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CHARACTERIZATION OF THE SET OF VALUES APPROACHED BY A MEROMORPHIC FUNCTION ON SEQUENCES OF JORDAN CURVES

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§ 1. Introduction

1. Let Ω denote the extended complex plane (or the Riemann sphere), and let C be the unit circle and D be the open unit disk.

A sequence A of distinct Jordan curves $J_1, J_2, \ldots, J_n, \ldots$ in D will be called an *annulation* if

1) J_n lies in the interior of J_{n+1} $(n = 1, 2, 3, \ldots)$

and

2) given any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ such that $n > n_0$ implies that J_n lies in the region $1 - \varepsilon < |z| < 1$.

If, furthermore,

3) every J_n is a circle with the origin as center, then the sequence A will be called a *strict annulation.*

Suppose that the function $f(z)$ is meromorphic in D. Then we define the sets $\Theta(f)$ and $\Theta^\circ(f)$ as follows: The point $c \in \Omega$ belongs to $\Theta(f)$, resp. $\Theta^\circ(f)$, provided that, for some annulation, resp. strict annulation, $A = \{J_n\}$, given any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ such that $n > n_0$ implies that, for every $z \in J_n$, we have $|f(z) - c| < \varepsilon$ or $|1/f(z)| < \varepsilon$ according as c is finite or the point at infinity. Under these conditions we say that the function f *tends to the value* c *on the annulation* A.

2. An immediate consequence of the foregoing definition is that $\Theta^\circ(f) \subseteq \Theta(f)$. In § 6 we shall give an example of a function f for which $\Theta^\circ(f) \neq \Theta(f)$; in fact, we shall show that Θ° and Θ can be prescribed arbitrarily except for the natural restrictions that they be closed sets and that $\Theta^\circ \subseteq \Theta$.

The main question that arises at once, however, is: What kind of point sets are Θ° and Θ ? It is again obvious that Θ° and Θ are closed sets. We are going to establish the following characterization:

Theorem 1. *Let F be an arbitrary closed subset of Ω. Then there exists a function $f(z)$, meromorphic in D, such that $\Theta^\circ(f) = \Theta(f) = F$.*

Moreover, if F is not empty, and A is a given annulation, then there exists an f such that $\Theta(f) = F$ and every value $c \in F$ is approached by f on a subannulation of A.

§ 2. Relations between $R(f)$ and $\Theta(f)$

3. It is well known that although a function $f(z)$ that is holomorphic in D cannot tend uniformly to infinity as z approaches C, nevertheless there exist holomorphic functions $f(z)$ in D for which $\{\infty\} = \Theta^\circ(f) = \Theta(f)$ (see, e.g., [1]). For any holomorphic function f, we have $\infty \notin R(f)$, where, as is customary, $R(f)$ stands for the range of values of f in D [5, p. 48].

Now suppose that $f(z)$ is meromorphic in D and that $R(f)$ omits at least three values of Ω. Then (see [4, p. 97, Theorem 6] or [5, pp. 52—53]) $\Theta(f)$ is empty if $f(z)$ is not identically constant.

If $R(f)$ omits just two values of Ω, then (see [4, p. 112, Theorem 9, (ii)] or [5, pp. 50—51, Theorem 1, (ii)]) $f(z)$ possesses at least two asymptotic values, and hence $\Theta(f)$ is empty.

If the complement of $R(f)$ consists of the sole value $c \in \Omega$, then either, as in the preceding case, $\Theta(f)$ is empty, or c is an asymptotic value of f, so that if $\Theta(f)$ is not empty, then $\Theta(f) = \{c\}$. For example, let $h(z) = \cos \dfrac{1}{1-z}$, and set $f(z)$ equal to $h(z)$, $1/h(z)$, $(1/h(z)) + b$ according as c is equal to ∞, 0, b, where $b \in \Omega - \{0, \infty\}$. Then $f(z)$ is meromorphic in D, $R(f)$ omits the sole value c, and $\Theta(f)$ is empty. On the other hand, let $h(z)$ be the function described as an infinite product in [2, p. 79], and then define $f(z)$ as in the preceding sentence. The function $f(z)$ is meromorphic in D, $R(f)$ omits the sole value c (cf. [5, p. 75, Remark, (v)]), and $\Theta^\circ(f) = \Theta(f) = \{c\}$.

