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ON THE COMPLETION OF UNIFORM SPACES

BY

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On the completion of uniform spaces

One of the principal results in the theory of uniform spaces is the possibility of imbedding an arbitrary uniform space in a complete uniform space as a dense subspace. This problem has been treated by BOURBAKI ([1], [2]) in two different ways. To the present author it seems that the situation becomes more lucid, if the construction of Bourbaki is modified by using a trick of KOWALSKY ([3], p. 159). This modification yields the following, more precise result:

Theorem: *For every uniform space* E *there exists a complete uniform space* $\tilde{E}$*, containing* E *as a dense subspace, and such that every point of* $\tilde{E} - E$ *is* T_0*-separated from every other point of* $\tilde{E}$*. Two complete spaces* $\tilde{E}_1$ *and* $\tilde{E}_2$*, satisfying these conditions, are always uniformly isomorphic and even in such a way that the identity mapping* $E \to E$ *has a unique extension to a uniform isomorphism* $\tilde{E}_1 \to \tilde{E}_2$.

Corollary: *For every uniform* T_0*-space* E *there exists a complete uniform* T_0*-space* $\tilde{E}$*, containing* E *as a dense subspace. If* $\tilde{E}_1$ *and* $\tilde{E}_2$ *are two complete* T_0*-spaces, containing* E *as a dense subspace, then the identity mapping* $E \to E$ *has a unique extension to a uniform isomorphism* $\tilde{E}_1 \to \tilde{E}_2$.

The purpose of this paper is to prove the theorem stated above and deduce the corollary from it.

If $(E, \mathfrak{w})$ is a uniform space, $\mathfrak{w}$ being its uniformity, we denote the uniform topology of $(E, \mathfrak{w})$ by $\tau_{\mathfrak{w}}$ and the neighborhood filter of a point x of the topological space $(E, \tau_{\mathfrak{w}})$ by $\tau_{\mathfrak{w}}(x)$. If $\mathfrak{u}$ is a filter of E and M a subset of E, we denote the trace of $\mathfrak{u}$ in M by $\mathfrak{u}_M$. The induced uniformity of M is, however, denoted by $\mathfrak{w}_M$ instead of $\mathfrak{w}_{M \times M}$. If W is a set of $\mathfrak{w}$ and x a point of E, the set of all points y in E such that $(x, y) \in W$ is denoted by $W[x]$.

1. **Minimal Cauchy filters.** A filter $\mathfrak{u}$ of a uniform space $(E, \mathfrak{w})$ is called open, if it has a base every set of which is open in the topology $\tau_{\mathfrak{w}}$.

It is called a Cauchy filter, if for every set W of $\mathfrak{w}$ there exists a set A of $\mathfrak{u}$ such that $A \times A \subset W$. If $\mathfrak{u}$ is a Cauchy filter of $(E, \mathfrak{w})$, then its trace $\mathfrak{u}_M$ in an arbitrary subspace $(M, \mathfrak{w}_M)$ is a Cauchy filter of this subspace.

A uniform space $(E, \mathfrak{w})$ is said to be complete, if every Cauchy filter of $(E, \mathfrak{w})$ is convergent in the uniform topology $\tau_{\mathfrak{w}}$, i.e. if it contains, as a subfamily, at least one neighborhood filter $\tau_{\mathfrak{w}}(x)$.

The family of all Cauchy filters of a uniform space $(E, \mathfrak{w})$ is partially ordered by inclusion. The maximal element of this partially ordered family is the filter that consists of all subsets of E, including the empty set $\emptyset$. We denote this filter by $\mathfrak{o}$ and call it the zero filter of E. The minimal elements of the partially ordered family of all Cauchy filters are called *minimal* Cauchy filters. In other words, a Cauchy filter $\mathfrak{u}$ of $(E, \mathfrak{w})$ is minimal, if from the relation $\mathfrak{v} \subset \mathfrak{u}$ it necessarily follows that the Cauchy filter $\mathfrak{v}$ is $= \mathfrak{u}$.

