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# ON THE CHARACTERS OF THE FINITE UNITARY GROUPS

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## On the characters of the finite unitary groups

This paper is an attempt to extend the remarkable results of Green [6] on the characters of the finite general linear groups to the finite unitary groups. It seems highly probable that the system of irreducible characters of the finite unitary group $\mathrm{U}(n, q^2)$ coincides with the system of classfunctions what we have called irreducible $C$-functions. These functions are obtained from the irreducible characters of the general linear group $\mathrm{GL}(n, q)$ by the simple formal change that $q$ is everywhere replaced by $-q$. From the results of Green it follows immediately that the $C$-functions form an orthonormal basis for the vector space (with the usual inner product) of the classfunctions of $\mathrm{U}(n, q^2)$. However, it seems to be more difficult to prove that these functions are characters of $\mathrm{U}(n, q^2)$. We have tried to attack this question by using Brauer's fundamental characterization of characters [1] and we give some results in this direction in §§ 3—5. In §§ 6—7 we verify our conjecture to be true in the cases $n = 2,3$ and we also give complete tables of the conjugacy classes and characters.

I wish to express my gratitude to Prof. R. Brauer for discussions concerning this problem and to Prof. G. E. Wall for sending me a large unpublished manuscript in which he solves the conjugacy class problem for all the classical groups. His results were of great assistance to me.

1. Let $\mathfrak{F} = \mathrm{GF}(q^2)$ be the finite field consisting of $q^2$ elements, where $q$ is a power of a prime $p$. For $\alpha \in \mathfrak{F}$ we write $\bar{\alpha} = \alpha^q$. Then $\bar{\alpha}$ is called the *conjugate* of $\alpha$. We shall also consider the »universal» field $\mathfrak{F}^* = \mathrm{GF}(q^{2n!})$ for some fixed $n$. Then we shall always regard the roots of polynomials over $\mathfrak{F}$ of degree $\leqq n$ as elements of $\mathfrak{F}^*$. By $\Theta$ we denote a fixed isomorphism of the multiplicative group of $\mathfrak{F}^*$ into the multiplicative group of the field of complex numbers. Let $V_n$ be an $n$-dimensional vector space over $\mathfrak{F}$ with a non-degenerate scalar product $f: V_n \times V_n \to \mathfrak{F}$ satisfying the conditions (for $X, X_1, X_2, Y, Y_1, Y_2 \in V_n, \lambda \in \mathfrak{F}$)

$$
(1) \qquad \begin{cases}
f(X_1 + X_2, Y) = f(X_1, Y) + f(X_2, Y), \\
f(X, Y_1 + Y_2) = f(X, Y_1) + f(X, Y_2), \\
f(\lambda X, Y) = \lambda f(X, Y) \\
f(X, \lambda Y) = \bar{\lambda} f(X, Y) \\
f(Y, X) = \overline{f(X, Y)} \\
f(X, Y) = 0 \quad \text{for every} \quad Y \in V_n \Rightarrow X = 0 .
\end{cases}
$$

The group of all non-singular matrices with elements in $\mathrm{GF}(q^t)$, or equivalently, the group of all non-singular linear transformations of an $n$-dimensional vector space over $\mathrm{GF}(q^t)$ onto itself, is called the *general linear group* and is denoted by $\mathrm{GL}(n, q^t)$. Then *the group of unitary transformations* (with respect to $f$) of $V_n$ is defined by

$$\mathfrak{U}_n = \mathrm{U}(n, q^2) = \{G \in \mathrm{GL}(n, q^2) \mid f(G(X), G(Y)) = f(X, Y) \text{ for all } X, Y \in V_n\}.$$

We denote by $g(n, q^t)$ and $u(n, q^2)$ the number of elements in $\mathrm{GL}(n, q^t)$ and $\mathfrak{U}_n$, respectively. As is well known (see e.g. [2])

$$g(n, q^t) = q^{\frac{1}{2}n(n-1)t} \prod_{i=1}^{n} (q^{ti} - 1),$$

$$u(n, q^2) = q^{\frac{1}{2}n(n-1)} \prod_{i=1}^{n} (q^i - (-1)^i).$$

Considering $g(n, x)$ as a polynomial in $x$ we can write

$$u(n, q^2) = (-1)^n g(n, -q). \tag{2}$$

We can choose a basis $\{X_i\}$ $(i = 1, 2, \ldots, n)$ of $V_n$, which we call the *standard basis*, such that

$$f(X_i, X_j) = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j; \end{cases}$$

and also a basis $\{Y_i\}$ $(i = 1, 2, \ldots, n)$, which we call the *hyperbolic basis*, so that

$$f(Y_i, Y_j) = \begin{cases} 0, & \text{if } i + j \neq n + 1, \\ 1, & \text{if } i + j = n + 1. \end{cases}$$

(See [2].) If we take the standard basis, then a matrix $M$ corresponds to a unitary transformation if and only if $M M^* = M^* M = 1$ ($M^*$ denoting the conjugate transpose of $M$). If we speak of unitary matrices without explicitly mentioning the basis, then we always mean the standard basis.

If $\chi$ and $\psi$ are complex valued class functions on $\mathfrak{U}_n$, we define the scalar product

$$(\chi, \psi) = \frac{1}{u(n, q^2)} \sum_{G \in \mathfrak{U}_n} \chi(G) \overline{\psi}(G)$$

$$= (-1)^n \sum_{c} \frac{1}{a(c)} \chi(c) \overline{\psi}(c),$$

where in the latter sum $(-1)^n a(c)$ is the order of the centralizer of an element of $\mathfrak{U}_n$ belonging to the class $c$, and the summation is over all classes $c$. We also put

$$||\chi|| = (\chi , \chi),$$

the norm of $\chi$. By a *character* (or *generalized character*) we mean a classfunction on $\mathfrak{U}_n$ which can be written in the form

$$a_1\chi_1 + a_2\chi_2 + \dots + a_l\chi_l, \tag{3}$$

where $\chi_1, \chi_2, \dots, \chi_l$ are the irreducible characters of $\mathfrak{U}_n$ and $a_1, a_2, \dots, a_l$ are arbitrary integers. If all the integers $a_j$ are non-negative, then we call (3) a *proper character*. As is well known, a character $\chi$ is irreducible if and only if $||\chi|| = 1$ and $\chi(1) > 0$; and two irreducible characters $\chi$ and $\psi$ are distinct if and only if $(\chi , \psi) = 0$.

