ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

325

A DISTORTION THEOREM FOR FUNCTIONS UNIVALENT IN AN ANNULUS

BY

F. W. GEHRING and GUNNAR af HÄLLSTRÖM

HELSINKI 1963
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1963.325

Communicated September 14, 1962 by F. Nevanlinna and Olli Lehto

KESKUSKIRJAPAINO
HELSINKI 1963

A distortion theorem for functions univalent in an annulus[1])

1. Let A denote an annulus $r < |z| < 1$, let R denote a ring bounded by $|w| = 1$ and a continuum Γ in $|w| < 1$, and suppose that R does not contain the point $w = 0$. D. Gaier recently proved in [4] the following theorem.

If $w = f(z)$ *maps* A *conformally onto* R *so that* $f(1) = 1$, *then*

$$|f(z) - z| \leqq Cr \tag{1}$$

for $z \in A$, *where* C *is an absolute constant. If* C_0 *denotes the smallest such constant* C, *then* $4 \leqq C_0 \leqq 12.6$.

The aim of this paper is to give the exact value of C_0. We do this by establishing

Theorem 1. *If* $w = f(z)$ *maps* A *conformally onto* R *so that* $f(1) = 1$, *then*

$$|f(z) - z| < 8r \tag{2}$$

for $z \in A$. *The constant* 8 *cannot be replaced by any smaller number.*[2])

Since $|f(z) - z|$ satisfies the maximum principle in A, (2) will follow if we can show that

$$\limsup_{z \to \zeta} |f(z) - z| < 8r$$

for all ζ on the boundary of A. That is we need only determine how great the distortion is on each boundary component of A. In this direction we prove

[1]) This research was supported by a grant to one of us (F. W. G.) from the U. S. National Science Foundation, Contract NSF-G-18913.

[2]) After we had completed the research for this paper, we learned from Professor Warschawski of a Stanford Technical Report [3] in which Duren and Schiffer consider the same problem. They use variational methods to identify the function $f(z)$ which yields maximum distortion in A and then show that $C_0 \geq 8$.

The following two articles have appeared while this paper was in press: Duren, P. L. and Schiffer, M., *A variational method for functions schlicht in an annulus,* Arch. Rat. Mech. Anal. 9, (1962), pp. 260–272, and Gaier, D. and Huckemann, F., *Extremal problems for functions schlicht in an annulus,* Arch. Rat. Mech. Anal. 9, (1962), pp. 415–421.

Theorem 2. *If* $w = f(z)$ *maps* A *conformally onto* R *so that* $f(1) = 1$, *then*

$$\limsup_{|z| \to r} |f(z) - z| < 5r \tag{3}$$

and

$$\sup_{|z| = 1} |f(z) - z| < 8r . \tag{4}$$

The constants 5 *and* 8 *cannot be replaced by smaller numbers.*

The examples which show that 5 and 8 are best possible have the desired asymptotic behaviour as $r \to 0$. That is, for each r, let

$$C_1(r) = \sup_{f \in F} \left(\limsup_{|z| \to r} \frac{|f(z) - z|}{r} \right),$$

$$C_2(r) = \sup_{f \in F} \left(\sup_{|z| = 1} \frac{|f(z) - z|}{r} \right),$$

where F is the class of functions $f(z)$ which map A onto an R with $f(1) = 1$. Then $C_1(r) < 5$, $C_2(r) < 8$ and

$$\lim_{r \to 0} C_1(r) = 5 , \qquad \lim_{r \to 0} C_2(r) = 8 .$$

2. The constant C_0 can be decreased if one considers an appropriate subclass of rings R. As an example, we have investigated the case where R is symmetric with respect to the origin.