The reason for making a distinction between $\Theta^\circ(f)$ and $\Theta(f)$ is that, although it may be known for a specific f that $c \in \Theta(f)$, it may be by no means an easy matter to determine whether $c \in \Theta^\circ(f)$ — and a strict annulation is, after all, the neatest.

§ 3. The skeleton S and the continuous function $g(z)$ on S

4. To prove the theorem formulated in § 1, we shall assume that the given closed subset F of Ω is not empty, because if F is empty, the assertion of the theorem is obviously true.

Let the annulation $A = \{J_n\}$ be given. We shall show that there exists a meromorphic function $f(z)$ in D such that $\Theta(f) = F$ and every value $c \in F$ is approached by f on a subannulation of A; this will prove the second part of the theorem. If we then take A to be a strict annulation, we obtain the first part of the theorem.

5. To accomplish this, we first define a point set S in D, where $S \supset \bigcup_{n=1}^{\infty} J_n$. Let a_{00} and a_{10} be distinct points in the interior of J_1, and take B_0, B_1 to be mutually exclusive Jordan arcs with initial points a_{00} resp. a_{10}, and terminal points 1 resp. -1; we regard the initial points, but not the terminal points, as belonging to these arcs. We further require B_0 and B_1 to lie in D, and each of them to intersect every J_n in precisely one point, a_{0n} resp. a_{1n}, so that

$$B_0 \cap J_n = \{a_{0n}\}, \quad B_1 \cap J_n = \{a_{1n}\} \qquad (n = 1, 2, 3, \ldots).$$

We put

$$S = B_0 \cup B_1 \cup (\bigcup_{n=1}^{\infty} J_n)$$

and call S the *skeleton.*

6. Our next step is to define a certain continuous function $g(z)$ on S. Whereas the definition of the skeleton depended on the given annulation A, the definition of $g(z)$ depends on the given (nonempty) closed set F.

Suppose that F contains ∞ as an isolated point. Then there exists a finite point $\zeta \notin F$. The image of F under the transformation $z' = \dfrac{1}{z - \zeta}$ is a set F' that does not contain ∞. If we can prove that there exists a meromorphic function $h(z)$ in D such that $\Theta(h) = F'$ and every value in F' is approached by h on a subannulation of A, then the function $f(z) = \dfrac{1}{h(z)} + \zeta$ is meromorphic in D, $\Theta(f) = F$, and every value in F is approached by f on a subannulation of A. We may therefore assume, in what follows, that F does not contain ∞ as an isolated point.

7. Let

$$c_1, \ c_2, \ \ldots, \ c_n, \ \ldots$$

be an infinite sequence of finite complex numbers in F with the property that $\{c_n\}$ is everywhere dense in F in the sense that every isolated point of F occurs infinitely often as a term of the sequence.

We put

$$g(z) \equiv c_n \quad (z \in J_n; \ n = 1, 2, 3, \ldots). \tag{1}$$

For every natural number n, let A_{jn} $(j = 0, 1)$ be the subarc of B_j extending from $a_{j,n-1}$ to a_{jn}, including the two end points.

We set

$$g(z) \equiv c_1 \qquad (z \in A_{01} \cup A_{11}).$$

For every $n > 1$, consider c_n and c_{n-1}. If $c_n = c_{n-1}$, we put

$$g(z) \equiv c_n \qquad (z \in A_{0n} \cup A_{1n}) . \tag{2}$$

But if $c_n \neq c_{n-1}$, we take the circle whose diameter is the rectilinear segment joining c_n and c_{n-1}, and denote one of the semicircular arcs of this circle extending from c_{n-1} to c_n, including the end points, by A^*_{0n}, and the other one, also extending from c_{n-1} to c_n, by A^*_{1n}. We now define $g(z)$ on $A_{jn} (j = 0, 1)$ to be a homeomorphism of A_{jn} onto A^*_{jn} such that $g(a_{j,n-1}) = c_{n-1}$ and $g(a_{jn}) = c_n$.