If $\mathfrak{H}$ is a subgroup of $\mathfrak{U}_n$ and $\psi$ is a character of $\mathfrak{H}$, by a classical theorem of Frobenius, the classfunction $\psi^*$ on $\mathfrak{U}_n$, defined by

$$\psi^*(G) = \frac{1}{h} \sum_{T \in \mathfrak{U}_n} \psi_0(TGT^{-1}),$$

where $G \in \mathfrak{U}_n$ is arbitrary, $h$ is the number of elements in $\mathfrak{H}$, and $\psi_0(X) = \psi(X)$, if $X \in \mathfrak{H}$, and $\psi_0(X) = 0$, if $X \notin \mathfrak{H}$, is a character of $\mathfrak{U}_n$ and is called the *character induced by* $\psi$.

We use the following notations of Green [6]. Let $\lambda = \{l_1, l_2, \dots, l_p\}$ be a partition of a positive integer $n = \sum_{i=1}^{p} l_i$ into the $p$ parts $l_1 \geqq l_2 \geqq \dots \geqq l_p > 0$. Let $k_1 \geqq k_2 \geqq \dots \geqq k_s > 0$ be the parts of the partition conjugate to $\lambda$ and put $k_{s+1} = 0$. Then we write

$$|\lambda| = n\,; \quad n_\lambda = \sum_{i=1}^{s} \tfrac{1}{2} k_i(k_i + 1)\,;$$

$$\Phi_r(x) = (1-x)(1-x^2)\dots(1-x^r), \text{ if } r \geqq 1\,; \quad \Phi_0(x) = 1\,;$$

$$k(\lambda , x) = \Phi_{p-1}(x)\,; \quad a_\lambda(x) = x^{|\lambda| + 2n_\lambda} \sum_{i=1}^{s} \Phi_{k_i - k_{i+1}}\left(\frac{1}{x}\right);$$

$\{\lambda : x\}$ = the formal Schur function in the infinity of variables $1, x, x^2, \dots$

$$= x^{l_2 + 2l_3 + \dots} \prod_{1 \leqq r < s \leqq p} (1 - x^{l_r - l_s - r + s}) \left[\prod_{r=1}^{p} \Phi_{l_r + p - r}(x)\right]^{-1}.$$

Let $\varrho$ be another partition of $n$, which we write in the form

$$\varrho = \{1^{r_1} 2^{r_2} \dots n^{r_n}\}, \quad n = \sum_{i=1}^{n} i\, r_i\,.$$

Then we denote

$$z_\varrho = 1^{r_1}\, r_1!\ 2^{r_2}\, r_2! \ldots n^{r_n}\, r_n!\,; \qquad w_\varrho = r_1!\, r_2! \ldots r_n!\,;$$
$$c_\varrho(x) = (x-1)^{r_1}(x^2-1)^{r_2} \ldots (x^n-1)^{r_n}\ ;$$
$$e_\varrho(x) = (1-x)^{-r_1}(1-x^2)^{-r_2} \ldots (1-x^n)^{-r_n}\,.$$

If $\lambda_1, \lambda_2, \ldots, \lambda_s$ are partitions such that $\sum_{i=1}^{s} |\lambda_i| = |\lambda|$, let $g^{\lambda}_{\lambda_1\lambda_2\ldots\lambda_s}$ be the Hall polynomial, and

$$Q^{\lambda}_{\varrho}(x) = \sum g^{\lambda}_{\lambda_1\lambda_2\ldots\lambda_s}(x)\, k(\lambda_1, x)\, k(\lambda_2, x) \ldots k(\lambda_s, x)\,,$$

the sum being over all rows of partitions such that $\lambda_1, \ldots, \lambda_{r_1}$ are partitions of 1; $\lambda_{r_1+1}, \ldots, \lambda_{r_1+r_2}$ are partitions of 2, etc.

2. Let

$$f(t) = t^d + a_1 t^{d-1} + \ldots + a_d$$

be an arbitrary monic polynomial over $\mathfrak{F}$ with $a_d \neq 0$. Then we write

$$\tilde{f}(t) = \bar{a}_d^{-1}(\bar{a}_d t^d + \bar{a}_{d-1} t^{d-1} + \ldots + 1)\,.$$

**Definition 1.** We say that a monic polynomial $g(t)$ over $\mathfrak{F}$ is *U-irreducible*, if either $g(t)$ is irreducible and $g(t) = \tilde{g}(t)$, or $g(t) = f(t)\,\tilde{f}(t)$, where $f(t)$ is irreducible and $f(t) \neq \tilde{f}(t)$.

**Lemma 1.** *A monic polynomial $g(t)$ over $\mathfrak{F}$ is U-irreducible if and only if $g(t) = \tilde{g}(t)$ and $g(t)$ cannot be written in the form $g(t) = g_1(t)\, g_2(t)$, where $g_1(t)$ and $g_2(t)$ are non-constant polynomials over $\mathfrak{F}$ such that $g_1(t) = \tilde{g}_1(t)$ and $g_2(t) = \tilde{g}_2(t)$. If $g(t)$ is U-irreducible, then it is irreducible if and only if its degree is odd.*

*Proof.* The first sentence is clear. The second one follows from [3], Lemma 2.

