ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

324

REMARKS ON A PAPER OF TIENARI CONCERNING QUASICONFORMAL CONTINUATION

BY

JUSSI VÄISÄLÄ

HELSINKI 1962
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1963.324

Communicated 12 October 1962 by F. NEVANLINNA and OLLI LEHTO.

KESKUSKIRJAPAINO
HELSINKI 1962

Remarks on a paper of Tienari concerning quasiconformal continuation

1. *Introduction.* All sets considered in this paper are assumed to lie in the extended complex plane. Let γ be either an open Jordan arc[1]) or a closed Jordan curve. We say that γ is *quasiconformal at a point* $z \in \gamma$ if there exists an open subarc γ_z of γ and a quasiconformal mapping of a domain G such that $z \in \gamma_z \subset G$ and the image of γ_z is a line segment. γ is called *quasiconformal* if it is quasiconformal at each point.

Quasiconformal arcs and curves have been studied recently by Tienari [4] who called them »curves which allow a quasiconformal continuation»; see also Pfluger [3]. Tienari showed that the »local» quasiconformality defined above implies quasiconformality in the large, i.e., a closed quasiconformal curve can be mapped onto a circle and every compact subarc of an open quasiconformal arc can be mapped onto a line segment by a quasiconformal mapping of the whole plane.

Tienari arrived at the class of quasiconformal arcs and curves by considering the possibility of extending a quasiconformal mapping over a boundary arc. He considered only simply connected domains. We shall generalize his results for multiply connected domains in Section 3. Section 2 deals with general properties of quasiconformal arcs.

2. We first establish

Theorem 1. *Let f be a real-valued function, defined on the real interval $a < x < b$ and satisfying a Lipschitz condition*

$$|f(x_1) - f(x_2)| \leqq M\, |x_1 - x_2|\,. \tag{1}$$

Then the graph of f is a quasiconformal arc.

Proof. Let G be the strip domain $a < \operatorname{Re} z < b$. Define a mapping $\varphi\colon G \to G$ by

$$\varphi(x + iy) = x + i(y - f(x))\,.$$

φ is clearly a homeomorphism. By (1), φ is absolutely continuous on every horizontal and vertical line. At a point $z = x + iy$ where φ is differentiable, the dilatation has the value

[1] An open arc is an arc minus its endpoints.

$$\tfrac{1}{2}(2 + f'(x)^2 + \sqrt{4 + f'(x)^2}) \leqq \tfrac{1}{2}(2 + M^2 + \sqrt{4 + M^2}) .$$

By the analytic definition of quasiconformal mappings, φ is quasiconformal (see e.g. Pfluger [2]). Since φ maps the graph of f onto the segment $a < x < b$, the theorem follows.

Tienari [4] proved that an arc is quasiconformal if it has a continuous curvature. With the aid of Theorem 1 this result can be sharpened as follows.

Theorem 2. *If an arc γ has a continuous tangent, it is quasiconformal.*

Proof. We must prove the quasiconformality of γ at an arbitrary point $z_0 \in \gamma$. We may assume that $z_0 = 0$ and the tangent of γ at z_0 is the real axis. Because of the continuity of the tangent, there exists a subarc γ_0 of γ which has the representation $z = x + i f(x)$, $|x| < a$, where f is continuously differentiable for $|x| \leqq a$. By Theorem 1, γ_0 is quasiconformal at z_0 , q.e.d.

Remark. Using Theorem 16 of [4] one can weaken the condition of Theorem 2 and allow a finite number of exceptional points where γ has only two half-tangents which make an angle $\neq 0$.

3. In this section we shall discuss the extension of a quasiconformal mapping over a boundary arc or curve. Let G_1 and G_2 be two disjoint domains such that G_2 is simply connected and there exists an open quasiconformal arc γ which is a free boundary arc of both G_1 and G_2 . Let G_1' , G_2' and γ' satisfy the same conditions. Assume that f is a quasiconformal mapping of G_1 onto G_1' which carries γ onto γ' . Tienari's result ([4], Theorem 10) can then be formulated as follows: If G_1 is also simply connected and if γ_0 is a compact subarc of γ with image γ_0' , then f can be extended to a quasiconformal mapping $f^*\colon G_1 \cup \gamma_0 \cup G_2 \to G_1' \cup \gamma_0' \cup G_2'$. However, it is almost trivial to show that the simple connectedness of G_1 is an unnecessary restriction. For, since γ is a free boundary arc of G_1 , we can find a simply connected subdomain D of G_1 such that D has a free open boundary arc γ_1 , $\gamma_0 \subset \gamma_1 \subset \gamma$, and then extend the mapping $f|D$ to $D \cup \gamma_0 \cup G_2$.

