partial D'_n) < \infty .$$

Since $y_n(O) = O$, we can use Lemma 4 and Ascoli's theorem to obtain a subsequence $\{ y_{n_i}(x) \}$ which converges to a finite limit in Δ. Now D can be expressed as the union of an expanding sequence of such domains Δ, and by means of a well known diagonal process, we can find a subsequence $\{n_j\}$ such that (19) holds in D. Finally, since the D_{n_j} converge to D, the first half of Theorem 3 implies that D' is the image of D under $y(x)$ and that $y(x)$ is either a constant or a K-quasiconformal mapping. This completes the proof for the second half of Theorem 3.

In Theorem 3 we established a relation between the convergence of the $y_n(x)$ to a finite function $y(x)$ and the convergence of the domains D'_n to a kernel D' which has a finite boundary point. The connection established here, between these two kinds of convergence, is not as close as in the usual forms of Carathéodory's convergence theorem. For though the convergence of the $y_n(x)$ implies convergence of the D'_n, convergence of the D'_n only implies convergence of some subsequence of the $y_n(x)$. To obtain the stronger conclusion that convergence of the D'_n implies convergence of the $y_n(x)$, we would have to know that all limit functions of the $y_n(x)$ are identical. This is clearly so in the case where D' is degenerate. However in the case where D' is a domain, we would have to include some additional normalization for the $y_n(x)$ which would guarantee that there exists at most one normalized K-quasiconformal mapping of D onto D' with $y(0) = 0$.

8. *Θ-mappings.* The proofs of the convergence theorems in § 7 are based on two important properties of sequences of K-quasiconformal mappings. These are the uniform boundedness and equicontinuity property given in Lemma 4, and the Hurwitz property given in Lemma 7. The proofs of these two lemmas follow, in turn, from the fact that a K-quasiconformal mapping of a domain D, as well as its restriction to any subdomain Δ, satisfies the distortion property given in Lemma 2. This suggests that it might be of interest to consider what more can be said about the class of homeomorphisms which have this distortion property.

Definition. *A homeomorphism $y(x)$ of a domain D onto D' is said to be a Θ-mapping if there exists a function $\Theta(t)$, which is continuous and increasing in $0 \leqq t < 1$ with $\Theta(0) = 0$, such that the following are true.*

(i) *If $P \in D$ and $|Q-P| < \varrho(P, \partial D)$, then*

$$\frac{|y(Q) - y(P)|}{\varrho(y(P), \partial D')} \leqq \Theta\left(\frac{|Q-P|}{\varrho(P, \partial D)}\right).$$

(ii) *The restriction of $y(x)$ to any subdomain Δ satisfies* (i).

It is now readily seen that Lemmas 1, 3, 4, 5, 6, and 7 can be reformulated so that they hold for sequences of Θ-mappings which have the same distortion function $\Theta(t)$. Since these lemmas include, with one exception, all the properties of quasiconformal mappings which we have used so far in this paper, it is reasonable to conjecture that Theorems 2 and 3 are also valid for such sequences of mappings. As a matter of fact, much more is true, and we have the following result.

Theorem 4. *A homeomorphism is a Θ-mapping if and only if it is a quasiconformal mapping.*

Proof. The sufficiency is a consequence of Lemma 2. For the necessity let $y(x)$ be a homeomorphism of a domain D onto D'. It will be sufficient to show that $y(x)$ is a quasiconformal mapping under the assumption that the inverse mapping $x(y)$ is a Θ-mapping. For this fix a point $P \in D$ and choose a, $0 < a < \varrho(P, \partial D)$, so that

$$|y(Q) - y(P)| < \varrho(y(P), \partial D') \tag{22}$$

whenever $|Q-P| < a$. Next for each fixed r, $0 < r < a$, choose Q_1 and Q_2 so that $|Q_1-P| = |Q_2-P| = r$ and so that

$$\begin{aligned} L(P, r) &= \max_{|x-P|=r} |y(x) - y(P)| = |y(Q_1) - y(P)|, \\ l(P, r) &= \min_{|x-P|=r} |y(x) - y(P)| = |y(Q_2) - y(P)|. \end{aligned} \tag{23}$$