Let $\mathscr{F}$ be the set of $U$-irreducible polynomials over $\mathfrak{F}$, excepting the polynomial $t$. Write $d(f)$ for the degree of $f \in \mathscr{F}$. Suppose $\gamma \in \mathfrak{F}^*$. Then there is a unique $U$-irreducible polynomial $f$ having $\gamma$ as a root and we put $d(\gamma) = d(f)$.

Let $f$ be a $U$-irreducible polynomial of degree $d$ and let $\gamma$ be a root of $f$. Then it is easy to see that

$$\gamma\,, \gamma^{-q}\,, \gamma^{q^2}\,, \ldots\,, \gamma^{(-q)^{d-1}} \tag{4}$$

are all the roots of $f$ (cf. [3], Lemma 2). We can call the elements (4) the *U-conjugates* of $\gamma$.

From the results of Wall ([11], cf. also [3]) it follows that every conjugacy class $c$ of $\mathfrak{U}_n$ can be uniquely described by a partition valued function $\nu(f) = \nu_c(f)$ on $\mathcal{F}$ satisfying the condition

$$\sum_{f \in \mathcal{F}} |\nu(f)| \, d(f) = n \, ,$$

and, conversely, to every such function $\nu(f)$ there corresponds a class $c$ of $\mathfrak{U}_n$. The characteristic polynomial of a unitary matrix belonging to $c$ is

$$F(t) = \prod_{f \in \mathcal{F}} f^{|\nu(f)|} \, .$$

We shall denote the class $c$ by the symbol

$$c = (\ldots f^{\nu(f)} \ldots) \, .$$

A class $c$ is *principal*, if $\nu(f)$ is $\{1\}$ or 0 for every $f \in \mathcal{F}$. If in this case $F(t)$ has $r_d$ factors of degree $d$ $(d = 1, 2, \ldots, n)$, we say that $c$ is of *principal type* $\varrho$, $\varrho$ being the partition $\{1^{r_1} 2^{r_2} \ldots n^{r_n}\}$ of $n$. A class is *primary*, if $\nu(f) = 0$ except for one particular $f \in \mathcal{F}$; $c$ has then the form $(f^\nu)$ for some partition $\nu$. Primary classes $(f^\lambda)$ and $(g^\mu)$ $(f, g \in \mathcal{F})$ are said to have the *same type* if $d(f) = d(g)$ and $\lambda = \mu$. Note that a primary class of $\mathfrak{U}_n$ is not necessarily primary in $\mathrm{GL}(n, q^2)$.

**Definition 2.** If $c = (\ldots f^{\nu(f)} \ldots)$ is a class of $\mathfrak{U}_n$, put

$$a(c) = \prod_{f \in \mathcal{F}} a_{\nu(f)}\left((-q)^{d(f)}\right) .$$

Then $a(c)$ has the sign $(-1)^n$. By Wall's results, $(-1)^n a(c)$ is the order of the centralizer of an element of $\mathfrak{U}_n$ belonging to the class $c$. Hence the number of elements in the class $c$ is

$$\frac{u(n, q^2)}{(-1)^n a(c)} = \frac{g(n, -q)}{a(c)} \, .$$

**Definition 3.** If $c = (\ldots f^{\nu(f)} \ldots)$ is a class of $\mathfrak{U}_n$, $c_i = (\ldots f^{\nu_i(f)} \ldots)$ is a class of $\mathfrak{U}_{s_i}$ $(i = 1, 2, \ldots, k)$, where $s_1, \ldots, s_k$ are natural numbers such that $\sum_{i=1}^{k} s_i = n$, then we denote

$$g^c_{c_1 c_2 \ldots c_k} = \prod_{f \in \mathcal{F}} g^{\nu(f)}_{\nu_1(f)\, \nu_2(f) \ldots \nu_k(f)}\left((-q)^{d(f)}\right) .$$

**Definition 4.** Let $\alpha_1, \ldots, \alpha_k$ be classfunctions of $\mathfrak{U}_{s_1}, \mathfrak{U}_{s_2}, \ldots, \mathfrak{U}_{s_k}$, respectively, where $s_1, \ldots, s_k$ are positive integers such that $\sum_{i=1}^{k} s_i = n$. Then by $\alpha = \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_k$ we mean the classfunction on $\mathfrak{U}_n$, whose value at a class $c$ is

$$\alpha(c) = \sum g^c_{c_1 c_2 \ldots c_k} \alpha_1(c_1) \alpha_2(c_2) \ldots \alpha_k(c_k) ,$$

where the summation is over all rows $c_1, c_2, \ldots, c_k$ of classes respectively of $\mathfrak{U}_{s_1}, \mathfrak{U}_{s_2}, \ldots, \mathfrak{U}_{s_k}$.

From the corresponding lemma of Green it then follows immediately

**Lemma 2.** *With the assumptions of Definition 4 we have*

$$\alpha(1) = \frac{\Phi_n(-q)}{\Phi_{s_1}(-q) \ldots \Phi_{s_k}(-q)} \alpha_1(1) \alpha_2(1) \ldots \alpha_k(1) .$$

If, in Definition 4, the functions $\alpha_i$ are characters of $\mathfrak{U}_{s_i}$ $(i = 1, 2, \ldots, k)$, we don't know if $\alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_k$ is a character of $\mathfrak{U}_n$. If we could show this, then the machinery of Green would work and would give us a proof of our conjecture.

As in the paper of Green, making only the required trivial modifications, we can now define the following concepts:

the set of *$\varrho$-variables* $X^\varrho = \{x_{di}\}$ $(i = 1, 2, \ldots, r_d;\ d = 1, 2, \ldots, n)$ $\varrho$ being the partition $\{1^{r_1} 2^{r_2} \ldots n^{r_n}\}$;

the set $\Xi^\varrho$ of *$\varrho$-roots*;

*substitution* of $X^\varrho$;

*equivalence* of two substitutions and the *mode* of substitution;

the partition $\varrho(m, f)$ which describes the mode $m$;

*$\varrho$-function* $U_\varrho$;

substitution of the $\varrho$-variables *into a class* $c$;

*isobaric* classes.