Theorem 3. *If* R *is symmetric in the origin and if* $w = f(z)$ *maps* A *conformally onto* R *so that* $f(1) = 1$, *then*

$$|f(z) - z| < 3r \tag{5}$$

for $z \in A$. *The constant* 3 *cannot be replaced by any smaller number.*

Again it is only necessary to examine what happens on each boundary component of A and hence Theorem 3 is an immediate consequence of

Theorem 4. *If* R *is symmetric in the origin and if* $w = f(z)$ *maps* A *conformally onto* R *so that* $f(1) = 1$, *then*

$$\limsup_{|z| \to r} |f(z) - z| < 3r \tag{6}$$

and

$$\sup_{|z| = 1} |f(z) - z| < 2\sqrt{2}\, r. \tag{7}$$

The constant 3 *cannot be replaced by any smaller number.*

For each fixed r let F_S denote the class of functions $f(z)$ which map A onto an R, symmetric in the origin, with $f(1) = 1$. Then set

$$C_{1,S}(r) = \sup_{f \in F_S} \left(\limsup_{|z| \to r} \frac{|f(z) - z|}{r} \right),$$

$$C_{2,S}(r) = \sup_{f \in F_S} \left(\sup_{|z|=1} \frac{|f(z) - z|}{r} \right).$$

Here $C_{1,S}(r) < 3$ and

$$\lim_{r \to 0} C_{1,S}(r) = 3 .$$

On the other hand, $C_{2,S}(r)$ assumes its maximum value at an r between 0 and 1. We have not obtained an explicit expression for this maximum and so the bound in (7) is not best possible. Direct computation shows, however, that up to four decimals

$$\sup_{0<r<1} C_{2,S}(r) = 2.1736 ,$$

and that this value is attained roughly for $r = 0.78$.

3. We establish Theorem 2 in the following way. First the inequality (3) is an easy consequence of a well known estimate (Lemma 1) for the maximal diameter of the inner boundary component Γ. The fact that the constant 5 is best possible follows from an example (Lemma 5) suggested to us by P. P. Belinskiĭ. Another result on the moduli of rings (Lemma 2) shows that the maximum distortion on $|z| = 1$ occurs when Γ is a segment with one endpoint at the origin. This extremal mapping can then be expressed in terms of elliptic functions and the distortion on $|z| = 1$ computed by means of an alternating series (Lemma 6). This series yields (4) as well as an example to show that the constant 8 is best possible.

The proof for Theorem 4 is reduced to the first case by observing that w^2, as a function of z^2, induces a mapping of the previous kind. We can then use the estimates required for the proof of Theorem 2 to deduce (6) and (7), and an example similar to Belinskiĭ's shows that the constant 3 is best possible.

We begin with some preliminary remarks on the moduli of rings.

4. *Rings.* A *ring* is by definition a doubly-connected domain. Each ring R can be mapped conformally onto an annulus $a < |z| < b$ and the *modulus* of R is defined as the conformal invariant

$$\operatorname{mod} R = \log \frac{b}{a} .$$

A ring R is said to *separate* two sets E_1 and E_2 if E_1 and E_2 lie in different components of the complement of R.

The modulus is monotone in the following sense. If R and R' are rings and if R' separates the boundary components of R, then $R' \subseteq R$ and

$$\text{mod } R' \leq \text{mod } R . \tag{8}$$

Equality holds only if $R' = R$. (See [9], p. 626.)

We use the following result, due to Grötzsch, to estimate the distortion on the inner boundary component of A. (See [9], pp. 631—635).

Lemma 1. *Suppose that* $0 < t < 1$, *that* R *is a ring in* $|w| < 1$ *which separates* $|w| = 1$ *from the points* $w = 0$ *and* $w = te^{i\Theta}$, *and that* R_1 *is the ring bounded by* $|w| = 1$ *and the segment* $-t \leq w \leq 0$. *Then*

$$\text{mod } R \leq \text{mod } R_1 < \log \frac{4}{t} , \tag{9}$$

and

$$\lim_{t \to 0} \left(\log \frac{4}{t} - \text{mod } R_1 \right) = 0 . \tag{10}$$

We need the following result, essentially due to Mori [6], to obtain an upper bound for the distortion on the outer boundary of A.

Lemma 2. *Suppose that* β *is an arc of* $|w| = 1$ *with midpoint at* $w = 1$, *that* S *is a ring which separates* β *from the points* $w = 0$ *and* $w = \infty$, *and that* S_1 *is the ring bounded by* β *and the ray* $-\infty \leq w \leq 0$. *Then*

$$\text{mod } S \leq \text{mod } S_1 \tag{11}$$

and mod S_1 *is a strictly decreasing function of the length of* β.