This completes the definition of $g(z)$ on S; $g(z)$ is obviously a continuous function on S.

§ 4. The relation between $g(z)$ and $f(z)$

8. For every natural number n, choose a point $b_{jn} (j = 0, 1)$ on B_j between a_{jn} and $a_{j,n+1}$, and let B_{jn} be the subarc of B_j that extends from $b_{j,n-1}$ to b_{jn}, including the end points, where we set $b_{j0} = a_{j0}$. Then put

$$S_n = B_{0n} \cup B_{1n} \cup J_n \qquad (n = 1, 2, 3, \ldots) . \tag{3}$$

We propose to demonstrate in § 5 the existence of a meromorphic function $f(z)$ in D, with no poles on the skeleton, such that

$$\lim_{n \to \infty} \max_{z \in S_n} |f(z) - g(z)| = 0 . \tag{4}$$

9. Suppose for a moment that this has already been accomplished. We wish to show how the conclusion of our theorem follows.

We first prove that $F \subseteq \Theta(f)$. Let $c \in F$. Then there exists an infinite subsequence $\{c_{n_k}\}$ of c_n such that $\lim_{k \to \infty} c_{n_k} = c$. Let $\varepsilon > 0$ be given. If c is finite, then there exists a $k_1 = k_1(\varepsilon)$ such that, for every $k > k_1$,

$$|c_{n_k} - c| < \frac{\varepsilon}{2} . \tag{5}$$

According to (1), for every $z \in J_{n_k}$ we have

$$g(z) \equiv c_{n_k} . \tag{6}$$

It follows from (5) and (6) that, for every $k > k_1$,

$$|g(z) - c| < \frac{\varepsilon}{2} \qquad (z \in J_{n_k}) . \tag{7}$$

By (3) and (4) there exists a $k_2 = k_2(\varepsilon)$ such that, for every $k > k_2$,

(8) $$|f(z) - g(z)| < \frac{\varepsilon}{2} \qquad (z \in J_{n_k}).$$

Now for every $k > \max(k_1, k_2)$, (8) and (7) yield

(9) $$|f(z) - c| < \varepsilon \qquad (z \in J_{n_k}).$$

This implies that $f(z)$ tends to c along the subannulation $\{J_{n_k}\}$ of A. In case $c = \infty$, replace inequalities (5), (7), (8), (9) by $|c_{n_k}| > \frac{2}{\varepsilon}$, $|g(z)| > \frac{2}{\varepsilon}$, $|f(z) - g(z)| < \frac{1}{\varepsilon}$, $\left|\frac{1}{f(z)}\right| < \varepsilon$.

10. It remains to be proved that $\Theta(f) \subseteq F$. This is accomplished by showing that if $c \notin F$ then $c \notin \Theta(f)$.

If c is finite, it is a positive distance d from the nonempty closed set F. Suppose that $c \in \Theta(f)$. Then there exists an annulation $A^* = \{J_n^*\}$ on which $f(z)$ tends to c. Take $\varepsilon = \frac{d\sqrt{2}}{2\sqrt{2}+4}$. According to (4), (3), and (1), there exists an n_0 such that, for every $n > n_0$,

(10) $$|f(z) - g(z)| < \varepsilon \qquad (z \in S_n),$$

and hence, in particular,

(11) $$|f(z) - c_n| < \varepsilon \qquad (z \in J_n).$$

Let δ be the positive distance between J_{n_0+1} and C. By 2) in the definition of annulation, there exists an m_1 such that J_m^* lies in the region $1 - \delta < |z| < 1$ for every $m > m_1$. There also exists an m_2 such that, for every $m > m_2$,