**Definition 5.** (Definition of uniform function) For each partition $\varrho$ of $n$ let there be given a $\varrho$-function $U_\varrho(X^\varrho)$. Then the uniform function $U = (U_\varrho)$ on $\mathfrak{U}_n$ is the classfunction defined at the class $c$ by

$$U(c) = \sum_\varrho \sum_m Q(m, c)\, U_\varrho(X^\varrho m) \tag{5}$$

summed over partitions $\varrho$ of $n$, and all modes $m$ of substitution of $X^\varrho$ into $c$; and

$$Q(m, c) = \prod_{f \in \mathcal{F}} \frac{1}{z_{\varrho(m,f)}} Q^{\nu_c(f)}_{\varrho(m,f)} \left((-q)^{d(f)}\right) .$$

The functions $U_\varrho(X^\varrho)$ are called the *principal parts* of $U$, $U_\varrho$ being the $\varrho$-part. Formula (5) is called the *degeneracy rule*. A uniform function whose principal parts are all zero except for $U_\varrho$ is called a *basic uniform function* of type $\varrho$. Then, clearly, the analogues to Green's theorems 7 and 8 are valid.

**Definition 6.** (Definition of $s$-simplex) Let $k$ be an arbitrary integer. Suppose that

$$k\,,\quad -kq\,,\ kq^2\,,\ \ldots\,,\ k(-q)^{s-1} \tag{6}$$

are distinct residues $\left(\operatorname{mod} q^s - (-1)^s\right)$. In this case we say that each of the integers (6) is an *s-primitive*, and that the set (6) is an *s-simplex* $g$ (or a simplex $g$ of degree $s$) with $k, -kq, kq^2, \ldots, k(-q)^{s-1}$ as its roots.

We use the following notation. Let $\omega$ be a primitive root of $\mathfrak{F}^*$. Write

$$\omega_s = \omega^{\frac{q^{2n!}-1}{(-q)^s-1}}\,. \tag{7}$$

Then there is an $1-1$ correspondence between the $s$-simplex (6) and the $U$-irreducible polynomial having

$$\omega_s^k\,,\ \omega_s^{-kq}\,,\ \omega_s^{kq^2}\,,\ \ldots\,,\ \omega_s^{k(-q)^{s-1}}$$

as its roots. Hence we have

**Lemma 3.** *There are exactly as many simplexes of degree $s$ as there are $U$-irreducible polynomials $f \in \mathcal{F}$ of degree $s$.*

Let $\mathcal{G}$ be the set of all $s$-simplexes for $s \geqq 1$ and let $d(g)$ denote the degree of $g \in \mathcal{G}$. Let $\nu(g)$ be a partition valued function on $\mathcal{G}$ such that

$$\sum_{g \in \mathcal{G}} |\nu(g)|\, d(g) = n\,.$$

Then we call $(\ldots g^{\nu(g)} \ldots)$ the symbol of a *dual class* $e$. It follows from lemma 2 that the number of classes of $\mathfrak{U}_n$ is the same as the number of dual classes.

We can then define the dual concepts:
the set $Y^\varrho$ of *dual $\varrho$-variables*;
*substitution* of $Y^\varrho$;
*equivalence* of two substitutions;
*mode* $m$ of substitution;
the partition $\varrho(m, g)$;
substitution *into a dual class*;
*isobaric* dual classes;
*dual $\varrho$-function.*

The set of *dual $\varrho$-roots* $H^\varrho$ is defined to be a set of $n$ symbols

$$h_{di}\,,\ h_{di}(-q)\,,\ \ldots\,,\ h_{di}(-q)^{d-1}\,,\quad (i = 1\,,2\,,\ldots\,,r_d\,;\ d = 1\,,2\,,\ldots\,,n)\,,$$

$\varrho$ being the partition $\{1^{r_1}\, 2^{r_2} \ldots n^{r_n}\}$.

If $\alpha$ is a substitution of $Y^\varrho$, we define $\alpha$ as a mapping of $H^\varrho$ into the rational integers by

$$h_{di}\alpha = c_{di}\,\frac{(-q)^d - 1}{(-q)^{s_{di}} - 1},$$

where $c_{di}$ is a root of the simplex $y_{di}\alpha$ and $s_{di}$ is the degree of $y_{di}\alpha$.

**Definition 7.** Let $J_d(k)$ be a basic uniform function on $\mathfrak{U}_d$ of type $d$ with $d$-part $S_d(k : \xi_{d1}) = \Theta^k(\xi_{d1}) + \Theta^{-kq}(\xi_{d1}) + \ldots + \Theta^{k(-q)^{d-1}}(\xi_{d1})$. Let $\varrho = \{1^{r_1} 2^{r_2} \ldots n^{r_n}\}$ be a partition of $n$ and let $h^\varrho$ stand for the row $(h_{11}, \ldots, h_{1r_1}; h_{21}, \ldots, h_{2r_2}; \ldots)$ of integers $h_{di}$, one integer for each part of $\varrho$. Then we denote

$$B^\varrho(h^\varrho) = J_1(h_{11}) \circ \ldots \circ J_1(h_{1r_1}) \circ J_2(h_{21}) \circ \ldots \circ J_2(h_{2r_2}) \circ \ldots .$$

From Green's results it follows that $B^\varrho(h^\varrho)$ is a basic uniform function on $\mathfrak{U}_n$ of type $\varrho$, with $\varrho$-part

$$B_\varrho = B_\varrho(h^\varrho : \xi^\varrho) = \prod_d \{ \sum_{1' \ldots r_d'} S_d(h_{d1} : \xi_{d1'}) S_d(h_{d2} : \xi_{d2'}) \ldots S_d(h_{dr_d} : \xi_{dr_d'})\},$$

where the summation is over all permutations $1'\,2' \ldots r_d'$ of $1\,2 \ldots r_d$ and

$$S_d(h : \xi) = \Theta^h(\xi) + \Theta^{-hq}(\xi) + \Theta^{hq^2}(\xi) + \ldots + \Theta^{h(-q)^{d-1}}(\xi).$$

We shall call $B^\varrho(h^\varrho)$ a *basic C-function of type* $\varrho$.