Proof. The inequality (11) can be established by means of extremal lengths. (See, for example, [1], p. 91.) Alternatively we can use the reflection principle to obtain a conformal mapping $z = h(w)$ of the exterior of β onto the exterior of the segment $-1 \leq z \leq 0$ such that $h(\infty) = \infty$ and $h(0) = b > 0$. Then S and S_1 are mapped onto rings R and R_1, where R separates the segment $-1 \leq z \leq 0$ from the points $z = b$ and $z = \infty$, and where S_1 is the ring bounded by the above segment and the ray $b \leq z \leq \infty$. Then by a theorem due to Teichmüller ([9], pp. 637—639) we have

$$\text{mod } S = \text{mod } R \leq \text{mod } R_1 = \text{mod } S_1 .$$

The second statement of Lemma 2 is an immediate consequence of the above mentioned monotoneity property of the modulus.

5. We require a result on the convergence of conformal mappings of rings for the examples which show that the constants 5 and 3 in (3) and (6) are best possible.

Suppose that R is a ring bounded by $|w| = 1$ and a Jordan curve Γ in $|w| < 1$, that $P_1, \ldots, P_m$ are fixed points on Γ, and that $\{R_n\}$ is a sequence of rings, bounded by $|w| = 1$ and by continua Γ_n in $|w| < 1$, with the following properties:

(a) $R \subset R_n$,

(b) $\Gamma \cap R_n$ is contained in the union of m disks with radii $1/n$ and centers at $P_1, \ldots, P_m$.

Lemma 3. *If R and R_n are as above and if $z = g(w)$ and $z = g_n(w)$ map R and R_n onto the annuli $r < |z| < 1$ and $r_n < |z| < 1$ so that $g(1) = g_n(1) = 1$, then*

$$\lim_{n \to \infty} g_n(w) = g(w) \tag{12}$$

for each $w \in \bar{R}$, $w \neq P_1, \ldots, P_m$, where we define $g_n(w)$ at each point $w_0 \in \bar{R} \cap \Gamma_n$ as the limit of $g_n(w)$ as $w \to w_0$ in R.

Proof. Because $R \subset R_n$, R separates the boundary components of R_n and hence by (8)

$$\text{mod } R \leq \text{mod } R_n \quad \text{or} \quad r \geq r_n$$

for all n. Next (a) and (b) imply that each point of Γ lies within a distance $2/n$ of Γ_n for large n. By a theorem in [5],

$$\text{mod } R \geq \limsup_{n \to \infty} \text{mod } R_n ,$$

and hence we have

$$\lim_{n \to \infty} r_n = r . \tag{13}$$

Let E_n denote the image of $\Gamma \cap R_n$ under $z = g(w)$ and let $\omega_n(z)$ denote the harmonic measure of E_n taken with respect to A, the annulus $r < |z| < 1$. Since $g(w)$ is continuous in $\bar{R}$, we can find a sequence of positive numbers $\{d_n\}$ which converge to 0 such that E_n is contained in the union of m arcs of $|z| = r$ with lengths d_n and centers at $g(P_1), \ldots, g(P_m)$. It is easy to see that

$$\lim_{n \to \infty} \omega_n(z) = 0 \tag{14}$$

for each $z \in A$.

Now set $h_n(z) = g_n(f(z))$, where $w = f(z)$ is the inverse of $z = g(w)$. Then $h_n(z)$ is analytic and univalent in A and

$$\frac{r_n}{r} \leq \lim_{z \to \zeta} \left| \frac{h_n(z)}{z} \right| \leq 1$$

for $|\zeta| = 1$ and for $|\zeta| = r$, $\zeta \notin E_n$. Since

$$r_n \leq \left|\frac{h_n(z)}{z}\right| \leq r^{-1}$$

in A, the two constant theorem ([7], § 36) applied to $\frac{h_n(z)}{z}$ and its reciprocal yields

$$\frac{r_n}{r} r^{\omega_n(z)} \leq \left|\frac{h_n(z)}{z}\right| \leq r^{-\omega_n(z)},$$

and we conclude from (13) and (14) that

$$\lim_{n \to \infty} \left|\frac{h_n(z)}{z}\right| = 1 \tag{15}$$

for $z \in A$.