Now suppose that

$$l(P, r) < L(P, r), \tag{24}$$

let Δ be the domain D minus the point Q_1, and let Δ' be the image of Δ under $y(x)$. Then Δ' is D' minus the point $y(Q_1)$, and we see from (22), (23) and (24) that

$$|y(Q_2) - y(P)| = l(P, r) < L(P, r) = \varrho(y(P), \partial\Delta').$$

Since $x(y)$ was assumed to be a Θ-mapping, we have

$$1 = \frac{|x(y(Q_2)) - x(y(P))|}{\varrho(x(y(P)), \partial\Delta)} \leqq \Theta\left(\frac{|y(Q_2) - y(P)|}{\varrho(y(P), \partial\Delta')}\right) = \Theta\left(\frac{l(P, r)}{L(P, r)}\right),$$

and hence it follows that

$$\frac{L(P, r)}{l(P, r)} \leqq \left(\Theta^{-1}(1)\right)^{-1} = K, \qquad K > 1,$$

where Θ^{-1} is the inverse function for Θ. If (24) does not hold, $L(P, r) = l(P, r)$, and so in either case we have

$$\frac{L(P, r)}{l(P, r)} \leqq K$$

for $0 < r < a$. Thus

$$H(P) = \limsup_{r \to 0+} \frac{L(P, r)}{l(P, r)} \leqq K$$

for each $P \in D$ and $y(x)$ is K-quasiconformal by Corollary 3 of [5].

Hence Theorem 4 gives us still another way of defining the notion of quasiconformality in space, and of course also in the plane. It further shows that the distortion property, given in Lemma 2, is not just a lucky accident, but rather a property which characterizes the class of quasiconformal mappings.

9. *Normal families.* Suppose that $\mathscr{F}$ is a family of homeomorphisms of a fixed domain D. We say that $\mathscr{F}$ is a *normal* family if each sequence of homeomorphisms in $\mathscr{F}$ contains a subsequence which either converges to a finite function or diverges to ∞, uniformly on each compact subset of D. Next we say that $\mathscr{F}$ has the *Hurwitz property* if each finite function, which is the limit of homeomorphisms in $\mathscr{F}$, is either a homeomorphism or a constant.

Lemma 7 shows that a normal family $\mathscr{F}$ has the Hurwitz property if all of the homeomorphisms in $\mathscr{F}$ are K-quasiconformal for some fixed K, and it is natural to ask for how large a class of homeomorphisms is this result true. That is, suppose that $\mathscr{F}$ is a normal family which has the Hurwitz property. What can we say about the homeomorphisms in $\mathscr{F}$? To obtain a meaningful answer to this question, we must make some further assumption about the structure of $\mathscr{F}$. For example, we must rule out the trivial case where $\mathscr{F}$ contains only a finite number of homeomorphisms. We say that $\mathscr{F}$ is *complete with respect to similarity mappings* if, given any pair of similarity mappings $S(x)$ and $T(x)$ such that $T(x)$ maps D into itself, the composite homeomorphism $S(y(T(x)))$ is in $\mathscr{F}$ whenever $y(x)$ is. We then have the following result.

Theorem 5. *Suppose that $\mathscr{F}$ is a family of homeomorphisms of a bounded domain D and that $\mathscr{F}$ is normal and complete with respect to similarity mappings. Then $\mathscr{F}$ has the Hurwitz property if and only if each homeomorphism in $\mathscr{F}$ is K-quasiconformal for some fixed K.*

Proof. The sufficiency is a consequence of Lemma 7. For the necessity we may assume, by performing a preliminary change of variables, that D contains the closed unit sphere $|x| \leqq 1$. Then for each homeomorphism $y(x) \in \mathscr{F}$ we set

$$K(y) = \frac{\max\limits_{|x|=1} |y(x) - y(O)|}{\min\limits_{|x|=1} |y(x) - y(O)|}.$$

Now the fact that $\mathscr{F}$ has the Hurwitz property implies that

$$K = \sup_{y \in \mathscr{F}} K(y) < \infty . \tag{25}$$

For if (25) does not hold, we can find a sequence of homeomorphisms $y_n(x) \in \mathscr{F}$ such that

$$n\, l_n = n \min_{|x|=1} |y_n(x) - y(O)| \leqq \max_{|x|=1} |y_n(x) - y_n(O)| = L_n .$$