For any classfunction $\chi$ on $\mathfrak{U}_n$ we shall call $\chi(1)$ the *degree* of $\chi$ (even if $\chi(1)$ is negative).

We now state our final

**Definition 8.** Let $\nu(g)$ be a partition valued function on the set $\mathcal{G}$ of simplexes, which satisfies the condition

$$\sum_{g \in \mathcal{G}} |\nu(g)|\, d(g) = n,$$

and let $e = (\ldots g^{\nu(g)} \ldots)$ be the dual class determined by $\nu(g)$. By the corresponding *irreducible C-function* (for which we use the same notation as for the dual class) we mean the classfunction on $\mathfrak{U}_n$ which is given by

$$(\ldots g^{\nu(g)} \ldots) = \pm \sum_{\varrho, m} \chi(m, e)\, B^\varrho(h^\varrho m), \tag{8}$$

summed over all partitions $\varrho$ of $n$ and all modes $m$ of substitution of $Y^\varrho$ into $e$ and

$$\chi(m, e) = \prod_{g \in \mathcal{G}} \frac{1}{z_{\varrho(m,g)}}\, \chi^{\nu(g)}_{\varrho(m,g)},$$

where $\chi_\varrho^\nu$ stands for the character of the symmetric group of appropriate degree, and the sign in (8) is to be so chosen that the degree of $(\ldots g^{\nu(g)} \ldots)$ is positive.

Then we have

**Theorem 1.** *The irreducible C-functions form an orthonormal basis for the vector space of the class functions on* $\mathfrak{U}_n$, *i.e. the number of distinct irreducible C-functions equals the number of classes of* $\mathfrak{U}_n$ *and*

$$\big((\ldots g^{\nu(g)} \ldots), (\ldots g^{\nu'(g)} \ldots)\big) = \begin{cases} 0, & \text{if } \nu(g) \neq \nu'(g) \text{ for some } g, \\ 1, & \text{if } \nu(g) = \nu'(g) \text{ for all } g. \end{cases}$$

*The degree of* $(\ldots g^{\nu(g)} \ldots)$ *is*

$$|\Phi_n(-q) \prod_{g \in \mathcal{G}} \{\nu(g) : (-q)^{d(g)}\}| . \tag{9}$$

*Furthermore,* $(\ldots g^{\nu(g)} \ldots)$ *can be written as*

$$(\ldots g^{\nu(g)} \ldots) = \prod_{g \in \mathcal{G}} (g^{\nu(g)}) , \tag{10}$$

*where* $\prod$ *denotes the* $\circ$*-product (the factor for which* $\nu(g) = 0$ *is omitted).*

**Conjecture.** *The system of irreducible C-functions coincides with the system of irreducible characters of* $\mathfrak{U}_n$.

As an important example we shall now show that the linear irreducible $C$-functions are characters of $\mathfrak{U}_n$. Suppose that $\chi = (\ldots g^{\nu(g)} \ldots)$ is linear, i.e. $\chi(1) = 1$. If $\nu(g) \neq 0$ for at least two $g$, then we can decide, by (10) and lemma 2, that the only possibility is $n = 2, q = 2, \nu(g_1) = \nu(g_2) = \{1\}$, where $g_1$ and $g_2$ are two simplexes of degree 1. This case will be considered in § 6, where we prove our conjecture for $n = 2$. Hence we may assume that $\chi$ is *primary*, i.e. $\chi = (g^{\nu(g)})$. Put $\nu(g) = \{l_1, l_2, \ldots, l_p\}$, where $l_1 \geqq l_2 \ldots \geqq l_p > 0$. Then from the formula for the Schur function we see that necessarily $p = 1$. Put $l = l_1, s = d(g)$. Then we have

$$|\Phi_{sl}(-q)| = |\Phi_l\big((-q)^s\big)| .$$

Clearly, this is the case if and only if $s = 1$. Hence we get the linear $C$-functions $(g^{\{n\}})$, where $g$ goes all the $q + 1$ simplexes of degree 1. Now let $g$ be an arbitrary simplex of degree 1 and let $k$ be a root of $g$. Then it follows from our definitions that $\chi = (g^{\{n\}})$ has the value $\Theta^k(\det A)$ at an element $A$ of $\mathfrak{U}_n$ and hence $\chi$ is a linear character of $\mathfrak{U}_n$.

3. We shall now discuss the problem how to prove that the $C$-functions are characters of $\mathfrak{U}_n$. The most straightforward idea is to apply the following fundamental theorem of Brauer ([1]).

*A classfunction $\chi$ on a finite group $\mathfrak{G}$ is a character of $\mathfrak{G}$ if and only if the restriction of $\chi$ to $\mathfrak{E}$ is a character of $\mathfrak{E}$ for every elementary subgroup $\mathfrak{E}$ of $\mathfrak{G}$.*

A subgroup $\mathfrak{E}$ of $\mathfrak{G}$ is called *elementary* if it is a direct product of a cyclic group and another group whose order is a power of a prime.

Hence, in order to show that $\chi$ is a character of $\mathfrak{G}$, it is enough to show that $\chi$ is a character of a system $\mathcal{S}$ of subgroups of $\mathfrak{G}$, which has the property that $\mathcal{S}$ together with all its conjugates covers every elementary subgroup of $\mathfrak{G}$.

In our case we claim that this property is possessed by the following system of subgroups

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 ,$$

the $\mathcal{S}_i$'s being defined as follows.