By reflecting in the circles $|z| = r$ and $|z| = 1$, we can extend $h_n(z)$ to be analytic in a domain D_n, where $D_n \subset D_{n+1}$ and D_n contains all points of $\bar{A}$ which are at least at a distance d_n from the points $g(P_1), \ldots, g(P_m)$. For each n_0, the functions $h_n(z)$ with $n \geq n_0$ form a normal family in D_{n_0}. Since $h_n(1) = 1$ for all n, (15) implies that

$$\lim_{n \to \infty} h_n(z) = z$$

for $z \in D_{n_0}$ and hence for $z \in \bar{A}$, $z \neq g(P_1), \ldots, g(P_m)$. This means that (12) holds for $w \in \bar{R}$, $w \neq P_1, \ldots, P_m$, and the proof for Lemma 3 is complete.

6. *Proof of Theorem 2 — Inner boundary.* We turn now to the proof of Theorem 2. We assume here and in what follows that A is an annulus $r < |z| < 1$, that R is a ring bounded by $|w| = 1$ and a continuum Γ in $|w| < 1$, and that R does not contain the point $w = 0$.

Lemma 4. *If $w = f(z)$ maps A conformally onto R so that the outer boundaries correspond, then*

$$\limsup_{|z| \to r} |f(z) - z| < 5r. \tag{16}$$

Proof. Let $w = te^{i\Theta}$ be one of the points of Γ which lie farthest from the origin, and let R_1 denote the ring bounded by $|w| = 1$ and the segment $-t \leq w \leq 0$. Then R separates $|w| = 1$ from $w = 0$ and $w = te^{i\Theta}$, and (9) implies that

$$\log \frac{1}{r} = \operatorname{mod} R < \log \frac{4}{t}.$$

Hence $t < 4r$, and since $|z| = r$ corresponds to Γ, we conclude that

$$\limsup_{|z| \to r} |f(z) - z| \leq \limsup_{|z| \to r} |f(z)| + r = t + r < 5r .$$

The following example, suggested by P. P. Belinskiĭ, shows that the constant 5 cannot be replaced by any smaller number, even under the more restrictive hypothesis that $f(1) = 1$.

Lemma 5. *For each* $\varepsilon > 0$ *there exists a conformal mapping* $w = f(z)$ *of an* A *onto an* R *such that* $f(1) = 1$ *and*

$$\limsup_{|z| \to r} |f(z) - z| > (5 - \varepsilon) r . \tag{17}$$

Proof. Fix $0 < \varepsilon < 1$. For each set of positive numbers t, h and d with $t^2 + h^2 < 1$ and $d < h$, let R_1, R_2 and R denote the rings bounded by $|w| = 1$ and by the continua Γ_1, Γ_2 and Γ, where Γ_1 is the segment $-t \leq w \leq 0$, Γ_2 is the rectangle with vertices at $w = \pm ih$ and $w = -t \pm ih$, and Γ is $\Gamma_1 \cup \Gamma_2$ minus the two vertical segments of lengths d with centers at $w = \pm ih/2$. Next let $g_1(w)$, $g_2(w)$ and $g(w)$ map R_1, R_2 and R conformally onto the annuli $r_1 < |z| < 1$, $r_2 < |z| < 1$ and $r < |z| < 1$ so that $g_1(1) = g_2(1) = g(1) = 1$.

Since $\operatorname{mod} R_1 = \log \dfrac{1}{r_1}$, (10) implies that $\dfrac{t}{4r_1} \to 1$ as $t \to 0$, and hence we may choose t so that

$$t > (4 - \varepsilon) r_1 . \tag{18}$$

By symmetry we have

$$g_2(0) = r_2 , \tag{19}$$

and it is easy to see that the segment joining $w = 0$ to $w = ih$ corresponds under $z = g_2(w)$ to an arc of $|z| = r_2$ whose length tends to 0 as $h \to 0$. Since $r_2 \to r_1$ as $h \to 0$ [5], we may choose h so that

$$|g_2(ih) - g_2(0)| < \varepsilon r_1 , \qquad r_2 < (1 + \varepsilon) r_1 . \tag{20}$$

The monotoneity property (8) and (20) then yield

$$r_1 < r < r_2 \qquad \text{whence} \qquad r_1 > (1 - \varepsilon) r . \tag{21}$$

Finally Lemma 3 implies that we may choose d so that

$$|g(0) - g_2(0)| < \varepsilon r_1 , \qquad |g(ih) - g_2(ih)| < \varepsilon r_1 , \tag{22}$$

where $g(0)$ and $g(ih)$ are defined as the limits of $g(w)$ as $w \to 0$ and $w \to ih$ in R_2.

