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CLUSTER SET THEOREMS FOR ARBITRARY FUNCTIONS WITH APPLICATIONS TO FUNCTION THEORY

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1. *Introduction.* At first sight a function of which we know no more than that it has a value at every point of its domain looks rather unpromising material to work with. There are obvious limitations; but it has been found that quite striking statements can be made about such functions and if we select them to fit situations that arise, for example, in the theory of functions these statements may turn out to be useful theorems. Or it may be that a theorem about an arbitrary function can be significantly sharpened if some condition such as continuity is imposed. In any case »arbitrariness» is qualified by the properties of the spaces one of which is mapped into the other by the function. Here we shall for the most part be concerned with mappings from the real line or the plane disc to the 2-sphere or a great circle upon it, thus generalising a familiar function — theoretic situation. The theorems we obtain are inherent in the topology and geometry of the spaces involved and, as J. D. Weston has shown [18], are capable of further generalisation by removing restrictions on these spaces. Important as these developments are we shall not attempt to enter into them here.

2. The theorems we shall discuss are in terms of cluster sets; but these are not the only kind of theorems about arbitrary functions that are known. We denote the functions by $f(z)$, $|z| < 1$; $f(x)$, $0 < x < 1$; $f(P)$ or simply f. The cluster set of f at a point P of the closure of its domain, which we denote by $C(f, P)$, is the set of all the limits of f as the variable z or x within the domain D of f approaches P in all possible ways. In precise terms, $C(f,P)$, $P \in \bar{D}$, may be defined in either of the two following equivalent ways:

(a) $C(f, P)$ is the set of points α of the w-sphere S into which $w = f(z)$ or $w = f(x)$ maps D such that there exists a sequence $\{z_n\} \subset D \diagdown P$ satisfying $\lim_{n\to\infty} z_n = P$ and $\lim_{n\to\infty} f(z_n) = \alpha$.

(b) $C(f, P) = \bigcap \bar{\Delta}_n$ where $\Delta_n = f(\mathcal{D}_n \diagdown P)$ and $\mathcal{D}_n$ is the intersection of D with a neighbourhood $|x - P| < \delta_n$, or $|z - P| < \delta_n$, $\delta_n \downarrow 0$ as $n \to \infty$. The open sets $\mathcal{D}_n$, $n = 1, 2, \ldots$, thus have P as their only common interior or frontier point. The bar denotes closure.

It follows at once from both definitions that $C(f, P)$ *is non-empty and closed*. The complement $\complement C(f, P)$ is thus always an open set.

We note that if $f(z)$ is defined at P then $f(P)$ does not necessarily belong to $C(f, P)$ unless f is continuous at P. But if $f(z)$ is continuous in the deleted set $D \setminus P$ then $C(f, P)$ is either a *continuum* (compact connected set) or a *point*. In the latter case we say that $C(f, P)$ is *degenerate;* and in the former we say that $C(f, P)$ is *total* or *sub-total* according as $\complement C(f, P)$ is empty or not.

The cluster set as we have defined it is *complete*, being relative to the entire neighbourhood of P belonging to the domain D. But *restricted* cluster sets (these are sometimes called *partial*) are important in the theory. Examples are the *radial* cluster set $C_\varrho(f, P)$, where D is a disc and P a point of its circumference, defined by restricting $\{z_n\}$ to the radius to P, and the right and left-hand cluster sets $C_R(f, P)$ and $C_L(f, P)$, where x_n approaches P from the right (left) for f defined on the real line and $\arg z_n$ approaches $\arg P$ from below (above) for f defined in a disc and P a point of the circumference.

Symmetry theorems

3. In 1907 W. H. Young [19] proved the following theorem.

If $f(x)$ is an arbitrary real function of the real variable x then

$$(1) \qquad \overline{\lim}\, f_L(x) = \overline{\lim}\, f_R(x) \qquad \textit{and} \qquad \underline{\lim}\, f_L(x) = \underline{\lim}\, f_R(x)\,,$$

where the R and L signs denote limits to the right and left of x respectively, for all values of x except perhaps for a countable set.