The case of a closed curve is slightly more complicated.

Theorem 3. *Let G be a domain bounded by a closed quasiconformal curve γ and a compact set which does not meet γ . Denote by D the complementary domain of γ which does not meet G . Let G', γ', D' satisfy the same conditions. Then every quasiconformal mapping $f\colon G \to G'$ which maps γ onto γ' can be extended to a quasiconformal mapping $f^*\colon G \cup \gamma \cup D \to G' \cup \gamma' \cup D'$.*

The proof is based on the following

Lemma. *Let α and β be two disjoint closed quasiconformal curves. Then there exists a quasiconformal mapping of the whole plane which maps α and β onto concentric circles.*

Proof. We first map the complementary domains of α onto the upper and lower half-planes by conformal mappings S_1 and S_2, respectively, so that $S_1^{-1}(\infty) = S_2^{-1}(\infty)$ and β is mapped into the upper half-plane. Let G be the doubly connected domain bounded by α and β. We next map G by a conformal mapping f onto a domain bounded by the real axis and a circle C in the upper half-plane so that $f^{-1}(\infty) = S_1^{-1}(\infty)$. The mapping fS_1^{-1} is conformal and maps the real axis onto itself and ∞ into ∞. By the reflection principle, the derivative of fS_1^{-1} is bounded from 0 and ∞ on the real axis. Consequently, fS_1^{-1} satisfies the so-called ϱ-condition (see [1]). By Theorem 3 of [4], the mapping $S_1S_2^{-1}$ of the real axis also satisfies a ϱ-condition. Thus the composite mapping $(fS_1^{-1})(S_1S_2^{-1}) = fS_2^{-1}$ satisfies a ϱ-condition. By a well-known result of Beurling and Ahlfors [1], there exists a quasiconformal mapping T of the lower half-plane onto itself such that $T = fS_2^{-1}$ on the real axis. The mappings f and TS_2 define a quasiconformal mapping of the α-component of the complement of β onto the unbounded complementary component of C. Since β is quasiconformal and Theorem 3 holds for simply connected domains ([4], Theorem 5), this mapping can be extended to a quasiconformal mapping h of the whole plane. The desired mapping is then gh where g is the linear mapping which maps C and the real axis onto two concentric circles.

Proof for Theorem 3. Let h be a quasiconformal mapping of the whole plane such that h maps D onto the unit disc $|z| < 1$. h maps G onto a domain which contains an annulus $1 < |z| < R$. Let α' be the image of the circle $|z| = R$ under the mapping fh^{-1}. By the above Lemma, there exists a quasiconformal mapping g of the whole plane which maps γ' and α' onto two concentric circles $|z| = 1$ and $|z| = R' > 1$, respectively. Then gfh^{-1} is a quasiconformal mapping of $1 < |z| < R$ onto $1 < |z| < R'$. By repeated reflections, it can be extended to a quasiconformal mapping φ of $|z| < R$ onto $|z| < R'$. Then $g^{-1}\varphi h$ maps D onto D' and $g^{-1}\varphi h(z) = f(z)$ for $z \in \gamma$. We have thus obtained the desired extension of f.

Corollary 1. *Let G and G' be two domains, each bounded by a finite number of disjoint closed quasiconformal curves. Then every quasiconformal mapping of G onto G' can be extended to a quasiconformal mapping of the whole plane.*

Corollary 2. *Let G and G' be two subdomains of the upper half-plane, each bounded by the real axis and a compact subset of the upper half-plane. Let $f: G \rightarrow G'$ be a quasiconformal mapping which maps the real axis onto itself and ∞ into ∞. Then f satisfies a ϱ-condition on the real axis.*

References

[1] A. Beurling—L. Ahlfors: The boundary correspondence under quasiconformal mappings. - Acta Math. 96, 1956, 125—142.

[2] A. Pfluger: Über die Äquivalenz der geometrischen und der analytischen Definition quasikonformer Abbildungen. - Comment. Math. Helv. 33, 1959, 23—33.

[3] —»— Ueber die Konstruktion Riemannscher Flächen durch Verheftung. - J. Indian Math. Soc. 24, 1960, 401—412.

[4] M. Tienari: Fortsetzung einer quasikonformen Abbildung über einen Jordanbogen. - Ann. Acad. Sci. Fenn. A I 321, 1962, 1—32.

University of Helsinki

Printed October 1962.