Now let α denote the arc described as z moves in the positive sense along $|z| = r$ from $g(0)$ to $g(ih)$. By symmetry the angular measure of α is less than π, and we see from (19) — (22) that

$$|z - r_1| \leq |g(ih) - g(0)| + |g(0) - r_1| < 5\,\varepsilon\, r_1 \tag{23}$$

for $z \in \alpha$. Next let $w = f(z)$ denote the inverse of $z = g(w)$. To each point $w_0 \in \Gamma$ with $\mathrm{Im}(w_0) \geq 0$ there corresponds at least one point $z_0 \in \alpha$ such that $f(z) \to w_0$ as $z \to z_0$. In particular if we take $w_0 = -t$, we obtain

$$\lim_{z \to z_0} |f(z) - z| = |t + z_0| \geqq t + r_1 - |z_0 - r_1| > (5 - 6\varepsilon) r_1 > (5 - 11\varepsilon) r$$

from (18), (21) and (23). Replacing ε by $\varepsilon/11$ then yields (17).

7. *Proof of Theorem 2 — Outer boundary.* We investigate next the distortion on the outer boundary of A. We consider first the case where Γ is a segment with an endpoint at $w = 0$, and then show how the general problem can be reduced to this special case.

Lemma 6. *Suppose that* $w = f_1(z)$ *is a conformal mapping of* A *onto a ring* R_1, *bounded by* $|w| = 1$ *and the segment* $-t \leq w \leq 0$, *and that* $f_1(1) = 1$. *Then each point* $z = e^{i\varphi}$ *with* $0 < \varphi < \pi$ *corresponds to a point* $w = e^{i\psi}$ *with* $0 < \psi < \pi$ *and*

$$\psi = \varphi + 4 \sum_{n=0}^{\infty} (-1)^n \arg(1 - r^{2n+1} e^{-i\varphi}), \tag{24}$$

where arg denotes the principal branch. In particular

$$0 < \psi - \varphi < 4 \arg(1 - re^{-i\varphi}) \leq 4 \arcsin r. \tag{25}$$

Proof. It is easy to show that the restriction of $w = f_1(z)$ to the upper half annulus $r \leq |z| \leq 1$, $\mathrm{Im}(z) \geq 0$ can be expressed as the composition of the following mappings:

$$\zeta = \log z, \quad \omega = \mathrm{sn}(a\zeta, k), \quad w = \frac{1 + \omega}{1 - \omega}. \tag{26}$$

Here $\log z$ is the principal branch of the logarithm, $\mathrm{sn}(a\zeta, k)$ is the Jacobi elliptic sine function with modulus k, and

$$k = \frac{1 - t}{1 + t}, \qquad a = \frac{K}{\log \frac{1}{r}} = \frac{K'}{\pi}, \tag{27}$$

where

$$\begin{cases} K = K(k) = \displaystyle\int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2\beta)^{-1/2}\, d\beta, \\ K' = K(k'), \qquad k' = (1 - k^2)^{1/2}. \end{cases} \tag{28}$$

If we set $z = e^{i\varphi}$ and $w = e^{i\psi}$ in (26), we obtain

$$i \tan \frac{\psi}{2} = \mathrm{sn}(ia\varphi \, , k) \, .$$

If we let $\mathrm{tn}(a\varphi \, , k')$ and $\mathrm{am}(a\varphi \, , k')$ denote the Jacobi elliptic tangent[3]) and amplitude functions with modulus k', then

$$\mathrm{sn}(ia\varphi \, , k) = i \, \mathrm{tn}(a\varphi \, , k') = i \tan(\mathrm{am}(a\varphi \, , k')) \, .$$