This appears to have been the first theorem to be explicitly stated for arbitrary functions. Young announced it at the meeting of the British Association held at Leicester in 1907. He used to refer to it as the Leicester theorem. A year later, at the Rome Congress of 1908, he announced what he called the Rome theorem [20]. Stated in modern notation, this is the Leicester theorem with (1) replaced by

$$(2) \qquad C_L(f, x) = C_R(f, x)\,.$$

Young extended his investigations to the plane and to euclidean n-space in a series of papers [19]—[24] extending over twenty years; but he confined himself to real functions and his results passed unnoticed by complex variable analysts (see also [16]).

4. What turns out to be a useful extension of Young's Rome theorem to an arbitrary complex function defined in the disc is, however, almost

immediate. Denote the unit disc by U and its boundary by $\varkappa$. Writing $z_n = r_n e^{i\vartheta_n}$ the *right (left) cluster set* of $f(z)$ at $P \in \varkappa$ is defined as the set of values α such that for some sequence $\{z_n\} \subset U$, $\lim_{n\to\infty} z_n = P$, $\vartheta_n \uparrow \arg P$ $(\vartheta_n \downarrow \arg P)$, we have $\lim_{n\to\infty} f(z_n) = \alpha$. We can now prove

Theorem 1. *For an arbitrary real or complex function $f(z)$ defined in U*

$$C_R(f, e^{i\vartheta}) = C_L(f, e^{i\vartheta}) = C(f, e^{i\vartheta}) \tag{3}$$

except perhaps for a countable set of points $e^{i\vartheta} \in \varkappa$.

The proof, which also suffices, without any adaptation, to prove Young's Rome theorem, is simple.

We cover the Riemann sphere S with a succession of nets N_k, $k = 1, 2, \ldots$, each with a finite number of closed triangular meshes m_{kn}, $n = 1, 2, \ldots, n_k$, of diameter $< 1/\mathrm{k}$. We arrange these in a sequence in any convenient way. Call the sequence $\{m_j\}$.

For each index j we define the set E_j of points $e^{i\vartheta}$ for which m_j meets $C(f, e^{i\vartheta})$ but does not meet $C_R(f, e^{i\vartheta})$. If $e^{i\vartheta} \in E_j$, then $e^{i\vartheta}$ is the left-hand end-point of an arc $I_j(\vartheta)$ such that for $e^{i\psi} \in I_j(\vartheta)$ the cluster set $C(f, e^{i\psi})$ does not meet m_j; for otherwise m_j would meet $C_R(f, e^{i\vartheta})$. $I_j(\vartheta)$ therefore contains no point of E_j. The set E_j thus consists of end-points of non-overlapping intervals and is therefore countable. Plainly, the set E_R of points $e^{i\vartheta}$ for which $C(f, e^{i\vartheta}) \setminus C_R(f, e^{i\vartheta})$ is not empty is the union of the sets E_j, $j = 1, 2, \ldots$, and this proves the theorem.

5. Exactly the same argument proves the local symmetry, except perhaps at a countable set, of the right and left *boundary cluster sets*. The right hand boundary cluster set $C_{BR}(f, e^{i\vartheta})$ at $e^{i\vartheta}$ is simply the cluster set defined along the arc of $\varkappa$ to the right of $e^{i\vartheta}$, only, since f is not necessarily defined at points of $\varkappa$, we use the cluster sets of f at points of the arc as the values. Any appearance of sophistication in this definition is deceptive. Similarly for the left hand boundary cluster set $C_{BL}(f, e^{i\vartheta})$. We then have

Theorem 2. *For an arbitrary function $f(z)$ in U*

$$C_{BR}(f, e^{i\vartheta}) = C_{BL}(f, e^{i\vartheta}) = C(f, e^{i\vartheta}) \tag{4}$$

except perhaps for a countable set of points $e^{i\vartheta} \in \varkappa$.

Since evidently $C_{BL}(f, e^{i\vartheta}) \subset C_L(f, e^{i\vartheta})$ and $C_{BR}(f, e^{i\vartheta}) \subset C_R(f, e^{i\vartheta})$ Theorem 1 is contained in Theorem 2.

It will be seen that Theorems 1 and 2 and Young's theorems quoted above depend essentially on the topology of the real line or circumference of the disc. However, for functions defined in a ball we can, by the same argument, establish bilateral symmetry of the *global cluster sets* and *global boundary cluster sets* to the right and left of a meridian, say, or parallel of latitude (looking respectively eastwards or northwards, say) except for a countable set of values of the longitude or latitude. By the global cluster set to the right of a meridian we mean the union of the cluster sets at all points of the meridian approached from the right, and similarly for the other one-sided global cluster sets. Patterns of symmetry at points of the spherical boundary of the ball, equality in opposite quadrants defined by the meridian and parallel through the point, for example, may be shown by an obvious adaptation of Young's method for the plane [23] to hold except for a set of points of first category on a countable set of meridians or parallels. In contrast, Bagemihl's ambiguous point theorem for an arbitrary function in the disc is essentially 2-dimensional and has no analogue in the ball.