Every neighborhood filter $\tau_{\mathfrak{w}}(x)$ is a minimal Cauchy filter and every minimal Cauchy filter is open. Furthermore, for every Cauchy filter $\mathfrak{u} \neq \mathfrak{o}$ there exists a unique minimal Cauchy filter $\mathfrak{v}$ such that $\mathfrak{v} \subset \mathfrak{u}$. It is readily seen that if $\mathfrak{a}$ and $\mathfrak{b}$ are two minimal Cauchy filters, such that $A \cap B \neq \emptyset$ for every A in $\mathfrak{a}$ and every B in $\mathfrak{b}$, then $\mathfrak{a} = \mathfrak{b}$. In fact, the family $\mathfrak{c} = \{A \cup B : A \in \mathfrak{a}, B \in \mathfrak{b}\}$ is then a Cauchy filter, satisfying $\mathfrak{c} \subset \mathfrak{a}$ and $\mathfrak{c} \subset \mathfrak{b}$, and from this it follows that $\mathfrak{a} = \mathfrak{c} = \mathfrak{b}$. Finally we observe that in a complete uniform space $(E, \mathfrak{w})$ the family of all minimal Cauchy filters coincides with the family of all neighborhood filters $\tau_{\mathfrak{w}}(x)$.

2. **The construction of $\tilde{E}$.** Let $(E, \mathfrak{w})$ be an arbitrary uniform space. We denote by $\tilde{E}$ the set whose elements are

(a) the points of E,

(b) those minimal Cauchy filters of $(E, \mathfrak{w})$ that do not coincide with any neighborhood filter $\tau_{\mathfrak{w}}(x)$.

We define a mapping φ of $\tilde{E}$ onto the family of all minimal Cauchy filters of $(E, \mathfrak{w})$ by setting $\varphi(x) = \tau_{\mathfrak{w}}(x)$ for every x in E and $\varphi(\tilde{x}) = \tilde{x}$ for every $\tilde{x}$ in the class (b).

With every set W of $\mathfrak{w}$ we associate the set $\tilde{W}$ consisting of all pairs $(\tilde{x}, \tilde{y})$, for which there exists a set $A \in \varphi(\tilde{x}) \cap \varphi(\tilde{y})$ such that $A \times A \subset W$. It is easy to see that

$$\widetilde{V \cap W} = \tilde{V} \cap \tilde{W}$$

for every pair (V, W) of sets of $\mathfrak{w}$. From this it immediately follows that the family

$$\tilde{\mathfrak{W}} = \{ \tilde{W} : W \in \mathfrak{w} \}$$

is a base of a filter $\tilde{\mathfrak{w}}$ in $\tilde{E} \times \tilde{E}$. We will show that $\tilde{\mathfrak{w}}$ is a uniformity for $\tilde{E}$.

Since each $\varphi(\tilde{x})$ is a Cauchy filter, every pair $(\tilde{x}, \tilde{x})$ belongs to every set of the family $\tilde{\mathfrak{W}}$. Thus every set of $\tilde{\mathfrak{W}}$ contains the diagonal of $\tilde{E} \times \tilde{E}$.

From the definition of the sets $\tilde{W}$ it immediately follows that each of these sets is symmetric.

Finally, let $\tilde{W}$ be an arbitrary set of $\tilde{\mathfrak{W}}$, and let V be a set of $\mathfrak{w}$ such that $V \circ V \subset W$. We consider an arbitrary element $(\tilde{x}, \tilde{y})$ of $\tilde{V} \circ \tilde{V}$. If $\tilde{z}$ is an element of $\tilde{E}$ such that $(\tilde{x}, \tilde{z})$ and $(\tilde{z}, \tilde{y})$ belong to $\tilde{V}$, then there exists a set A in $\varphi(\tilde{x}) \cap \varphi(\tilde{z})$ and a set B in $\varphi(\tilde{z}) \cap \varphi(\tilde{y})$ such that $A \times A \subset V$ and $B \times B \subset V$. Since $A \cap B$ belongs to $\varphi(\tilde{z}) \neq \mathfrak{o}$, $A \cap B$ is non-empty. From this it readily follows that $A \times B \subset V \circ V \subset W$ and, consequently,

$$(A \cup B) \times (A \cup B) \subset W ,$$

if W is symmetric. Since, on the other hand, $A \cup B$ belongs to $\varphi(\tilde{x}) \cap \varphi(\tilde{y})$, the pair $(\tilde{x}, \tilde{y})$ thus belongs to $\tilde{W}$, and we have

$$\tilde{V} \circ \tilde{V} \subset \tilde{W} .$$

The filter $\tilde{\mathfrak{w}}$ is therefore a uniformity for $\tilde{E}$.