(See [2], p. 24 or [8], p. 24, and see [10], p. 494.) Thus we have

$$\frac{\psi}{2} = \mathrm{am}(a\varphi \, , k') \, . \tag{29}$$

Now we can express the function $\mathrm{am}(\Theta \, , k')$ by means of its Fourier series as follows:

$$\mathrm{am}(\Theta \, , k') = \frac{\pi\Theta}{2K'} + \sum_{m=1}^{\infty} \frac{1}{m} \frac{2(q')^m}{1 + (q')^{2m}} \sin \left(\frac{m\pi\Theta}{K'} \right) \, ,$$

where by (27)

$$q' = \exp\left(- \frac{\pi K}{K'} \right) = r \, .$$

(See [2], p. 303, [8], p. 20 or [10], p. 511.) This, together with (27) and (29), yields

$$\psi = \varphi + 4 \sum_{m=1}^{\infty} \frac{r^m}{1 + r^{2m}} \frac{\sin m\varphi}{m} \, .$$

We see that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{r^m}{1 + r^{2m}} \frac{\sin m\varphi}{m} &= \sum_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} (-1)^n \, r^{m+2mn} \right) \frac{\sin m\varphi}{m} \\ &= \sum_{n=0}^{\infty} (-1)^n \, \mathrm{Im} \left(\sum_{m=1}^{\infty} r^{(2n+1)m} \frac{e^{im\varphi}}{m} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \arg(1 - r^{2n+1} e^{-i\varphi}) \, , \end{aligned}$$

and hence (24) follows.

Finally since $0 < \varphi < \pi$, it is easy to show by elementary geometry that $\arg(1 - r^{2n+1} e^{-i\varphi})$ is strictly decreasing in n and that

$$0 < \arg(1 - r^{2n+1} e^{-i\varphi}) \leq \arcsin r^{2n+1} \, . \tag{30}$$

Thus (25) follows by virtue of a well known theorem on alternating series.

---

[3]) This function is often denoted as $\mathrm{sc}(a\varphi \, , k')$.

We now use Lemmas 2 and 6 to prove the following result which, in turn, implies (4) of Theorem 2.

Lemma 7. *If* $w = f(z)$ *maps* A *conformally onto* R *so that an arc* α *of* $|z| = 1$ *corresponds to an arc* β *of* $|w| = 1$, *then*

$$|\text{length}\,\beta - \text{length}\,\alpha| < 8 \arcsin r\,. \tag{31}$$

Proof. We may clearly assume that α and β have their midpoints at $z = 1$ and $w = 1$. Let γ be the image of α under $w = f_1(z)$, where $f_1(z)$ is the mapping of Lemma 6. By reflecting in the circles $|z| = 1$ and $|w| = 1$ we can extend $f(z)$ and $f_1(z)$ to be univalent in $r < |z| < r^{-1}$.

Let B denote the ring bounded by α and the continuum consisting of the circles $|z| = r$, $|z| = r^{-1}$ plus the segment $-r^{-1} \leq z \leq -r$. Then $f(z)$ and $f_1(z)$ map B onto rings S and T_1, where S separates the arc β from the points $w = 0$ and $w = \infty$ and where T_1 is bounded by γ and the ray $-\infty \leq w \leq 0$. If S_1 denotes the ring bounded by β and the above ray, then (11) yields

$$\text{mod}\,T_1 = \text{mod}\,S \leq \text{mod}\,S_1\,,$$

and the second part of Lemma 2 implies that

$$\text{length}\,\beta \leq \text{length}\,\gamma\,. \tag{32}$$

Next let $e^{i\varphi}$ and $e^{i\psi}$ denote the upper endpoints of α and γ, where $0 < \varphi$, $\psi < \pi$. Then we have

$$\text{length}\,\beta - \text{length}\,\alpha \leqq 2(\psi - \varphi) < 8 \arcsin r$$

from (25) and (32). Applying this inequality to the complementary arcs in $|z| = 1$ and $|w| = 1$ gives

$$\text{length}\,\alpha - \text{length}\,\beta < 8 \arcsin r$$

and we obtain (31).