Then, since $\mathcal{F}$ is complete with respect to similarity mappings,

$$z_n(x) = \frac{y_n(x) - y_n(O)}{L_n} \in \mathcal{F} .$$

Now $|z_n(x)| \leqq 1$ for $|x| \leqq 1$, and because $\mathcal{F}$ is a normal family, we can find a subsequence $\{n_j\}$ such that

$$\lim_{j\to\infty} z_{n_j}(x) = z(x) , \qquad |z(x)| < \infty , \tag{26}$$

uniformly on each compact subset of D. Next for each n, there exist points P_n and Q_n on $|x| = 1$ such that

$$|z_n(P_n)| = 1 , \qquad |z_n(Q_n)| = \frac{l_n}{L_n} \leqq \frac{1}{n} . \tag{27}$$

Because $|x| = 1$ is compact, we may assume, by choosing a second subsequence and then relabeling, that

$$\lim_{j\to\infty} P_{n_j} = P , \qquad \lim_{j\to\infty} Q_{n_j} = Q ,$$

and by virtue of the uniform convergence in (26), we conclude from (27) that

$$|z(P)| = \lim_{j\to\infty} |z_{n_j}(P_{n_j})| = 1, \qquad |z(Q)| = \lim_{j\to\infty} |z_{n_j}(Q_{n_j})| = 0 .$$

Now $z(O) = O$ and $Q \neq O$. Thus $z(x)$ is neither a homeomorphism nor a constant, and we have a contradiction.

We complete the proof of Theorem 5 by showing that each homeomorphism in $\mathcal{F}$ is K-quasiconformal, where K is the finite constant given in (25). For this, fix $y(x) \in \mathcal{F}$ and let $P \in D$. Since D is bounded, we can choose $a < \infty$ so that D is contained in $|x| \leqq a$. Then for $0 < a\, r < \varrho(P , \partial D)$, the similarity mapping

$$T(x) = P + r\, x$$

maps D into itself, and hence

$$z_r(x) = y(P + r\, x) \in \mathcal{F} .$$

If we now apply (25) to $z_r(x)$, we obtain

$$\begin{aligned} L(P , r) &= \max_{|x-P|=r} |y(x) - y(P)| = \max_{|x|=1} |z_r(x) - z_r(O)| \\ &\leqq K \min_{|x|=1} |z_r(x) - z_r(O)| = K \min_{|x-P|=r} |y(x) - y(P)| = K\, l(P , r) \end{aligned}$$

for $0 < a\, r < \varrho(P , \partial D)$. We conclude that

$$H(P) = \limsup_{r \to 0+} \frac{L(P, r)}{l(P, r)} \leqq K$$

for all $P \in D$, and hence $y(x)$ is K-quasiconformal by Corollary 3 of [5].

10. *Lindelöf's theorem.* We conclude this paper by showing how Lemma 2 can be used to obtain a space analogue of a theorem due to Lindelöf on the boundary behaviour of conformal mappings of a disk. We require first the following result.

Lemma 9. *Suppose that* $y(x)$ *is a* K*-quasiconformal mapping of the hemisphere* $|x| < c$, $x_3 > 0$, *that* Δ *is the half spherical annulus* $a < |x| < b < c$, $x_3 > 0$, *and that* Δ' *is the image of* Δ *under* $y(x)$. *Then*

$$\int_a^b \left(\operatorname{osc}_S y(x)\right)^3 \frac{dr}{r} \leqq A\, m(\Delta'),$$

where $S = S(r)$ *is the hemispherical surface* $|x| = r$, $x_3 > 0$,

$$\operatorname{osc}_S y(x) = \sup_{P,Q \in S} |y(P) - y(Q)|,$$

and A *is a finite constant which depends only on* K.

