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SOME APPROXIMATION THEOREMS FOR NORMAL FUNCTIONS

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Some approximation theorems for normal functions [1)]

1. Suppose that $f(z)$ is a nonconstant normal meromorphic function (see [2, p. 86]) in the open unit disk D. We are going to show, among other things, that if the set $A(f)$ of asymptotic values of f is of harmonic measure zero, then there exists a set S of points on the unit circle Γ, where S is both of Lebesgue measure 2π and a residual set in the sense of Baire category, with the following property: the function $f(z)$ approximates every complex value arbitrarily closely on every continuous curve in D that intersects Γ in a point of S at an angle different from zero.

This generalizes in several directions certain classical results of P. J. Myrberg ([3], [4]). (A very special case to which our theorem applies is that of the elliptic modular function.)

2. We denote by Ω the extended complex plane. If z and z' are points of D, then $\varrho(z, z')$ is the non-Euclidean hyperbolic distance between them. A Stolz angle with vertex $\zeta \in \Gamma$ will be called a Stolz angle at ζ. By an arc at $\zeta \in \Gamma$ we mean a continuous curve $\Lambda : z = z(t)$ $(0 \leqq t < 1)$ such that $|z(t)| < 1$ for $0 \leqq t < 1$ and $\lim_{t \to 1} z(t) = \zeta$. A terminal subarc of an arc Λ at ζ is a subarc of Λ of the form $z = z(t)$ $(t_0 \leqq t < 1)$, where $0 \leqq t_0 < 1$. By an admissible arc at ζ we mean an arc at ζ having a tangent at ζ different from the tangent to Γ at ζ.

If $f(z)$ is meromorphic in D, and $\zeta \in \Gamma$, the cluster set of f at ζ is denoted by $C(f, \zeta)$; the cluster set of f at ζ on an arc Λ at ζ, or on a Stolz angle Δ at ζ, is denoted by $C_\Lambda(f, \zeta)$, $C_\Delta(f, \zeta)$, respectively. We write $C_{\mathscr{A}}(f, \zeta) = \bigcup_{\Delta} C_\Delta(f, \zeta)$, where the union is taken over all Stolz angles Δ at ζ. We define $\Pi_T(f, \zeta) = \bigcap_{\Lambda} C_\Lambda(f, \zeta)$, where the intersection

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is taken over all admissible arcs Λ at ζ. The set of points $\zeta \in \Gamma$ such that $C_{\Delta}(f, \zeta) = C_{\Delta'}(f, \zeta)$ for every pair of Stolz angles Δ, Δ' at ζ is called $K(f)$. (The set $K(f)$ is always a residual subset of Γ of measure 2π [2, p. 68].)

3. Our theorems are based upon the following general

Lemma. *Let $f(z)$ be a normal meromorphic function in D, and suppose that $\zeta \in K(f)$. Then $\Pi_T(f, \zeta) = C_{\mathcal{A}}(f, \zeta)$.*

Proof: Suppose that $\omega \in \Pi_T(f, \zeta)$. Then $\omega \in C_{\Lambda}(f, \zeta)$ for every admissible arc Λ at ζ. Since Λ intersects Γ at a nonzero angle, there exists a Stolz angle Δ at ζ containing a terminal subarc of Λ. Clearly $C_{\Lambda}(f, \zeta) \subseteq C_{\Delta}(f, \zeta)$, so that $\omega \in C_{\mathcal{A}}(f, \zeta)$.

Now suppose that $\omega \in C_{\mathcal{A}}(f, \zeta)$. Let Λ be any admissible arc at ζ. Since $\zeta \in K(f)$, we have $\omega \in C_{\Delta}(f, \zeta)$ for *every* Stolz angle Δ at ζ, and hence there exists a sequence of points $\{z_n'\}$ in D, where $\lim_{n\to\infty} z_n' = \zeta$ and $\lim_{n\to\infty} f(z_n') = \omega$, such that, for an appropriate sequence of points $\{z_n\}$ on Λ with $\lim_{n\to\infty} z_n = \zeta$, we have $\lim_{n\to\infty} \varrho(z_n, z_n') = 0$. From the fact that $f(z)$ is a normal meromorphic function in D, we infer [1, p. 10, Lemma 1] that $\lim_{n\to\infty} f(z_n) = \omega$, and hence $\omega \in C_{\Lambda}(f, \zeta)$. This holds for an arbitrary admissible arc Λ at ζ, and therefore $\omega \in \Pi_T(f, \zeta)$.

4. Theorem 1. *Let $f(z)$ be a nonconstant normal meromorphic function in D, and suppose that $A(f)$ is of harmonic measure zero. Then there exists a residual subset S of Γ of measure 2π such that, for every $\zeta \in S$, $\Pi_T(f, \zeta) = \Omega$.*

Proof: According to Plessner's theorem [2, p. 70], almost every point of Γ is either a Fatou point or a Plessner point of f. The set of Fatou points of f, however, must be of measure zero, otherwise Privalov's theorem [2, p. 72] would imply a contradiction to the fact that f is nonconstant and $A(f)$ is of harmonic measure zero. Hence, the set $I(f)$ of Plessner points of f is of measure 2π. It follows [2, p. 65] that $I(f)$ is also a residual subset of Γ. If $\zeta \in I(f)$, then $C_{\mathcal{A}}(f, \zeta) = \Omega$, and since $I(f) \subseteq K(f)$, our Lemma yields $\Pi_T(f, \zeta) = C_{\mathcal{A}}(f, \zeta)$. Setting $S = I(f)$, we obtain Theorem 1.

A set of linear measure zero may be of positive harmonic measure [2, p. 7]. In case $A(f)$ is known merely to be of linear measure zero, we have nevertheless the following result.

Theorem 2. *Let $f(z)$ be a nonconstant normal meromorphic function in D, and suppose that $A(f)$ is of linear measure zero. Then there exists a residual subset R of Γ such that, for every $\zeta \in R$, $\Pi_T(f, \zeta) = \Omega$.*

Proof: Since $f(z)$ is nonconstant and $A(f)$ is of linear measure zero, we have $C(f, \zeta) = \Omega$ for every $\zeta \in \Gamma$ [2, p. 51]. It follows [2, III, § 3] that $I(f)$ is a residual subset of Γ. As in the proof of Theorem 1, $\zeta \in I(f)$ implies that $\Pi_T(f, \zeta) = \Omega$, and setting $R = I(f)$, we obtain Theorem 2.

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References

[1] F. BAGEMIHL and W. SEIDEL, Sequential and continuous limits of meromorphic functions.- Ann. Acad. Sci. Fennicæ A I 280 (1960), 1—17.
[2] K. NOSHIRO, Cluster sets.- Berlin, 1960.
[3] P. J. MYRBERG, Ein Satz über die fuchsschen Gruppen und seine Anwendung in der Funktionentheorie.- Ann. Acad. Sci. Fennicæ A 32 (1929) No. 9, 1—35.
[4] —»— Ein Approximationssatz für die fuchsschen Gruppen.- Acta Math. 57 (1931), 389—409.

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