3. ***E* as a subspace of $\tilde{E}$.** We consider the trace $\tilde{\mathfrak{w}}_E$ of the uniformity $\tilde{\mathfrak{w}}$ in $E \times E$. Let W be a set of $\mathfrak{w}$. If (x, y) belongs to $\tilde{W} \cap (E \times E)$, then there exists a set A in $\varphi(x) \cap \varphi(y) = \tau_{\mathfrak{w}}(x) \cap \tau_{\mathfrak{w}}(y)$ such that $A \times A \subset W$. The points x and y then belong to A and, consequently, (x, y) belongs to $A \times A \subset W$. Thus we have

$$\tilde{W} \cap (E \times E) \subset W \tag{1}$$

for every set W in $\mathfrak{w}$.

Conversely, let (x, y) be an arbitrary element of W. Since $\tau_{\mathfrak{w}}(x)$ and $\tau_{\mathfrak{w}}(y)$ are Cauchy filters, there exists a set A in $\tau_{\mathfrak{w}}(x)$ and a set B in $\tau_{\mathfrak{w}}(y)$ such that $A \times A \subset W$ and $B \times B \subset W$. For an arbitrary element (u, v) of $A \times B$ we then get $(u, x) \in A \times A \subset W$, $(x, y) \in W$ and $(y, v) \in B \times B \subset W$ and, consequently, $(u, v) \in W^3$. We thus see that $A \times B \subset W^3$ and therefore

$$(A \cup B) \times (A \cup B) \subset W^3 ,$$

if W is symmetric. Since $A \cup B \in \tau_{\mathfrak{w}}(x) \cap \tau_{\mathfrak{w}}(y)$, the pair (x, y) therefore belongs to $\widetilde{W^3}$. This indicates that

$$W \subset \widetilde{W^3} \cap (E \times E) \tag{2}$$

for every symmetric set W in $\mathfrak{w}$. From the relations (1) and (2) it finally follows that $\tilde{\mathfrak{w}}_E = \mathfrak{w}$, and $(E, \mathfrak{w})$ is thus a subspace of $(\tilde{E}, \tilde{\mathfrak{w}})$.

Now we show that E is dense in $(\tilde{E}, \tau_{\tilde{\mathfrak{w}}})$. In order to do this we consider an arbitrary symmetric set W of $\mathfrak{w}$ and an arbitrary point $\tilde{x}$ of $\tilde{E}$. Since $\varphi(\tilde{x})$ is a Cauchy filter of $(E, \mathfrak{w})$, there exists a set A of this filter satisfying $A \times A \subset W$. Since $\tilde{x}$ is not the zero filter, the set A is non-empty. Let a be a point of A. From the Cauchy filter $\varphi(a) = \tau_{\mathfrak{w}}(a)$ we choose a set B such that $B \times B \subset W$. For an arbitrary element (u, v) of $A \times B$ we then get $(u, a) \in A \times A \subset W$ and $(a, v) \in B \times B \subset W$ and, consequently, $(u, v) \in W^2$. From this it follows, as before, that

$$(A \cup B) \times (A \cup B) \subset W^2 .$$

Since $A \cup B \in \varphi(\tilde{x}) \cap \varphi(a)$, the pair $(\tilde{x}, a)$ thus belongs to $\widetilde{W^2}$. Therefore we have

$$\widetilde{W^2}[\tilde{x}] \cap E \neq \emptyset ,$$

which shows that E is dense in $(\tilde{E}, \tilde{\mathfrak{w}})$.