Proof. Let $y_i(x)$ be the i-th coordinate function for $y(x)$. Then $y_i(x)$ is continuous and ACL, and an elementary adaptation of the proof of Lemma 12 in [5] gives

$$\int_a^b \left(\operatorname{osc}_S y_i(x)\right)^3 \frac{dr}{r} \leqq B \int_\Delta |\nabla y_i(x)|^3\, d\omega,$$

where B is an absolute constant. Since $y(x)$ is a K-quasiconformal mapping, $y(x)$ is differentiable with $|\nabla y_i(x)|^3 \leqq K^2 J(x)$ a. e., where $J(x)$ denotes the absolute value of the Jacobian of the mapping. Now

$$\left(\operatorname{osc}_S y(x)\right)^2 \leqq \sum_{i=1}^{3} \left(\operatorname{osc}_S y_i(x)\right)^2,$$

and applying Hölder's inequality we obtain

$$\int_a^b \left(\operatorname{osc}_S y(x)\right)^3 \frac{dr}{r} \leqq A \int_\Delta J(x)\, d\omega = A\, m(\Delta'),$$

where $A = 3\sqrt{3}\, B\, K^2$. (See either [5] or [11] for the analytic properties of quasiconformal mappings used in the above argument.)

Now suppose that $y(x)$ is a homeomorphism of a domain D and that $P \in \partial D$. We say that a sequence of points $\{P_n\}$ in D *converges in a cone* to P if the P_n converge to P and there exists a constant a, $1 \leqq a < \infty$, such that

$$|P_n - P| \leqq a\,\varrho(P_n, \partial D)$$

for all n. We then denote by $C_A(P)$ the set of all points P', including possibly the point at infinity, for which there exists a sequence $\{P_n\}$ converging in a cone to P such that

$$\lim_{n \to \infty} y(P_n) = P'. \tag{28}$$

Next let γ denote any arc which has P as an endpoint and lies, except for this endpoint, in D. We say that γ is an *endcut* of D from P, and we denote by $C_\gamma(P)$ the set of all P' for which there exists a sequence of points $\{P_n\}$ converging to P along γ such that (28) holds. Finally we set

$$\Pi(P) = \bigcap_{\gamma} C_\gamma(P),$$

where the intersection is taken over all endcuts γ of D from P.

The following space analogue of a theorem due to Lindelöf (p. 28 in [7]) gives us a relation between the sets $\Pi(P)$ and $C_A(P)$ when $y(x)$ is a quasiconformal mapping.

Theorem 6. *If $y(x)$ is a quasiconformal mapping of a sphere D, then*

$$\Pi(P) = C_A(P) \tag{29}$$

for all $P \in \partial D$.

Proof. Fix $P \in \partial D$. By performing a preliminary Möbius transformation in the x-space, we may assume that D is the half space $x_3 > 0$ and that P is the origin O. If γ is any segment which joins O to a point of D, then γ is an endcut of D from O which lies in a cone and hence

$$\Pi(O) \subset C_\gamma(O) \subset C_A(O).$$

To complete the proof for (29) we must show that, given $P' \in C_A(O)$ and any endcut γ of D from O, there exists a sequence of points $Q_n \in \gamma \cap D$ which converge to P such that

$$\lim_{n \to \infty} y(Q_n) = P'. \tag{30}$$

Choose $P' \in C_A(O)$ and let γ be an endcut of D from O. Then there exists a sequence of points $\{P_n\}$ in D which converge to O and a constant a, $1 \leqq a < \infty$, such that

$$\lim_{n \to \infty} y(P_n) = P' \quad \text{and} \quad |P_n| \leqq a\,\varrho(P_n, \partial D) \tag{31}$$

for all n. Pick $c > 0$ so that Δ, the hemisphere $|x| < c$, $x_3 > 0$ contains the P_n, and assume for the moment that $y(x)$ maps Δ K-quasiconformally onto a bounded domain Δ'. Next set $b = 1 - 1/(2a)$, let Δ_n denote the half spherical annulus $b|P_n| < |x| < |P_n|$, $x_3 > 0$, and let Δ_n' denote the image of Δ_n under $y(x)$. Since $\Delta_n' \subset \Delta'$ and $m(\Delta') < \infty$, it follows that

$$\lim_{n\to\infty} m(\Delta_n') = 0 . \tag{32}$$

Lemma 9 implies that for each n there exists an r_n, $b|P_n| < r_n < |P_n|$, such that

$$\operatorname*{osc}_{S(r_n)} y(x) \leqq \left(\frac{A\, m(\Delta_n')}{-\log b}\right)^{1/3} . \tag{33}$$