(12) $$|f(z) - c| < \varepsilon \qquad (z \in J_m^*).$$

If $m_0 = \max(m_1, m_2)$, then $m > m_0$ implies that J_m^* intersects no J_n with $n > n_0$, for otherwise it would follow from (11) and (12) that $|c - c_n| < d\,\frac{2\sqrt{2}}{2\sqrt{2}+4} < d$, contradicting the definition of d. Therefore there exists an $n_1 > n_0$ such that J_{n_1} is in the interior of J_m^* and J_m^* is in the interior of J_{n_1+1}. Consequently J_m^* intersects A_{0,n_1+1} in at least one (interior) point t_0 and A_{1,n_1+1} in at least one point t_1. Let

$$g(t_j) = t_j^* \qquad (j = 0, 1).$$

11. Suppose first that $c_{n_1} \neq c_{n_1+1}$. It then follows from the definition of $g(z)$ that

(13) $$t_j^* \in A_{j,\, n_1+1}^* \qquad (j = 0\,, 1)\,.$$

In view of (3) and (10) for $n = n_1$, $n_1 + 1$, we have

(14) $$|f(t_j) - t_j^*| < \varepsilon \qquad (j = 0\,, 1)\,.$$

From (12), for $z = t_j\,(j = 0\,,\ 1)$, and (14), we obtain

(15) $$|t_j^* - c| < 2\,\varepsilon \qquad (j = 0\,, 1)\,,$$

and hence

(16) $$|\,t_0^* - t_1^*| < 4\,\varepsilon = d\,\frac{4\sqrt{2}}{2\sqrt{2} + 4}\,.$$

Now

(17) $$|t_j^* - c_{n_1}| > d - 2\,\varepsilon \qquad (j = 0\,, 1)\,,$$

because otherwise, in view of (15), we should have $|c - c_{n_1}| < d$, contrary to the definition of d. Likewise we must have

(18) $$|t_j^* - c_{n_1+1}| > d - 2\,\varepsilon \qquad (j = 0\,, 1)\,.$$

Because of (17), (18), and (13), $|c_{n_1} - c_{n_1+1}| > (d - 2\,\varepsilon)\sqrt{2}$ and

(19) $$|t_0^* - t_1^*| > (d - 2\,\varepsilon)\sqrt{2}\,.$$

According to (16) and (19),

$$(d - 2\,\varepsilon)\sqrt{2} < d\,\frac{4\sqrt{2}}{2\sqrt{2} + 4}\,,$$

which implies that $\varepsilon > \dfrac{d\sqrt{2}}{2\sqrt{2} + 4}$, contrary to our choice of ε. Hence, if $c_{n_1} \neq c_{n_1+1}$, then $c \notin \Theta(f)$.

12. Suppose next that $c_{n_1} = c_{n_1+1}$. It then follows from (2) that

$$t_j^* = c_{n_1+1} \qquad (j = 0\,, 1)\,,$$

and from (3) and (10) that

(20) $$|f(t_j) - c_{n_1+1}| < \varepsilon \qquad (j = 0\,, 1)\,.$$

On the other hand, (12) implies that

(21) $$|f(t_j) - c| < \varepsilon \qquad (j = 0\,, 1)\,.$$

From (20) and (21) we infer that $|c - c_{n_1+1}| < 2\varepsilon < d$, contradicting the definition of d, and hence $c \notin \Theta(f)$.

This disposes of the case that c is finite.

13. Now suppose that $c = \infty$. Then F is a bounded set. It follows from this and the way in which $g(z)$ was defined, that $g(z)$ is a bounded function on S, and (4) now implies that $c \notin \Theta(f)$.

§ 5. The construction of $f(z)$

14. To complete the proof of the theorem all, that remains is the demonstration of the existence of a function $f(z)$, meromorphic in D, satisfying (4). This is accomplished by means of approximation and interpolation by rational functions. We use a modification of a method devised in [3].