4. **The completeness of $\tilde{E}$.** In this section we first prove the following lemma, which will primarily be used in section 6, but which will also to some extent shorten the proof of the completeness of $\tilde{E}$:

Lemma: *If M is a dense subspace of a uniform space E, then $\mathfrak{a} \to \mathfrak{a}_M$ is a one-to-one mapping from the family of all minimal Cauchy filters of E onto the family of all minimal Cauchy filters of M.*

Let first $\mathfrak{a}$ be a minimal Cauchy filter of E. Then $\mathfrak{a}_M$ is a Cauchy filter in M and there exists, consequently, a minimal Cauchy filter $\mathfrak{u}$ of M for which $\mathfrak{u} \subset \mathfrak{a}_M$. Let $\overline{\mathfrak{u}}$ be the filter of E, with $\mathfrak{u}$ as a base. If A were a set of $\mathfrak{a}$, such that $A \cap M$ would not belong to $\mathfrak{u}$, then A could not belong to $\overline{\mathfrak{u}}$, and thus $\overline{\mathfrak{u}} \cap \mathfrak{a}$ were a Cauchy filter of E properly contained in the minimal Cauchy filter $\mathfrak{a}$. This contradiction shows that $\mathfrak{u} = \mathfrak{a}_M$, and thus the trace in M of every minimal Cauchy filter of E is a minimal Cauchy filter of M.

Conversely, let $\mathfrak{u}$ be a minimal Cauchy filter of M. This filter $\mathfrak{u}$ is a base of a Cauchy filter $\overline{\mathfrak{u}}$ in E. Let $\mathfrak{v}$ be a minimal Cauchy filter of E such that $\mathfrak{v} \subset \overline{\mathfrak{u}}$. Then we have $\mathfrak{v}_M \subset \overline{\mathfrak{u}}_M = \mathfrak{u}$ and therefore $\mathfrak{v}_M = \mathfrak{u}$, since $\mathfrak{v}_M$ is a Cauchy filter and $\mathfrak{u}$ is minimal. Thus every minimal Cauchy filter of M is the trace of at least one minimal Cauchy filter of E.

At last we see that if $\mathfrak{a}$ and $\mathfrak{b}$ are two minimal Cauchy filters of E such that $\mathfrak{a}_M = \mathfrak{b}_M$ ($\neq \mathfrak{o}$ since M is dense), then every set of $\mathfrak{a}$ intersects every set of $\mathfrak{b}$ and, consequently, $\mathfrak{a} = \mathfrak{b}$. Our lemma is proved.

In order to prove that $(\tilde{E}, \tilde{\mathfrak{w}})$ is complete, we consider an arbitrary non-zero Cauchy filter $\tilde{\mathfrak{u}}$ of $(\tilde{E}, \tilde{\mathfrak{w}})$. We have to show that there exists at least one point $\tilde{x}$ of $\tilde{E}$ such that $\tau_{\tilde{\mathfrak{w}}}(\tilde{x}) \subset \tilde{\mathfrak{u}}$. Since there exists a minimal Cauchy filter $\tilde{\mathfrak{v}}$ satisfying $\tilde{\mathfrak{v}} \subset \tilde{\mathfrak{u}}$, we may assume that $\tilde{\mathfrak{u}}$ is a minimal Cauchy filter. From our lemma it then follows that the trace $\tilde{\mathfrak{u}}_E$ of $\tilde{\mathfrak{u}}$ is a minimal Cauchy filter of $(E, \mathfrak{w})$. Let $\tilde{x}$ be a point of $\tilde{E}$ such that $\varphi(\tilde{x}) = \tilde{\mathfrak{u}}_E$. We show that

$$\tilde{\mathfrak{u}} = \tau_{\tilde{\mathfrak{w}}}(\tilde{x}) \, . \tag{3}$$

Since $\tilde{\mathfrak{u}}$ and $\tau_{\tilde{\mathfrak{w}}}(\tilde{x})$ are minimal Cauchy filters, the assertion is proved, if we show that every set of $\tilde{\mathfrak{u}}$ intersects every set of $\tau_{\tilde{\mathfrak{w}}}(\tilde{x})$. To see this we consider an arbitrary set $\tilde{A}$ of $\tilde{\mathfrak{u}}$ and an arbitrary set W of $\mathfrak{w}$. Let V and U be symmetric sets of $\mathfrak{w}$ such that $V^2 \subset W$ and $U^3 \subset V$. Since E is dense in $(\tilde{E}, \tilde{\mathfrak{w}})$, we can choose a point x from $\tilde{U}[\tilde{x}] \cap E$. From the relation $(\tilde{x}, x) \in \tilde{U}$ it follows that there exists a set B in $\varphi(\tilde{x}) \cap \varphi(x) = \tilde{\mathfrak{u}}_E \cap \tau_{\mathfrak{w}}(x)$ satisfying $B \times B \subset U$. Since both $\tilde{A} \cap E$ and B belong to the non-zero filter $\tilde{\mathfrak{u}}_E$, their intersection $\tilde{A} \cap B$ is non-empty. If y is a point of $\tilde{A} \cap B$, then by (2)

$$(x, y) \in B \times B \subset U \subset \widetilde{U^3} \subset \tilde{V} \, .$$