Maximality theorems

6. It follows from (4) that the cluster sets at $e^{i\vartheta}$ in the two domains in U lying outside chords intersecting at $e^{i\vartheta}$ and to the right and left of the radius are both *maximal*, i. e. equal to $C(f, e^{i\vartheta})$, except for a countable set of values of ϑ. This suggests the question: What can we say about the cluster set in a small angle Δ at $e^{i\vartheta}$ between two chords? To fix ideas suppose that Δ is bisected by the radius ϱ. In this situation the kind of argument by which we proved Theorem 1 is not available and the set of points $e^{i\vartheta} \in \varkappa$ at which $C_\Delta(f, e^{i\vartheta}) \neq C(f, e^{i\vartheta})$ is no longer countable but is of first category (we shall write this category I) in the sense of Baire. The theorem [10] is as follows:

Theorem 3. *For an arbitrary function* $f(z)$ *in* U *and an angle* $\Delta(\vartheta)$ *of any fixed magnitude bisected by the radius to* $e^{i\vartheta}$

$$C_{\Delta(\vartheta)}(f, e^{i\vartheta}) = C(f, e^{i\vartheta}) \tag{5}$$

except perhaps for a set of points of category I on $\varkappa$.

Suppose the contrary so that the set E of points $e^{i\vartheta}$ at which $C_{\Delta(\vartheta)}(f, e^{i\vartheta}) \neq C(f, e^{i\vartheta})$ is of category II. In three steps we shall select a subset E_0 of E, also of category II and therefore dense in an arc β of $\varkappa$, in which a certain closed subset T_0 of the 2-sphere S meets $C(f, e^{i\vartheta})$ but is uniformly bounded away from $f(\Delta(\vartheta))$. These properties are inconsistent. For, since E_0 is dense in β every point z of U sufficieutly near to β is contained in a $\Delta(\vartheta)$, $e^{i\vartheta} \in E_0$, so that $f(z)$ is uniformly bounded away from T_0 which therefore does not meet $C(f, e^{i\vartheta})$ for any $e^{i\vartheta} \in \beta$. This contradiction will prove the theorem.

The selection is carried out as follows.

(i) For any point of E, since $C_{\Delta(\vartheta)}(f, e^{i\vartheta})$ is closed and $\subset C(f, e^{i\vartheta})$, there is a positive number ε such that the intersection

$$C(f, e^{i\vartheta}) \cap \mathcal{C}(C_{\Delta(\vartheta)}(f, e^{i\vartheta})_{+\varepsilon})$$

is not empty, where $C_{\Delta(\vartheta)}(f, e^{i\vartheta})_{+\varepsilon}$ is the closed ε-neighbourhood of $C_{\Delta(\vartheta)}(f, e^{i\vartheta})$ (in the spherical metric) and $\mathcal{C}$ denotes the complement. Choose a sequence $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_n > \ldots$, $\lim_{n\to\infty} \varepsilon_n = 0$, and denote by E_n the subset of E for which

$$C(f, e^{i\vartheta}) \cap \mathcal{C}(C_{\Delta(\vartheta)}(f, e^{i\vartheta})_{+\varepsilon_n})$$

is not empty. Plainly, $E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$ (a finite number of these sets may be empty) and

$$E = \bigcup_n E_n .$$

We can therefore find N such that E_N (and every set of higher rank) is of category II.

(ii) To select from E_N we now cover S with a finite net of meshes $T_1, T_2, \ldots, T_m$ each of diameter less than $\varepsilon_N/4$. We treat the meshes T as closed. For a given $\mu \leqq m$ let $E_{N\mu}$ be the subset of E_N in which the set

$$T_\mu \cap C(f, e^{i\vartheta}) \cap \mathcal{C}(C_{\Delta(\vartheta)}(f, e^{i\vartheta})_{+\varepsilon_N})$$

is not empty (T_μ meets $C(f, e^{i\vartheta})$ but not $C_{\Delta(\vartheta)}(f, e^{i\vartheta})_{+\varepsilon_N}$). Since

$$E_N = \bigcup_{\mu \leqq m} E_{N\mu}$$

and E_N is of category II we can find $M \leqq m$ such that E_{NM} is of category II.