Now (4) is an immediate consequence of Lemma 7. For suppose that $w = f(z)$ maps A conformally onto R so that $f(1) = 1$ and $f(e^{i\varphi}) = e^{i\Theta}$ where $0 < \varphi, \Theta < 2\pi$. Then Lemma 7 implies that

$$|\Theta - \varphi| < 8 \arcsin r$$

and hence that

$$|f(e^{i\varphi}) - e^{i\varphi}| = 2 \sin \frac{|\Theta - \varphi|}{2} \leq 8 \sin \frac{|\Theta - \varphi|}{8} < 8r\,.$$

Finally we must show that the constant 8 in (4) cannot be replaced by any smaller number. Fix r, $0 < r < 1$, let $f_1(z)$ be the mapping of Lemma 6 and set $\varphi_1 = \arccos r$. Then

$$\arg(1 - re^{-i\varphi_1}) = \arcsin r$$

and from (24) and (30) it follows that

$$\psi_1 - \varphi_1 = 4 \arcsin r + O(r^3) ,$$

where $e^{i\psi_1} = f_1(e^{i\varphi_1})$. Now

$$w = f(z) = e^{i\psi_1} f_1(e^{-i\varphi_1} z)$$

maps A conformally onto a ring R so that $f(1) = 1$ and $f(e^{2i\varphi_1}) = e^{2i\psi_1}$. Hence

$$\sup_{|z|=1} |f(z) - z| \geq 2 \sin(\psi_1 - \varphi_1) = 8r + O(r^3) ,$$

and we see that, for each $\varepsilon > 0$, we can find a mapping $f(z)$ such that

$$\sup_{|z|=1} |f(z) - z| > (8 - \varepsilon) r .$$

This completes the proof of Theorem 2.

8. *Proof of Theorem 4.* We turn now to the case where the ring R is symmetric in the origin. We consider first the following analogue of Lemma 4.

Lemma 8. *If R is symmetric in the origin and if $w = f(z)$ maps A conformally onto R so that the outer boundaries correspond, then*

$$\limsup_{|z| \to r} |f(z) - z| < 3r . \tag{33}$$

Proof. Set $w^* = w^2$ and $z^* = z^2$. Then it is easy to see that $w = f(z)$ induces a conformal mapping $w^* = f^*(z^*)$ of $r^2 < |z^*| < 1$ onto a ring R^* , bounded by $|w^*| = 1$ and a continuum Γ^* in $|w^*| < 1$. Again R^* does not contain the origin, and if we let $w = te^{i\Theta}$ denote one of the points of Γ which lie farthest from the origin, then R^* separates $|w^*| = 1$ from $w^* = 0$ and $w^* = t^2 e^{2i\Theta}$. If we apply (9) to R^* , we conclude that $t < 2r$ and hence that

$$\limsup_{|z| \to r} |f(z) - z| \leq \limsup_{|z| \to r} |f(z)| + r = t + r < 3r .$$

The following example shows that the constant 3 in (33) cannot be replaced by any smaller number even under the more restrictive hypothesis that $f(1) = 1$.

Lemma 9. *For each $\varepsilon > 0$ there exists a conformal mapping $w = f(z)$ of an A onto an R , which is symmetric in the origin, such that $f(1) = 1$ and*

$$\limsup_{|z| \to r} |f(z) - z| > (3 - \varepsilon) r . \tag{34}$$

Proof. Fix $0 < \varepsilon < 1$. For each set of positive numbers t , h and d with $t^2 + h^2 < 1$ and $d < h$, let Γ_1 denote the segment $-t \leq w \leq t$,

Γ_2 the rectangle with vertices at $w = t \pm ih$ and $w = -t \pm ih$, and Γ the continuum which consists of $\Gamma_1 \cup \Gamma_2$ minus the two vertical segments of lengths d with centers at $w = \pm (t + ih/2)$. Next let R_1, R_2 and R denote the rings bounded by $|w| = 1$ and by Γ_1, Γ_2 and Γ, and let $g_1(w)$, $g_2(w)$ and $g(w)$ map these rings conformally onto the annuli $r_1 < |z| < 1$, $r_2 < |z| < 1$ and $r < |z| < 1$ so that $g_1(1) = g_2(1) = g(1) = 1$.