Let K_0 be a Jordan curve in the interior of J_1 having no point in common with $B_{01} \cup B_{11}$. For every natural number n, let K_n be a Jordan curve containing J_n in its interior and contained in the interior of J_{n+1}, such that

$$K_n \cap A_{0,\, n+1} = \{b_{0n}\}, \qquad K_n \cap A_{1,\, n+1} = \{b_{1n}\}.$$

Denote by D_n $(n = 0, 1, 2, \ldots)$ the set of all points lying either on K_n or in the interior of K_n, and put

$$E_n = D_n \cup S_{n+1}. \tag{22}$$

15. We now define, by induction on n, a function $\varphi_n(z)$ on E_n and a rational function $r_n(z)$.

Let

$$\varphi_0(z) = \begin{cases} 0, & z \in D_0; \\ g(z), & z \in S_1. \end{cases}$$

The function $\varphi_0(z)$ is evidently continuous on E_0 and holomorphic at all interior points of E_0. Because of the nature of E_0, there exists (cf. [6, pp. 260—261, 313]) a rational function $r_0(z)$ with no poles on $D_0 \cup S$, such that

$$|r_0(z) - \varphi_0(z)| < 1 \qquad (z \in E_0),$$

and

$$r_0(b_{j1}) = g(b_{j1}) \qquad (j = 0, 1).$$

Now suppose that $n > 0$ and that rational functions $r_0(z)$, $r_1(z)$, ..., $r_{n-1}(z)$ have been determined so that $r_0(z) + r_1(z) + \ldots + r_{n-1}(z)$ has no poles on S and

(23) $$r_{n-1}(b_{jn}) = g(b_{jn}) - [r_0(b_{jn}) + r_1(b_{jn}) + \ldots + r_{n-2}(b_{jn})] \quad (j = 0, 1),$$

where the expression in brackets is missing in case $n = 1$.

Let

(24), (25) $$\varphi_n(z) = \begin{cases} 0, & z \in D_n; \\ g(z) - [r_0(z) + r_1(z) + \ldots + r_{n-1}(z)], & z \in S_{n+1}. \end{cases}$$

It follows from (23), (24), and (25) that $\varphi_n(z)$ is continuous on E_n and holomorphic at all interior points of E_n. As before, there exists a rational function $r_n(z)$ with no poles on $D_n \cup S$, such that

(26) $$|r_n(z) - \varphi_n(z)| < \frac{1}{2^n} \quad (z \in E_n),$$

and

$$r_n(b_{j,n+1}) = g(b_{j,n+1}) - [r_0(b_{j,n+1}) + r_1(b_{j,n+1}) + \ldots + r_{n-1}(b_{j,n+1})] \quad (j = 0, 1).$$

This completes the induction.

16. Set

$$f(z) = \sum_{n=0}^{\infty} r_n(z).$$

Suppose that $z \in D_n$. It follows from (24) and (26) that

(27) $$|r_{n+k}(z)| < \frac{1}{2^{n+k}} \quad (k = 0, 1, 2, \ldots).$$

Since $r_{n+k}(z)$ $(k = 0, 1, 2, \ldots)$ is holomorphic on D_n, (27) implies that $\sum_{k=0}^{\infty} r_{n+k}(z)$ is holomorphic in the interior of D_n, and hence $f(z)$ is meromorphic in the interior of D_n. Every point of D, however, is in the interior of D_n for some value of n; this is so because of 2) in the definition of annulation, and because of the way in which K_n was defined. Hence, $f(z)$ is meromorphic in D.

17. Let $z \in S_n$, where $n > 1$. Then by (26) and (22), we have

(28) $$|r_{n-1}(z) - \varphi_{n-1}(z)| < \frac{1}{2^{n-1}}.$$

According to (25),

(29) $$\varphi_{n-1}(z) = g(z) - [r_0(z) + r_1(z) + \ldots + r_{n-2}(z)].$$

Combining (28) and (29), we obtain

$$(30) \qquad |r_0(z) + r_1(z) + \ldots + r_{n-1}(z) - g(z)| < \frac{1}{2^{n-1}} .$$

Now (30), (27), and the definition of $f(z)$ yield

$$\begin{aligned} &|f(z) - g(z)| \\ &\quad \leqq |r_0(z) + r_1(z) + \ldots + r_{n-1}(z) - g(z)| + |r_n(z)| + |r_{n+1}(z)| + \ldots \\ &\quad < \sum_{m=n-1}^{\infty} \frac{1}{2^m} , \end{aligned}$$

which implies (4).