(iii) Finally we have to select from E_{NM}. Let T_0 be any closed subset of T_M. Then for all $e^{i\vartheta} \in E_{NM}$ the distance of T_0 from $C_{\Delta(\vartheta)}(f, e^{i\vartheta})$ is greater than $3\,\varepsilon_N/4$. We define E_{NMq}, $q = 1, 2, \ldots$, as the subset of E_{NM} in which the (spherical) distance $[f(z), T_0]$ of $f(z)$ from T_0 satisfies the inequality $[f(z), T_0] > \varepsilon_N/2$ for all $z \in \Delta(\vartheta)$, $1 - 2^{-q} < |z| < 1$. Again,

$$E_{NM1} \subset E_{NM2} \subset \ldots \subset E_{NMq} \subset \ldots$$

and

$$E_{NM} = \bigcup_q E_{NMq} .$$

Since E_{NM} is of category II we can find Q such that E_{NMQ} is of category II and therefore dense in some arc $\beta \subset \varkappa$. To complete the proof of the theorem we have only to put $E_0 = E_{NMQ}$.

7. The argument by which we have proved Theorem 3 may be applied to any angle of fixed magnitude and orientation relative to the radius. We can specify a countable set of directions and magnitudes so that any angle with $e^{i\vartheta}$ as vertex contains one of the angles so specified. We therefore have

Theorem 4. *Let* $f(z)$ *be an arbitrary function in* U. *Then except perhaps for a set* E *of points* $e^{i\vartheta}$ *of category I on* $\varkappa$ *we have for every angle* $\Delta \subset U$ *with* $e^{i\vartheta}$ *as vertex*

$$C_\Delta(f, e^{i\vartheta}) = C(f, e^{i\vartheta}) . \tag{7}$$

At a point $e^{i\vartheta} \in \varkappa \setminus E$ at which $C(f, e^{i\vartheta})$ is *total* $f(z)$ has the cluster set property which a meromorphic function has at a *Plessner point* [9]. It must be emphasised, however, that Theorem 4 is in no way comparable with Plessner's well known theorem for meromorphic functions.

8. If we now impose the condition of continuity on $f(z)$ a quite trivial change in the proof of Theorem 3, working with $C_\varrho(f, e^{i\vartheta})$ instead of $C_{\Delta(\vartheta)}(f, e^{i\vartheta})$ gives the following *maximality theorem* [9]

Theorem 5. *If* $f(z)$ *is continuous in* U *then*

$$C_\varrho(f, e^{i\vartheta}) = C(f, e^{i\vartheta}) \tag{8}$$

except perhaps for a set of points of category I on $\varkappa$.

By the same process of selection as in the proof of Theorem 3 we obtain a set E_0 dense in an arc $\beta \subset \varkappa$ in which T_0 meets $C(f, e^{i\vartheta})$ but is uniformly bounded away from $f(r e^{i\vartheta})$, $e^{i\vartheta} \in E_0$, $1-2^{-Q} < r < 1$. By the continuity of $f(z)$ these properties are inconsistent and the theorem is proved.

Theorems 3 and 5 are special cases of more general results proved by essentially the same arguments [9]—[10].

Applications

9. An obvious application of Theorem 5 was to Caratheodory's problem of determining the distribution of the four kinds of prime ends which correspond under a conformal mapping of U onto a simply connected domain, not necessarily Jordan, to points of $\varkappa$. The correspondence is 1:1 and the *impression* of the prime end corresponding to a point $e^{i\vartheta} \in \varkappa$ under a conformal mapping $f(z)$ is the cluster set $C(f, e^{i\vartheta})$. If λ is any path in U terminating at $e^{i\vartheta}$ then plainly $C_\lambda(f, e^{i\vartheta}) \subseteq C(f, e^{i\vartheta})$. A point of $C(f, e^{i\vartheta})$ which belongs to every cluster set $C_\lambda(f, e^{i\vartheta})$ is a *principal point* of the prime end in question. Points of $C(f, e^{i\vartheta})$ which are not

principal are *subsidiary*. The set $\Pi(f, e^{i\vartheta})$ of principal points of the prime end is either degenerate, in which case the point is accessible from inside U and is the sole accessible point of the prime end, or is a continuum, in which case there are no accessible points of the prime end.

