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FIXED POINTS IN NON-NORMED SPACES

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Fixed points in non-normed spaces

1. Differential calculus in locally convex topological vector spaces which are not normed has been studied by many authors. de Lamadrid [5] gave a definition of the differential based on the idea of spaces of continuous mappings. More recently, Fischer [3] gave a different definition and proved several differential calculus theorems in addition to a fixed point theorem. The author [2] used the definition of de Lamadrid to prove the differential calculus theorems. The purpose of this note is to prove a fixed point theorem which is more general than that of Fischer. It should be noted that both theorems are relatively simple generalizations of the theorem of Hildebrandt and Graves [4][1]).

All notations and definitions, unless otherwise stated are those of Bourbaki [1].

2. Let X be a locally convex topological vector space, X_0 an open neighborhood of 0 in X, $x_0 \in X$, and $f\colon x_0 + X_0 \to X$. Let $k \geqq 0$. We shall say that the function f satisfies the condition $L(k)$ if there exists a convex, closed, bounded set $B \subset X_0$ such that $x, y \in x_0 + X_0$ and $y - x \in \lambda B$ implies

$$f(y) - f(x) \in \lambda k B .$$

If X is normable and f is Lipschitz in the ordinary sense with constant k, then f satisfies $L(k)$ with the unit sphere taking the place of B. On the other hand, suppose X is normable and f satisfies $L(k)$. Let $x, y \in x_0 + X_0$. If B happens to be smaller than a sphere, say a line segment, then $y - x$ may not be in any non-zero multiple of B so that we may not conclude that f is Lipschitz in the ordinary sense.

Let us now consider the definition given in [3]. Here we say, essentially, that f satisfies $L'(k)$ if for every seminorm, p, there is a number, r_p, such that if $x, y \in x_0 + X_0$ and $p(y - x_0) \leqq r_p$, $p(x - x_0) \leqq r_p$, then

$$p(f(y) - f(x)) \leqq k\, p(y - x) .$$

[1]) *Added in proof*: A sharper form of this theorem has been proved for Banach spaces by Rolf Nevanlinna [6].

If $L'(k)$ is satisfied, we can take B to be the set of $x \in X_0$ such that for all p, $p(x) \leqq r_p$. This is a closed, convex, bounded set. If $x, y \in x_0 + X_0$ and $y - x \in \lambda B$, then $p(y - x) \leqq \lambda r_p$, so $p(f(y) - f(x)) \leqq k \lambda r_p$. This means that $f(y) - f(x) \in \lambda k B$.

Thus we have shown that $L'(k) \Rightarrow L(k)$. The converse is false. This is most easily seen by noting that in the normable case, L' is equivalent to the usual Lipschitz condition, while L, as we have seen, is not. It should also be noted that in locally convex topological vector spaces, L' implies continuity but L does not.

3. We will now show that a fixed point theorem can be proven using L instead of L'.

Theorem. *Let X be complete. Let $f : x_0 + X_0 \to X$ such that f satisfies $L(k)$, $k < 1$, with corresponding bounded set B. Let*

$$f(x_0) - x_0 \in (1 - k) B .$$

Then there exists a unique $x \in x_0 + B$ such that $f(x) = x$.

Proof. We define the sequence (x_n) by

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots .$$

We must show that $x_n \in x_0 + X_0$ so that $f(x_n)$ is defined. Since $B \subset X_0$, we have

$$x_1 = f(x_0) \in X_0 + (1 - k) B \subset x_0 + X_0 .$$

Let us assume that $x_1, \ldots, x_n$ are in $x_0 + X_0$, and

$$x_n - x_0 \in (1 - k^n) B , \quad x_n - x_{n-1} \in k^{n-1} (1 - k) B .$$

This assumption is valid for $n = 1$. Let us prove it for $n - 1$.

We have,

$$x_{n+1} - x_n = f(x_n) - f(x_{n-1}) \in k^{n-1} (1 - k) k B = k^n (1 - k) B ,$$

and

$$x_{n+1} - x_0 = x_{n+1} - x_n + x_n - x_0 = k^n (1 - k) B + (1 - k^n) B$$
$$\subset [k^n (1 - k) + 1 - k^n] B = (1 - k^{n-1}) B .$$

Hence our assumptions are valid for all n. In particular, $x_n \in x_0 - X_0$ so that the sequence is well defined.

Further, for any integer $m \geqq 0$,

$$x_{n+m+1} - x_n = x_{n+m+1} - x_{n+m} + x_{n+m} - x_{n+m-1} + \ldots + x_{n+1} - x_n$$
$$\in (k^{n+m} + k^{n+m-1} + \ldots + k^n)(1 - k) B \subset k^n (1 - k^{m+1}) B \subset k^n B .$$

Hence if V is a convex neighborhood of 0, there exists $\mu > 0$ such that $\mu B \subset V$. For n sufficiently large, since $k < 1$ we have $k^n \leqq \mu$ so $x_{n+m+1} - x_n \in V$. Thus the sequence (x_n) is cauchy and has a limit, x. Since each x_n is in $x_0 + B$, so is x. In fact, for each n,

$$x - x_n \in k^n B .$$

This is so because for all m, $x_{n+m+1} - x_n \in k^n B$ and $k^n B$ is closed. Thus,

$$f(x) - x_n = f(x) - f(x_{n-1}) \in k^n B .$$

Therefore, $f(x)$ is the limit of (x_n), i.e., $f(x) = x$.

Finally let us suppose that $y \in x_0 + B$ and $f(y) = y$. Then

$$y - x = f(y) - f(x) \in k B .$$

This may be repeated to obtain $y - x \in k^n B$ for all n, or $y = x$.

4. Corresponding to our fixed point theorem, we have the following implicit function theorem.

Theorem. *Let T be a topological space and*

$$x_0 : T \rightarrow X_0 , \qquad f : (x_0(T) + X_0) \times T \rightarrow X$$

such that, for each $t \in T$, the map $f(\ , t) : x_0(t) + X_0 \rightarrow X$ satisfies the conditions of the previous theorem with k , B independent of t.

Then there exists a unique function $x : T \rightarrow x_0(t) + B$ such that

$$f(x(t) , t) = x(t) .$$

If x_0 and f are continuous, then x is continuous.

Proof. The existence and uniqueness of x follows by applying the previous theorem for each t.

Let us suppose that x_0 and f are continuous. If $x_n(t)$ is continuous, then $x_{n+1}(t) = f(x_n(t) , t)$ is the composition of two continuous functions so it is continuous. Hence $x_n(t)$ is continuous for all n. From the proof of the previous theorem, we have, given a neighborhood of 0, V, in X, that if n is sufficiently large (depending only on k , B) then $x_n(t) - x(t) \in V$. Thus the sequence $(x_n(t))$ converges uniformly to $x(t)$ so $x(t)$ is continuous.

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