From this and from $(\tilde{x}, x) \in \tilde{U} \subset \tilde{V}$ it then follows that

$$(\tilde{x}\, , y) \in \tilde{V} \circ \tilde{V} \subset \tilde{W} \, ,$$

i.e. that $y \in \tilde{W}[\tilde{x}]$. This shows that the intersection $\tilde{W}[\tilde{x}] \cap \tilde{A}$ is non-void and thus the equation (3) holds. From this equation it finally follows that $\tilde{\mathfrak{u}}$ converges to $\tilde{x}$, and thus $(\tilde{E}, \tilde{\mathfrak{w}})$ is complete.

5. **The postulate T_0.** Let $(\tilde{x}, \tilde{y})$ be an element of $\tilde{E} \times \tilde{E}$ belonging to every set $\tilde{W}$. For every set W of $\mathfrak{w}$ it then exists a set M of $\varphi(\tilde{x}) \cap \varphi(\tilde{y})$ satisfying $M \times M \subset W$. This shows that the family

$$\mathfrak{c} = \{A \cup B : A \in \varphi(\tilde{x}) \, , B \in \varphi(\tilde{y})\}$$

is a Cauchy filter in $(E, \mathfrak{w})$. Since it is included in both $\varphi(\tilde{x})$ and $\varphi(\tilde{y})$, we have thus $\varphi(\tilde{x}) = \mathfrak{c} = \varphi(\tilde{y})$. Therefore we see that if $(\tilde{x}, \tilde{y})$ is a pair of elements of $\tilde{E}$ for which $\varphi(\tilde{x}) \neq \varphi(\tilde{y})$, then there always exists a set $\tilde{W}$ of

$\tilde{\mathfrak{w}}$ such that $(\tilde{x}, \tilde{y}) \notin \tilde{W}$. The set $\tilde{W}[\tilde{x}]$ is then a neighborhood of $\tilde{x}$ that does not contain $\tilde{y}$. This indicates that every point of $\tilde{E} - E$ is T_1-separated from every other point of $\tilde{E}$.

6. **The uniqueness of the completion.** Let now $(\tilde{E}, \tilde{\mathfrak{w}})$ and $(\hat{E}, \hat{\mathfrak{w}})$ be two completions of $(E, \mathfrak{w})$ satisfying the conditions of our theorem. Let $\tilde{x}$ be an arbitrary point of $\tilde{E}$. Then the trace $\tau_{\tilde{\mathfrak{w}}}(\tilde{x})_E$ of $\tau_{\tilde{\mathfrak{w}}}(\tilde{x})$ is, according to the lemma of section 4, a minimal Cauchy filter of $(E, \mathfrak{w})$. After this, the same lemma shows that $\tau_{\tilde{\mathfrak{w}}}(\tilde{x})_E$ is the trace in E of a minimal Cauchy filter $\hat{\mathfrak{u}}$ of $(\hat{E}, \hat{\mathfrak{w}})$. Since $(\hat{E}, \hat{\mathfrak{w}})$ is complete, there exists a point $\hat{x}$ of $\hat{E}$ such that $\hat{\mathfrak{u}} = \tau_{\hat{\mathfrak{w}}}(\hat{x})$. Thus we see that with every point $\tilde{x}$ of $\tilde{E}$ we can associate a point $\hat{x}$ of $\hat{E}$ in such a way that

$$\tau_{\tilde{\mathfrak{w}}}(\tilde{x})_E = \tau_{\hat{\mathfrak{w}}}(\hat{x})_E \,. \tag{4}$$

If, in particular, $\tilde{x}$ lies in E, we may obviously suppose that $\hat{x} = \tilde{x}$.