A prime end is of the *first kind* if it consists of a single (necessarily) accessible point so that $C(f, e^{i\vartheta})$ is degenerate. It is of the *second kind* if it has one (accessible) principal point and an infinity of subsidiary points; of the *third kind* if it has no subsidiary points and an infinity of principal points; and of the *fourth kind* if it has an infinity of both principal and subsidiary points. We denote these four classes of prime ends by $\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \mathscr{E}_4$, and the corresponding sets on $\varkappa$ by e_1, e_2, e_3, e_4.

It was proved by Lindelöf that *for any conformal mapping* $f(z)$ *and any point* $e^{i\vartheta}$

$$C_\varrho(f, e^{i\vartheta}) = \Pi(f, e^{i\vartheta}) . \tag{9}$$

Combining (8) with (9) we see that

$$\Pi(f, e^{i\vartheta}) = C(f, e^{i\vartheta}) \tag{10}$$

except perhaps for a set of points $e^{i\vartheta}$ *of category I on* $\varkappa$. The points for which (10) is satisfied are precisely the points $e_1 \cup e_3$ corresponding to $\mathscr{E}_1 \cup \mathscr{E}_3$ so that the complementary sets $e_2 \cup e_4$ in $\varkappa$ and $\mathscr{E}_2 \cup \mathscr{E}_4$ in the metric space of prime ends, the metric being, for example, the chordal distance between corresponding points of $\varkappa$, are both of category I. This result is one of the keys to the solution of the distribution problem the implications of which have been described elsewhere [9] and need not be repeated here.

10. A recent refinement of the distribution problem concerns the relative frequency of asymmetrical prime ends. There can be no asymmetry about a prime end of the first or third kind. But a prime end of the second or fourth kind is asymmetrical if its right and left wings are not identical, which is equivalent to saying that $C_R(f, e^{i\vartheta}) \neq C_L(f, e^{i\vartheta})$. This question is solved by Theorem 1, which is so fundamental as to be valid for an arbitrary function. It shows that the asymmetrical prime ends of any domain are actually *countable;* and examples show that they may be infinite in number and their corresponding points dense on $\varkappa$ [11].

11. Unlike the symmetry theorems 1 and 2 the maximality theorems 3, 4 and 5 may be immediately extended to functions defined in a three-dimensional ball, where now $\Delta(P)$ will denote a cone having P, a point of the spherical boundary of the ball, as its vertex and the radius to this point as its axis. The 2-sphere being a complete metric space, the Baire category

theorem is again applicable and the arguments used to prove Theorem 3, with only the obvious verbal changes, give

Theorem 6. *For an arbitrary function f which maps the unit ball into 3-space we have*

$$C_{\Delta(P)}(f, P) = C(f, P) \tag{11}$$

except perhaps for a set of points P of category I on the spherical boundary S of the ball.

The definitions of the cluster sets featuring in this theorem are obvious.

Similarly, if f is continuous in the ball and $C_\varrho(f, P)$ is the radial cluster set at P we have

Theorem 7. *If the function f is continuous in the unit ball, then*

$$C_\varrho(f, P) = C(f, P) \tag{12}$$

except perhaps for a set of points P of category I on S.

This last result is a special case of a general theorem of Weston's on cluster sets of mappings from one topological space to another. For an abstract theory of cluster sets in a general topological setting we refer to Weston [18].

Theorem 7, being sufficiently fundamental to be valid for any continuous function is applicable, for example, to quasiconformal mappings of the ball in space.

12. Let $D(f)$ denote the set of points of $\varkappa$ for which $C_\varrho(f, e^{i\vartheta})$ is degenerate, i. e. $\lim_{r \to 1} f(r\, e^{i\vartheta})$ exists. We denote this limit by $f(e^{i\vartheta})$; and following Doob [12] we may call $\mathcal{F}(e^{i\vartheta}) = C(f, e^{i\vartheta})$ the cluster boundary function for f. The following theorem is due to Weniaminoff [17].

If $f(z)$ is analytic and bounded in U and if $\varkappa \subset D(f)$, then $C(f, e^{i\vartheta})$ is degenerate at every point of $\varkappa$ at which $f(e^{i\vartheta})$ is continuous and hence $e^{i\vartheta}$ is a point of continuity of $\mathcal{F}(e^{i\vartheta})$.