1) $\mathcal{S}_1 = \{\mathfrak{P} \times \mathfrak{Z}\}$, where $\mathfrak{P}$ is the $p$-Sylowgroup of $\mathfrak{U}_n$ ($q$ = power of $p$) and $\mathfrak{Z}$ is the center of $\mathfrak{U}_n$.

2) $\mathcal{S}_2 = \{\mathfrak{U}_i \times \mathfrak{U}_{n-i}\}$ $\left(i = 1, 2, \ldots, \left[\frac{n}{2}\right]\right)$, where $\mathfrak{U}_i \times \mathfrak{U}_{n-i}$ means the group of matrices of the form

$$\begin{pmatrix} U_i & 0 \\ 0 & U_{n-i} \end{pmatrix}$$

with $U_i \in \mathfrak{U}_i$, $U_{n-i} \in \mathfrak{U}_{n-i}$.

3) $\mathcal{S}_3$ is the set of centralizers of all the primary classes $\left(f^{\left\{1 \frac{n}{d(f)}\right\}}\right)$, where $f$ goes all the $U$-irreducible polynomials of degree $d(f) \geqq 2$, $d(f) \mid n$.

4) $\mathcal{S}_4 = \{\mathfrak{R}_i\, \mathfrak{Z}\}$, where $\mathfrak{R}_i$ goes all the $r$-Sylowgroups of $\mathfrak{U}_n$ corresponding to primes $r$ satisfying

$$\begin{cases} r \,\Big|\, \dfrac{(-q)^n - 1}{q+1} , \\ r \mid (-q)^i - 1 \text{ for at least one } i \text{ with } 1 \leqq i \leqq n-1 . \end{cases} \tag{11}$$

5) $\mathcal{S}_5$ is the set of centralizers of the classes $\left((t-1)^{\left\{\frac{n}{i}\, i\right\}}\right)$ for $i \mid n$, $2 \leqq i \leqq n - 1$.

Namely, suppose that $\mathfrak{E} = \mathfrak{Q} \times \{G\}$ is an elementary subgroup of $\mathfrak{U}_n$, where $\{G\}$ is the cyclic group generated by an element $G$ of $\mathfrak{U}_n$ and $\mathfrak{Q}$ is a subgroup of the centralizer $\mathfrak{C}(G)$ of $G$, whose order is a power of a prime. Let $c$ denote the class of $\mathfrak{U}_n$ to which $G$ belongs. If $c$ is not primary, i.e. $c = (\ldots f^{\nu(f)} \ldots)$, where $0 < |\nu(f)| < \dfrac{n}{d(f)}$ for at least one

$f$, then clearly some conjugate of $\mathfrak{C}$ is contained in $\mathfrak{U}_i \times \mathfrak{U}_{n-i}$ for $i = d(f)\ |\nu(f)|$. Suppose therefore that $c$ is primary, i.e. $c = (f^{\nu(f)})$. If now $d(f) \geqq 2$, then it is easy to see that a conjugate of $\mathfrak{C}$ is contained in a centralizer of an element belonging to the class $(f^{\{1^{\frac{n}{d(f)}}\}})$. Suppose then that $d(f) = 1$. If $G \in \mathfrak{Z}$, then a conjugate of $\mathfrak{C}$ is contained in $\mathfrak{T}\,\mathfrak{Z}$, where $\mathfrak{T}$ is a Sylowgroup of $\mathfrak{U}_n$. But a conjugate of $\mathfrak{T}\,\mathfrak{Z}$ is always contained in one of the groups $\mathfrak{P} \times \mathfrak{Z}$, $\mathfrak{U}_1 \times \mathfrak{U}_{n-1}$, $\mathfrak{R}_i\,\mathfrak{Z}$ or in the centralizer of an element belonging to a class $(f)$, where $d(f) = n$. Let $G \notin \mathfrak{Z}$ and $c = \big((t-\alpha)^{\varrho}\big)$, where $\varrho$ $(\neq \{1^n\})$ is a partition of $n$. If $\varrho = \{n\}$, then a conjugate of $\mathfrak{C}$ is contained in $\mathfrak{P} \times \mathfrak{Z}$. If $\varrho = \{i^{\frac{n}{i}}\}$ for $i \mid n$, $2 \leqq i \leqq n-1$, then a conjugate of $\mathfrak{C}$ is contained in a group belonging to $\mathscr{S}_5$. If $\varrho = \{1^{r_1}\, 2^{r_2} \ldots n^{r_n}\}$, where at least two $r_i$'s are $\neq 0$, then one can easily see that a conjugate of $\mathfrak{C}$ is contained in a suitable group of the system $\mathscr{S}_2$.

4. We shall now consider more closely a special subgroup of $\mathfrak{U}_n$ belonging to the system $\mathscr{S}_3$. Write

$$f(t) = \prod_{i=0}^{n-1} (t - \omega_n^{(-q)^i}),$$

where $\omega_n$ is defined by the equation (7). Then there are elements of $\mathfrak{U}_n$ having the characteristic polynomial $f(t)$. Let $A$ be such an arbitrary fixed element and let $\mathfrak{A}$ be the cyclic group generated by $A$. Then $\mathfrak{A}$ is the centralizer of $A$. Write $R(n)$ for the order of $A$ so that

$$R(n) = q^n - (-1)^n .$$

We shall prove the following

**Theorem 2.** *The restrictions of the irreducible C-functions to* $\mathfrak{A}$ *are characters of* $\mathfrak{A}$.

Let $k$ be an arbitrary natural number. We define $w(k)$ to be the least one of the natural numbers $w$ which satisfy the condition

$$k \equiv k\,(-q)^w \pmod{R(n)}.$$

Clearly $w(k) \mid n$.

Put

$$\varepsilon = \Theta(\omega_n).$$

Then $\varepsilon$ is a primitive $R(n)$-th root of unity.