Now we consider the mapping $f : (\tilde{E}, \tilde{\mathfrak{w}}) \to (\hat{E}, \hat{\mathfrak{w}})$ defined by

$$f(\tilde{x}) = \hat{x} \,.$$

If $\tilde{x}$ is in $\tilde{E} - E$, then $\hat{x}$ is in $\hat{E} - E$. In fact, if $\hat{x}$ were in E, then

$$\tau_{\tilde{\mathfrak{w}}}(\tilde{x})_E = \tau_{\hat{\mathfrak{w}}}(\hat{x})_E = \tau_{\tilde{\mathfrak{w}}}(\hat{x})_E$$

and, consequently, $\tau_{\tilde{\mathfrak{w}}}(\tilde{x}) = \tau_{\tilde{\mathfrak{w}}}(\hat{x})$. This is a contradiction, however, since the points of $\tilde{E} - E$ are T_1-separated from other points of $(\tilde{E}, \tilde{\mathfrak{w}})$. If $\hat{x}$ is an arbitrary point of $\hat{E} - E$, then $\tau_{\hat{\mathfrak{w}}}(\hat{x})_E$ is a minimal Cauchy filter of $(E, \mathfrak{w})$ and, as a consequence, there exists a point $\tilde{x}$ of $\tilde{E}$ such that

$$\tau_{\tilde{\mathfrak{w}}}(\tilde{x})_E = \tau_{\hat{\mathfrak{w}}}(\hat{x})_E \,.$$

By (4) we then have $\tau_{\hat{\mathfrak{w}}}(f(\tilde{x})) = \tau_{\hat{\mathfrak{w}}}(\hat{x})$, i.e. $f(\tilde{x}) = \hat{x}$. Since f is the identity in $E, \tilde{x}$ belongs to $\tilde{E} - E$. This shows that f is a mapping from $\tilde{E} - E$ onto $\hat{E} - E$. Finally we observe that if $f(\tilde{x}) = f(\tilde{y})$, then

$$\tau_{\tilde{\mathfrak{w}}}(\tilde{x})_E = \tau_{\hat{\mathfrak{w}}}(f(\tilde{x}))_E = \tau_{\hat{\mathfrak{w}}}(f(\tilde{y}))_E = \tau_{\tilde{\mathfrak{w}}}(\tilde{y})_E$$

and, consequently, $\tau_{\tilde{\mathfrak{w}}}(\tilde{x}) = \tau_{\tilde{\mathfrak{w}}}(\tilde{y})$. From this it follows, if $\tilde{x}$ and $\tilde{y}$ belong to $\tilde{E} - E$, that $\tilde{x} = \tilde{y}$. This indicates that f is a one-to-one mapping from $\tilde{E} - E$ onto $\hat{E} - E$. Since it coincides in E with the identity mapping, it is thus a one-to-one map from $\tilde{E}$ onto $\hat{E}$.

Next we prove that f is a uniform isomorphism. Because of the symmetry it suffices to show that f is uniformly continuous. Let $\hat{W}$ be an arbitrary symmetric set of $\hat{\mathfrak{w}}$. Since the restriction $f|E : (E, \mathfrak{w}) \to (\hat{E}, \hat{\mathfrak{w}})$ is uniformly continuous, there exists a set $\tilde{W}$ in $\tilde{\mathfrak{w}}$ such that

$$\tilde{W}^3 \cap (E \times E) \subset \hat{W} . \tag{5}$$

We consider an arbitrary point $(\tilde{x}, \tilde{y})$ of $\tilde{W}$. From (4) it follows that there exists a symmetric set $\tilde{V}$ of $\tilde{\mathfrak{w}}$ such that $\tilde{V} \subset \tilde{W}$ and

$$\tilde{V}[\tilde{x}] \cap E \subset \hat{W}[f(\tilde{x})] \cap E , \; \tilde{V}[\tilde{y}] \cap E \subset \hat{W}[f(\tilde{y})] \cap E . \tag{6}$$