Weniaminoff also proves (his Lemma 1 in [17]) that if $\varkappa \subset D(f)$ then for an analytic function (evidently in fact for a continuous function) $f(e^{i\vartheta})$ is of Baire class 1 and so, by a theorem of Baire, its discontinuities are a set of category I. From this it follows, by the theorem of Weniaminoff quoted above, that

If $f(z)$ is analytic and bounded in U and if $\varkappa \subset D(f)$, then $f(e^{i\vartheta})$ being of Baire class 1 is pointwise discontinuous on $\varkappa$, having a residual set of points of continuity on $\varkappa$ which are also points of continuity of $\mathcal{F}(e^{i\vartheta})$.

Theorem 5 enables us to generalise this theorem by dropping the requirements that $f(z)$ be bounded or analytic. It is enough that $f(z)$ should be continuous.

Suppose then that $f(z)$ is continuous in U and that $\varkappa \subset D(f)$. Then, by Theorem 5, $C(f, e^{i\vartheta})$ is degenerate on a residual set i. e. the complement of a set of category I on $\varkappa$. But a point at which $C(f, e^{i\vartheta})$ is degenerate is a point of continuity of both $\mathcal{F}(e^{i\vartheta})$ and $f(e^{i\vartheta})$. Thus both $\mathcal{F}(e^{i\vartheta})$ and $f(e^{i\vartheta})$ are pointwise discontinuous and the set of discontinuities is in each case of category I. By a well known theorem the set of points of continuity of a single valued function is a G_δ (Hausdorff [13], p. 251). The same argument shows that the points of continuity of $\mathcal{F}(e^{i\vartheta})$ and $f(e^{i\vartheta})$ are both of type G_δ. A precisely similar argument applies to a function continuous in the ball and so we have

Theorem 8. *If the function f is continuous in the disc (or the ball) and if $\varkappa \subset D(f)$ (or $S \subset D(f)$) then the cluster boundary function $\mathcal{F}(e^{i\vartheta})$ ($\mathcal{F}(P)$, $P \in S$) is pointwise discontinuous on $\varkappa$ (on S) and the set of its discontinuities is an F_σ of category I.*

13. We denote by $\Gamma_\varrho(f, A)$ the set $f(e^{i\vartheta})$, $e^{i\vartheta} \in D(f) \subset A$, of radial limits of $f(z)$ on an arc $A \subset \varkappa$. The following theorem (proved in [8]), related to Theorem 8, is also a consequence of Theorem 5.

Theorem 9. *If $f(z)$ is meromorphic in U and $D(f)$ is of category II on some arc $A \subset \varkappa$, then either $\Gamma_\varrho(f, A)$ is of positive linear measure or* $f(z) =$ constant.

This result is considerably stronger than an earlier theorem of Privalov ([15], pp. 231—232), The conclusion on the measure of $\Gamma_\varrho(f, A)$ depends upon a lemma of M. L. Cartwright's on meromorphic functions so that the theorem is deeper than the much more general Theorem 8.

Interior theorems

14. For the purpose of function-theoretic applications we fix our attention, in considering an arbitrary function, on its properties at the boundary of its domain of definition since it is only there that the regularity of an analytic function breaks down. There are, however, properties of an arbitrary function within its domain which are of independent interest. These are of the kind first brought to light by W. H. Young for real functions and further studied by H. Blumberg [2]—[7]. Variants of them, or some of them, are very easily proved by the methods of this paper for complex functions and for more general mappings.

Let $f(z)$ be an arbitrary function defined in a plane domain D which it maps into the 2-sphere S (or into the complete euclidean n-space) and let P be a point of D. We prove

Theorem 10. *If $f(z)$ is an arbitrary function mapping D into S then $f(P) \in C(f, P)$ for all P in D except perhaps for a set which is at most countable.*