Denote

$$f_k(t) = \prod_{i=0}^{w(k)-1} (t - \omega_n^{k(-q)^i}) \qquad (k = 1, 2, \ldots)$$

and let $c_k$ be the class of $\mathfrak{U}_n$ defined by

$$c_k = (f_k^{\left\{1^{\frac{n}{w(k)}}\right\}}) .$$

Then $A^k$ belongs to the class $c_k$.

Let $\psi^{(t)}$ be a character of $\mathfrak{A}$ defined by the condition

$$\psi^{(t)}(A) = \varepsilon^t \qquad \big(t = 0, 1, \ldots, R(n) - 1\big) .$$

If $a$ is any natural number and $\varrho$ is a partition, then the notation $a \mid \varrho$ means that $a$ divides all the parts of the partition $\varrho$. If then $\sigma$ is the partition whose parts are equal to the parts of $\varrho$ divided by $a$, then we denote $\varrho = a \cdot \sigma$ and $\sigma = \dfrac{1}{a}\, \varrho$.

Let $B^\varrho(h^\varrho)$ be a basic $C$-function of type $\varrho$ defined in the definition 7. Firstly, we shall compute the value of $B^\varrho(h^\varrho)$ at the class $c_k$, i.e. $B^\varrho(h^\varrho)(A^k)$.

If $w(k) \nmid \varrho$, then $B^\varrho(h^\varrho)(A^k) = 0$.

If $w(k) \mid \varrho$, then there is exactly one mode $m$ of substitution of the $\varrho$-variables $X^\varrho$ into the class $c_k$ and

$$Q(m, c_k) = \frac{1}{z_{\frac{1}{w(k)}\varrho}}\, Q_{\frac{1}{w(k)}\varrho}^{\left\{1^{\frac{n}{w(k)}}\right\}} \big((-q)^{w(k)}\big) .$$

Let $\dfrac{1}{w(k)}\, \varrho = \{1^{r_1} 2^{r_2} \ldots n^{r_n}\}$. Then we have

$$B^\varrho(h^\varrho)(A^k) = \frac{1}{z_{\frac{1}{w(k)}\varrho}}\, Q_{\frac{1}{w(k)}\varrho}^{\left\{1^{\frac{n}{w(k)}}\right\}} \big((-q)^{w(k)}\big)\, B_\varrho(h^\varrho : \xi^\varrho m)$$

$$= \frac{1}{z_{\frac{1}{w(k)}\varrho}}\, Q_{\frac{1}{w(k)}\varrho}^{\left\{1^{\frac{n}{w(k)}}\right\}} \big((-q)^{w(k)}\big) \prod_d \Big( \sum_{1' \ldots r'_d} S_{dw(k)}(h_{dw(k),1} : \xi_{dw(k),1'}\, m)$$

$$\ldots S_{dw(k)}(h_{dw(k),1} : \xi_{dw(k),r'_d}\, m)\Big)$$

$$= \Phi_{\frac{n}{w(k)}} \big((-q)^{w(k)}\big)\, e_\varrho(-q) \prod_d \prod_{i=1}^{r_d} \sum_{j=0}^{w(k)-1} \varepsilon^{h_{dw(k),i}\, k(-q)^j} .$$

In general, we denote by $p_\varrho$ the number of parts of a partition $\varrho$. We shall renumber the integers $h_{dw(k),i}$ and denote them more simply by

$$h_1, h_2, \ldots, h_{p_\varrho} .$$

Then we have

$$B^{\varrho}(h^{\varrho})(A^k) = \Phi_{\frac{n}{w(k)}}\left((-q)^{w(k)}\right) e_{\varrho}(-q) \sum_{j_1=0}^{w(k)-1} \sum_{j_2=0}^{w(k)-1} \cdots \sum_{j_{p_\varrho}=0}^{w(k)-1} \varepsilon^{k \sum_{i=1}^{p_\varrho} h_i (-q)^{j_i}} .$$

We denote

$$D_{\varrho}(h^{\varrho}, t) = \frac{1}{R(n)} \sum_{k=1}^{R(n)} B^{\varrho}(h^{\varrho})(A^k)\, \overline{\psi^{(t)}}(A^k) .$$

Then we have

$$D_{\varrho}(h^{\varrho}, t) = \frac{e_{\varrho}(-q)}{R(n)} \sum_{w \mid \varrho} \sum_{\substack{1 \leq k \leq R(n) \\ w(k)=w}} \Phi_{\frac{n}{w}}\left((-q)^{w}\right) \sum_{j_1=0}^{w-1} \sum_{j_2=0}^{w-1} \cdots \sum_{j_{p_\varrho}=0}^{w-1} \varepsilon^{k\left(\sum_{i=1}^{p_\varrho} h_i(-q)^{j_i} - t\right)} .$$

Clearly $k$ satisfies the condition $w(k) = w$ if and only if

$$k = l \frac{R(n)}{R(w)} ,$$

where the natural number $l$ is $w$-primitive.

We shall use the following notations:

$$\varepsilon_a = \varepsilon^{\frac{R(n)}{R(a)}} \qquad \text{for } a \mid n ;$$

$$\delta^{d}_{a,b} = \begin{cases} 0, & \text{if } a \not\equiv b \pmod{R(d)} . \\ 1, & \text{if } a \equiv b \pmod{R(d)} ; \end{cases}$$

$$\triangle^{d}_{a;\, b_1, b_2, \ldots, b_s} = \sum_{j_1=0}^{d-1} \sum_{j_2=0}^{d-1} \cdots \sum_{j_s=0}^{d-1} \delta^{d}_{a,\, \sum_{i=1}^{s} b_i(-q)^{j_i}} .$$