By setting $w^* = w^2$ and $z^* = z^2$ and arguing as in the proof of Lemma 8, we can apply (10) to show that $\dfrac{t}{2r_1} \to 1$ as $t \to 0$, and hence we may choose t so that

(35) $$t > (2 - \varepsilon) r_1 .$$

By symmetry we have

(36) $$g_2(t) = r_2 ,$$

and as in the proof of Lemma 5, we may choose h so that

(37) $$|g_2(t) - g_2(t + ih)| < \varepsilon\, r_1 , \qquad r_2 < (1 + \varepsilon) r_1 .$$

The monotoneity property and (37) then yield

(38) $$r_1 < r < r_2 \quad \text{whence} \quad r_1 > (1 - \varepsilon) r .$$

Finally Lemma 3 implies we may choose d so that

(39) $$|g(t) - g_2(t)| < \varepsilon r_1 , \qquad |g(t + ih) - g_2(t + ih)| < \varepsilon r_1 ,$$

where again $g(t)$ and $g(t + ih)$ are defined as the limits of $g(w)$ as $w \to t$ and $w \to t + ih$ in R_2.

Now let α denote the arc described as z moves in the positive sense along $|z| = r$ from $g(t)$ to $g(t + ih)$. Then the angular measure of α is less than π and we see from (36) — (39) that

(40) $$|z - r_1| \le \quad |g(t + ih) - g(t)| + |g(t) - r_1| < 5 \varepsilon r_1$$

for $z \in \alpha$. Next let $w = f(z)$ denote the inverse of $z = g(w)$. To each point $w_0 \in \Gamma$ with $\operatorname{Im}(w_0) \ge 0$ there corresponds at least one point $z_0 \in \alpha$ such that $f(z) \to w_0$ as $z \to z_0$. If we choose $w_0 = -t$, we obtain

$$\lim_{z \to z_0} |f(z) - z| = |t + z_0| \ge t + r_1 - |z_0 - r_1| > (3 - 6\varepsilon) r_1 > (3 - 9\varepsilon) r$$

from (35), (38) and (40). Replacing ε by $\varepsilon/9$ then yields (34).

If we apply Lemma 7 to the mapping $w^* = f^*(z^*)$, defined in the proof of Lemma 8, we obtain the following upper bound for the distortion on the outer boundary of A.

Lemma 10. *If* R *is symmetric in the origin and if* $w = f(z)$ *maps* A *conformally onto* R *so that an arc* α *of* $|z| = 1$ *corresponds to an arc* β *of* $|w| = 1$, *then*

$$|\text{length } \beta - \text{length } \alpha| < 4 \ \arcsin r^2 . \tag{41}$$

In particular if $f(1) = 1$, then (41) implies that

$$|f(e^{i\varphi}) - e^{i\varphi}| < 4 \ \sin(\arcsin r^2) = 4r^2$$

for $0 < \varphi < 2\pi$. Hence (7) follows for $r \leq 1/\sqrt{2}$, and since (7) clearly holds when $r > 1/\sqrt{2}$, the proof for Theorem 4 is complete.

References

[1] AHLFORS, L. V., *Lectures on conformal mapping,* Lecture notes from Oklahoma A. and M. College, Stillwater, 1951.

[2] BYRD, P. F. and FRIEDMAN, M. D., *Handbook of elliptic integrals for engineers and physicists,* Springer-Verlag, 1954.

[3] DUREN, P. L. and SCHIFFER, M., *A variational method for functions schlicht in an annulus,* Stanford University Technical Report No. 99, (1961), pp. 1–27.

[4] GAIER, D., *Untersuchungen zur Durchführung der konformen Abbildung mehrfach zusammenhängender Gebiete,* Arch. Rat. Mech. Anal. 3, (1959), pp. 149–178.

[5] GEHRING, F. W., *A remark on the moduli of rings,* Comment. Math. Helv. 36, (1961), pp. 42–46.

[6] MORI, A., *On an absolute constant in the theory of quasi-conformal mappings,* J. Math. Soc. Japan 8, (1956), pp. 156–166.

[7] NEVANLINNA, R., *Eindeutige analytische Funktionen,* Springer-Verlag, 1936 and 1953.

[8] OBERHETTINGER, F. and MAGNUS, W., *Anwendung der elliptischen Funktionen in Physik und Technik,* Springer-Verlag, 1949.

[9] TEICHMÜLLER, O., *Untersuchungen über konforme und quasikonforme Abbildung* Deutsche Math. 3, (1938), pp. 621–678.

[10] WHITTAKER, E. T. and WATSON, G. N., *A course of modern analysis,* Cambridge Univ. Press, 1946.

University of Michigan
Ann Arbor, Michigan

Åbo Akademi
Åbo, Finland

Printed January 1963