§ 6. A function $f(z)$ with prescribed $\Theta^\circ(f)$ and $\Theta(f)$

18. As we remarked in § 1, for every function $f(z)$ that is meromorphic in D, $\Theta^\circ(f)$ and $\Theta(f)$ are closed subsets of Ω such that $\Theta^\circ(f) \subseteqq \Theta(f)$. We are going to show that Θ° and Θ are subject to no further restrictions. Specifically, we shall prove the following:

Theorem 2. *Let F and F_0 be closed subsets of Ω such that $F_0 \subseteqq F$. Then there exists a function $f(z)$, meromorphic in D, such that $\Theta^\circ(f) = F_0$ and $\Theta(f) = F$.*

19. If $F_0 = F$, then Theorem 2 reduces to the first part of Theorem 1. We shall therefore assume that $F_0 \subset F$.

As in § 2.6, we may assume that F does not contain ∞ as an isolated point.

If F contains ∞, but not as an isolated point, whereas F_0 contains ∞ as an isolated point, we can reduce this case to the case that neither F nor F_0 contains ∞ as an isolated point by taking a finite point $\zeta \notin F_0$ such that ζ is not an isolated point of F and proceeding as in § 2.6.

20. We define an annulation A and, in case F_0 is not empty, a subannulation A_0, as follows.

Let

$$0 < \alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \alpha_n < \ldots < 1 ,$$

and

$$\lim_{n \to \infty} \alpha_n = 1 .$$

Put

$$\beta_n = \frac{\alpha_{n-1} + \alpha_n}{2} \qquad (n = 1, 2, 3, \ldots) .$$

If F_0 is not empty, we take J_{2n-1} $(n = 1, 2, 3, \ldots)$ to be the circle with center at the origin and radius β_{2n-1}, and J_{2n} $(n = 1, 2, 3, \ldots)$ to be the ellipse whose major axis extends from $\alpha_{2n} e^{\frac{-\pi i}{4}}$ to $\alpha_{2n} e^{\frac{3\pi i}{4}}$ and whose minor axis extends from $\alpha_{2n-1} e^{\frac{\pi i}{4}}$ to $\alpha_{2n-1} e^{\frac{5\pi i}{4}}$. We then set

$$A = \{J_n\}, \quad A_0 = \{J_{2n-1}\} \quad (n = 1, 2, 3, \ldots).$$

In case F_0 is empty, we set

$$A = \{J_{2n}\} \quad (n = 1, 2, 3, \ldots).$$

21. Our next step is the definition of a skeleton S.

Let $\beta_0 = \frac{\alpha_0}{2}$, and denote by B_0 resp. B_1 the rectilinear segment extending from β_0 resp. $-\beta_0$ to 1 resp. -1; the initial point is included, the terminal point not.

We take $T_{0,\,2n-1}$ resp. $T_{1,\,2n-1}$ $(n = 1, 2, 3, \ldots)$ to be the closed rectilinear segment extending from $\beta_{2n-1} e^{\frac{\pi i}{4}}$ resp. $\beta_{2n-1} e^{\frac{5\pi i}{4}}$ to $\beta_{2n-2} e^{\frac{\pi i}{4}}$ resp. $\beta_{2n-2} e^{\frac{5\pi i}{4}}$; we similarly define $T_{0,\,2n}$ resp. $T_{1,\,2n}$ $(n = 1, 2, 3, \ldots)$ to extend from $\beta_{2n} e^{\frac{3\pi i}{4}}$ resp. $\beta_{2n} e^{\frac{-\pi i}{4}}$ to $\beta_{2n-1} e^{\frac{3\pi i}{4}}$ resp. $\beta_{2n-1} e^{\frac{-\pi i}{4}}$.