The proof follows familiar lines. E is now the set of points $P \in D$ in which $f(P) \notin C(f, P)$ so that for all $P \in E$ there is a positive ε such that the distance $[f(P), C(f, P)]$ of $f(P)$ from $C(f, P)$ (in any appropriate metric) exceeds ε. Given a decreasing sequence $\{\varepsilon_n\}$, $\lim_{n\to\infty} \varepsilon_n = 0$, E_n is the set such that

$$[f(P), C(f, P)] > \varepsilon_n$$

so that $E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$ and

$$E = \cup E_n . \tag{13}$$

Let $T_1, T_2, \ldots, T_{mn}$ be a finite triangulation on S whose meshes are all of diameter less than $\varepsilon_n/4$; and let $E_{n\mu}$ be the subset of E_n at every point of which $f(P) \in T_\mu$, $\mu \leqq mn$. Then every $E_{n\mu}$ is an isolated set. For suppose a point P_0 of $E_{n\mu}$ to be a limit point of a sequence of points $P_\nu \in E_{n\mu}$ so that $C(f, P_0)$ meets T_μ. Then the distance of $f(P_0)$ from $C(f, P_0)$ is less than $\varepsilon_n/4$, contrary to the definition of E_n which contains $E_{n\mu}$. Thus every set $E_{n\mu}$ is countable; hence $E_n = \cup E_{n\mu}$ is countable and consequently $E = \cup E_n$ is countable, which proves the theorem.

In the case of a function of a real variable D is a segment of the real line and exactly the same argument applies and the exceptional set is again countable. The theorem is also true for an arbitrary mapping into any complete metric space.

This theorem generalises a theorem of Young's (Theorem 1 of [21] and 2° on page 5 of [22]). It is also easy, by our general method, to prove a rather stronger result of Blumberg's (Theorem 1 of [3]) which he proved for real functions but which is also valid for an arbitrary mapping into S (or into any complete metric space). He introduces the notion of dense approach to a value of the function. The definition is as follows. The function f is said to be *densely approached at the point* P if for every $\varepsilon > 0$ there exists a neighbourhood $G(P, \varepsilon)$ of P such that the points of this neighbourhood for which

$$[f(z), f(P)] < \varepsilon$$

are dense in $G(P, \varepsilon)$. With this definition we have

Theorem 11. *If $f(z)$ is an arbitrary function mapping D into S (or any complete metric space) then the set of points at which the value of f is densely approached is residual in D.*

The set of dense approach is of course a subset of the set for which $f(P) \in C(f, P)$ whose complement is countable. Now let E be the set in

D at which $f(P)$ is not densely approached. For a given ε_n, where again $\{\varepsilon_n\}$ is a decreasing sequence tending to zero, we denote by E_n the subset of E such that every neighbourhood of $P \in E_n$ contains a domain in which the distance from $f(P)$ to $f(z)$ exceeds ε_n. Evidently, $E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$ and $E = \cup E_n$. Repeating the triangulations $T_1, T_2, \ldots, T_{mn}$, the meshes being all of diameter $< \varepsilon_n/4$, the set $E_{n\mu}$ is again the subset of E_n in which $f(P) \in T_\mu$, $\mu \leqq mn$. If now the theorem is false the set E of exceptional points is of category II and it follows as before that there exist $n = N$ and $\mu = M$ such that E_{NM} is of category II and therefore dense in a domain $G \subset D$. Let P_0 be a point of $G \cap E_{NM}$. At every other point of $G \cap E_{NM}$ and at P_0 itself $f \in T_M$ so that the distance of $f(P_0)$ from $f(z)$, $z \in G \cap E_{NM}$, is less than $\varepsilon_N/4$ and so there is no sequence of domains having P_0 as a limit point in which the distance from $f(P_0)$ to $f(z)$ exceeds ε_N: Since $P_0 \in E_N$ this is a contradiction and the theorem is proved.

Concluding remarks

15. The earlier literature on arbitrary functions, which is quite considerable, was concerned exclusively with the interior theory. Sufficient references to trace this theory back to its sources are given in the list of references at the end of this paper. The more recent work has been almost as heavily biassed towards boundary theory owing to the discovery, largely through the development of the theory of cluster sets, of applications in the theory of functions and allied fields. The interior theory has, however, also attracted attention as has the extension of the theory to general topological spaces. The lack of a unifying idea, namely that of the cluster set, or of an accepted terminology and notation hampered development for a long time. As has been shown here the descriptive theory may be handled very easily by a uniform method which is applicable to very general situations. The part of the theory which we have not touched on is the metrical part to which both Young and Blumberg contributed, but this, again, was an interior theory not immediately applicable to boundary problems. The list of recent papers is confined to those actually referred to and is not complete.

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