Since E is dense in $\tilde{E}$, we can choose a point x from $\tilde{V}[\tilde{x}] \cap E$ and a point y from $\tilde{V}[\tilde{y}] \cap E$. From the relations $(x, \tilde{x}), (\tilde{y}, y) \in \tilde{V} \subset \tilde{W}$ and $(\tilde{x}, \tilde{y}) \in \tilde{W}$ it follows that $(x, y) \in \tilde{W}^3$ and therefore, by (5), that $(x, y) \in \hat{W}$. On the other hand, by (6) $x \in \hat{W}[f(\tilde{x})]$ and $y \in \hat{W}[f(\tilde{y})]$, i.e. $(f(\tilde{x}), x) \in \hat{W}$ and $(y, f(\tilde{y})) \in \hat{W}$ and, consequently, $(f(\tilde{x}), f(\tilde{y})) \in \hat{W}^3$. Thus we see that from $(\tilde{x}, \tilde{y}) \in \tilde{W}$ it always follows that $(f(\tilde{x}), f(\tilde{y})) \in \hat{W}^3$. This proves that f is a uniform isomorphism, as we wished to show.

We have seen that the identity mapping $i : (E, \mathfrak{w}) \to (E, \mathfrak{w})$ can be extended to a uniform isomorphism $f : (\tilde{E}, \tilde{\mathfrak{w}}) \to (\hat{E}, \hat{\mathfrak{w}})$. It remains to be shown that this extension is unique. In order to see this, we suppose that $g : (\tilde{E}, \tilde{\mathfrak{w}}) \to (\hat{E}, \hat{\mathfrak{w}})$ is another such extension and that $f(\tilde{x}) \neq g(\tilde{x})$. Then the points $f(\tilde{x})$ and $g(\tilde{x})$ belong to $\hat{E} - E$ and, as a consequence, $\tau_{\hat{\mathfrak{w}}}(f(\tilde{x})) \neq \tau_{\hat{\mathfrak{w}}}(g(\tilde{x}))$. From this it follows, since the neighborhood filters are minimal Cauchy filters, that there exist disjoint sets $\hat{A} \in \tau_{\hat{\mathfrak{w}}}(f(\tilde{x}))$ and $\hat{B} \in \tau_{\hat{\mathfrak{w}}}(g(\tilde{x}))$. On the other hand, we see that, since f and g are continuous, there exists a neighborhood $\tilde{U}$ of $\tilde{x}$ such that $f(\tilde{U}) \subset \hat{A}$ and $g(\tilde{U}) \subset \hat{B}$. Since E is dense in $\tilde{E}$, we can choose a point x from $\tilde{U} \cap E$. Then the point $x = f(x) = g(x)$ belongs to $f(\tilde{U}) \cap g(\tilde{U})$ and, consequently, to $\hat{A} \cap \hat{B}$. This contradiction shows that f is unique.

7. **Proof of the corollary.** Now we suppose that $(E, \mathfrak{w})$ is a uniform T_0-space and show that then $(\tilde{E}, \tilde{\mathfrak{w}})$ is a T_0-space, too. We have seen above that every point of $\tilde{E} - E$ is T_0-separated from every other point of $\tilde{E}$. Thus we only have to consider two points x and y, both belonging to E. If they are not T_0-separated, then $\tau_{\tilde{\mathfrak{w}}}(x) = \tau_{\tilde{\mathfrak{w}}}(y)$ and, consequently, $\tau_{\tilde{\mathfrak{w}}}(x)_E = \tau_{\tilde{\mathfrak{w}}}(y)_E$. On the other hand, since both $\tau_{\tilde{\mathfrak{w}}}(x)_E$ and $\tau_{\mathfrak{w}}(x)$ are minimal Cauchy filters in $(E, \mathfrak{w})$ and since every set of the former filter intersects every set of the latter filter (at x), we see that $\tau_{\tilde{\mathfrak{w}}}(x)_E = \tau_{\mathfrak{w}}(x)$ and, likewise, $\tau_{\tilde{\mathfrak{w}}}(y)_E = \tau_{\mathfrak{w}}(y)$. Thus $\tau_{\mathfrak{w}}(x) = \tau_{\mathfrak{w}}(y)$, from which it finally follows, since $(E, \mathfrak{w})$ is a T_0-space, that $x = y$. This completes the proof of the corollary.

References

[1] Bourbaki, N.: Topologie générale, Chap. II, second edition, Paris 1951.
[2] —»— Topologie générale, Chap. II, third edition, Paris 1961.
[3] Kowalsky, H.-J.: Topologische Räume, Basel und Stuttgart 1961.

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