If F_0 is not empty, put

$$S = B_0 \cup B_1 \cup \left(\bigcup_{n=1}^{\infty} J_n\right) \cup \left(\bigcup_{n=1}^{\infty} T_{0n}\right) \cup \left(\bigcup_{n=1}^{\infty} T_{1n}\right).$$

In case F_0 is empty, set

$$S = B_0 \cup B_1 \cup \left(\bigcup_{n=1}^{\infty} J_{2n}\right) \cup \left(\bigcup_{n=1}^{\infty} T_{0n}\right) \cup \left(\bigcup_{n=1}^{\infty} T_{1n}\right).$$

22. We now show how to define a continuous function $g(z)$ on S.

In case F_0 is not empty, we take

$$c_1, \; c_3, \; c_5, \; \ldots, \; c_{2n-1}, \; \ldots$$

to be an infinite sequence of finite complex numbers in F_0 such that $\{c_{2n-1}\}$ is everywhere dense in F_0, and we let

$$c_2, \; c_4, \; c_6, \; \ldots, \; c_{2n}, \; \ldots$$

be a similar sequence in $F - F_0$ that is everywhere dense in $F - F_0$.

If F_0 is empty, we consider merely a sequence

$$c_2, \quad c_4, \quad c_6, \quad \ldots, \quad c_{2n}, \quad \ldots$$

in F that is everywhere dense in F.

If F_0 is not empty, set

$$g(z) = \begin{cases} c_{2n-1}, & z \in J_{2n-1} \cup T_{0,2n-1} \cup T_{1,2n-1} \cup T_{0,2n} \cup T_{1,2n}; \\ c_{2n}, & z \in J_{2n}; \end{cases}$$

$$(n = 1, 2, 3, \ldots).$$

But if F_0 is empty, put

$$g(z) = \begin{cases} 0, & z \in \bigcup_{n=1}^{\infty} T_{0n}; \\ 1, & z \in \bigcup_{n=1}^{\infty} T_{1n}; \\ c_{2n}, & z \in J_{2n} \qquad (n = 1, 2, 3, \ldots). \end{cases}$$

On the segments of B_0 and B_1 where $g(z)$ has not yet been defined, we define $g(z)$ by means of homeomorphisms like those described in § 2.7, and thus obtain a continuous function $g(z)$ on S.

23. Now it is not difficult to see that a function $f(z)$, meromorphic in D, can be constructed after the pattern of § 5 so that

$$\lim_{n \to \infty} \max_{\substack{a_n < |z| < a_{n+1} \\ z \in S}} |f(z) - g(z)| = 0. \tag{31}$$

If F_0 is empty, then clearly $\Theta(f) = F$. Furthermore, $\Theta^\circ(f)$ is empty, because every circle Q in D with the origin as center and a sufficiently large radius, intersects both T_{0n} and T_{1n} for a suitable n. By the definition of $g(z)$ on T_{jn}, it follows from (31) that every Q sufficiently near C contains a point at which $f(z)$ is very close to zero as well as a point at which $f(z)$ is very close to one, and hence $f(z)$ cannot tend to a limit on any sequence of such circles Q tending to C.

If F_0 is not empty, it is again clear that $\Theta(f) = F$, and that every value $c \in F_0$ is approached by f on the annulation A_0. The way in which $g(z)$ was defined on T_{jn} in this case evidently guarantees that no value in the complement of F_0 can be approached by f on a sequence of circles Q with the origin as center and tending to C, and hence $\Theta^\circ(f) = F_0$.

24. *Remark.* We have also succeeded in characterizing the well-known sets $\Phi(f, e^{i\Theta})$ and $\Phi(f)$ for a function $f(z)$ meromorphic in D. Our results in this direction will appear elsewhere (Math. Z. 80 (1962), 230—238).

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