Now $\gamma \cap S(r_n) \neq \emptyset$ for $n \geqq n_0$; choose $Q_n \in \gamma \cap S(r_n)$ and let R_n denote the point where the radius from O to P_n meets $S(r_n)$. Then (32) and (33) imply that

$$\lim_{n\to\infty} |y(Q_n) - y(R_n)| = 0 . \tag{34}$$

On the other hand, we see from (31) that $|R_n - P_n| \leqq \frac{1}{2}\varrho(P_n, \partial D)$ and hence Lemma 2 yields

$$|y(R_n) - y(P_n)| \leqq \Theta(\tfrac{1}{2})\, \varrho(y(P_n), \partial D') , \tag{35}$$

where D' is the image of D under $y(x)$. Now $y(P_n) \in \Delta'$. Hence the $y(P_n)$ are bounded, and since the P_n converge to $O \in \partial D$,

$$\lim_{n\to\infty} \varrho(y(P_n), \partial D') = 0 , \tag{36}$$

and (30) follows from (31), (34), (35) and (36).

Now we have obtained (30) under the assumption that Δ' is a bounded domain. Suppose that this is not the case. Since $y(x)$ is a homeomorphism, there exists an open sphere U' such that $U' \cap \Delta' = \emptyset$. Let $z(y)$ denote inversion with respect to U'. Then $z(y(x))$ maps Δ K-quasiconformally onto a bounded domain, and arguing exactly as above, we can find a sequence of points $Q_n \in \gamma \cap D$ which converge to O such that

$$\lim_{n\to\infty} z(y(Q_n)) = z(P') . \tag{37}$$

Since $z(y)$ is an inversion, (37) implies (30) and the proof of Theorem 6 is complete.

The following immediate consequence of Theorem 6 is an analogue of a very well known theorem due to Lindelöf (p. 10 in [7]) on bounded analytic functions.

Corollary. *If* $y(x)$ *is a quasiconformal mapping of a sphere* D *and if* $y(x)$ *converges to* P' *as* x *converges to* $P \in \partial D$ *along some endcut* γ *of* D, *then* $y(x)$ *converges to* P' *as* x *converges to* P *in a cone.*

University of Michigan
Ann Arbor, Michigan, U.S.A.

References

[1] Белинский, П. П.: О непрерывности пространственных квазиконформных отображений и о теореме Лиувилля. - Докл. Акад. Наук СССР (Новая Серия) 147, 1962, pp. 1003—1004.

[2] Carathéodory, C.: Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten. - Math. Ann. 72, 1912, pp. 107—144.

[3] —»— Conformal representation. - 2nd ed., Cambridge Tracts in Mathematics and Mathematical Physics 28, University Press, Cambridge, 1952.

[4] Gehring, F. W.: Symmetrization of rings in space. - Trans. Amer. Math. Soc. 101, 1961, pp. 499—519.

[5] —»— Rings and quasiconformal mappings in space. - Trans. Amer. Math. Soc. 103, 1962, pp. 353—393.

[6] —»— Quasiconformal mappings in space. - Bull. Amer. Math. Soc. 69, 1963, pp. 146—164.

[7] Lindelöf, E.: Sur un principe général de l'analyse et ses applications à la théorie de la représentation conforme. - Acta Soc. Scient. Fennicæ 46:4, 1915.

[8] Loewner, C.: On the conformal capacity in space. - J. Math. Mech. 8, 1959, pp. 411—414.

[9] MacLane, G. R.: Riemann surfaces and asymptotic values associated with real entire functions. - Rice Inst. Pamphlet, Special Issue: Monograph in Mathematics, 1952.

[10] Newman, M. H. A.: Elements of the topology of plane sets of points. - 2nd ed., University Press, Cambridge, 1954.

[11] Väisälä, J.: On quasiconformal mappings in space. - Ann. Acad. Scient. Fennicæ A. I. 298, 1961.

[12] —»— On quasiconformal mappings of a ball. - Ann. Acad. Scient. Fennicæ A. I. 304, 1961.

Printed November 1963.