We consider in general a sum of the form

$$S(a) = \sum_{\substack{1 \leq l \leq R(w) \\ l\ w\text{-primit.}}} \varepsilon_w^{al} ,$$

where the summation is over the $w$-primitive residue classes $(\bmod\, R(w))$. We claim that

$$S(a) = \sum_{u \mid w} \sum_{m=1}^{R(u)} \mu\left(\frac{w}{u}\right) \varepsilon_u^{am} , \tag{12}$$

where $\mu$ is the Möbius function. Namely, we can write the right-hand side of (12) in the form

$$\sum_{l=1}^{R(w)} \left( \sum_{\substack{u \mid w \\ \frac{R(w)}{R(u)} \mid l}} \mu\left(\frac{w}{u}\right)\right) \varepsilon_w^{al} . \tag{13}$$

Let $u_0$ be the least natural number which satisfies the condition

$$l \equiv l\,(-q)^{u_0} \qquad \bigl(\mathrm{mod}\ R(w)\bigr)\,.$$

Then

$$\frac{R(w)}{R(u)} \Bigm| l$$

if and only if $u_0 \mid u$. Hence the inner sum in (13) equals

$$\sum_{v \mid \frac{w}{u_0}} \mu(v) = \begin{cases} 0\,, & \text{if } u_0 < w\,, \\ 1\,, & \text{if } u_0 = w\,. \end{cases}$$

This implies (12). Furthermore, we can compute the inner sum in (12), and so we get

$$S(a) = \sum_{u \mid w} \delta^u_{a,0}\, \mu\left(\frac{w}{u}\right) R(u)\,. \tag{12'}$$

Using (12′) we can write the above expression for $D_\varrho(h^\varrho, t)$ in the form

$$D_\varrho(h^\varrho, t) = \frac{e_\varrho(-q)}{R(n)} \sum_{w \mid \varrho} \Phi_{\frac{n}{w}}\bigl((-q)^w\bigr) \sum_{u \mid w} \sum_{j_1=0}^{w-1} \sum_{j_2=0}^{w-1} \cdots \sum_{j_{p_\varrho}=0}^{w-1} \mu\left(\frac{w}{u}\right) R(u)\, \delta^u_{t,\, \sum_{i=1}^{p_\varrho} h_i (-q)^{j_i}}\,.$$

From this we get the final expression for $D_\varrho(h^\varrho, t)$

$$D_\varrho(h^\varrho, t) = \frac{e_\varrho(-q)}{R(n)} \sum_{w \mid \varrho} \sum_{u \mid w} \Phi_{\frac{n}{w}}\bigl((-q)^w\bigr)\, \mu\left(\frac{w}{u}\right)\left(\frac{w}{u}\right)^{p_\varrho} R(u)\, \triangle^u_{t;\,h_1 h_2, \ldots, h_{p_\varrho}}\,. \tag{14}$$

We put $x = -q$ and we consider (14) as a rational function in the variable $x$. We assert that (14) is, in fact, a polynomial in $x$ with integral coefficients.

We distinguish between two cases. Suppose first that all parts of the partition $\varrho$ are not equal. It is clearly enough to show that

$$\frac{1}{1 - x^n}\, \Phi_{\frac{n}{w}}(x^w)\, e_\varrho(x) \tag{15}$$

is a polynomial in $x$ for all natural numbers $w$, which satisfy the condition $w \mid \varrho$. Put $m = \dfrac{n}{w}$, $\dfrac{1}{w}\varrho = \{1^{r_1} 2^{r_2} \ldots (m-1)^{r_{m-1}}\}$, $z = x^w$. Then the expression (15) is

$$\frac{(1 - z)\,(1 - z^2) \ldots (1 - z^{m-1})}{(1 - z)^{r_1}\,(1 - z^2)^{r_2} \ldots (1 - z^{m-1})^{r_{m-1}}}\,. \tag{16}$$

Let $\nu_d$ denote the value of (16) in the normed exponential valuation of $Q(z)$ ($Q$ = rationals) associated with the $d$-th cyclotomic polynomial. We have to show that $\nu_d \geqq 0$ for all natural numbers $d \leqq m - 1$.

We have

$$\nu_d = \left[\frac{m-1}{d}\right] - \sum_{i=1}^{\left[\frac{m-1}{d}\right]} r_{di} .$$

In any case, by our assumption, at least two of the numbers $r_i$ are $\neq 0$. We have three possibilities. Firstly, if all the $r_{di}$'s are $= 0$, then clearly $\nu_d \geqq 0$. Secondly, if exactly one $r_{di}$, say $r_{di_0}$, is $\neq 0$, then the assumption $\nu_d < 0$ implies

$$r_{di_0} \geqq \left[\frac{m-1}{d}\right] + 1 ,$$

so that

$$m = \sum_{j=1}^{m-1} j\, r_j > d\, i_0\, r_{di_0} \geqq d\, r_{di_0} > m - 1 ,$$

which is impossible. Thirdly, suppose that at least two $r_{di}$'s, say $r_{di_0}$ and $r_{di_1}$ are $\neq 0$. Let $i_1 > 1$. Then we have

$$d\, \nu_d > m - 1 - d - d\, r_{di_0} - d\, r_{di_1} \geqq m - 1 - d\, i_0\, r_{di_0} - d\, i_1\, r_{di_1} \geqq -1 ,$$

so that $\nu_d \geqq 0$.

We suppose now that all parts of the partition $\varrho$ are equal, i.e.

$$\varrho = \left\{ \left(\frac{n}{r}\right)^r \right\} .$$

In this case our proof will depend on the following lemma.

**Lemma 4.** *Let $r$ and $s$ be arbitrary natural numbers. Put*

$$M(z) = \sum_{v \mid s} v^r \mu(v)\, (1 - z^v)\, (1 - z^{2v})\, (1 - z^{3v}) \ldots (1 - z^{rs}) .$$

*Then*

$$\frac{1 - z}{(1 - z^{rs})\, (1 - z^s)^r}\, M(z)$$

*is a polynomial in $z$ with integral coefficients.*

*Proof of the lemma.* The case $s = 1$ is trivial. Hence assume that $s > 1$. Denote

$$M_1(z) = M(z) \prod_{i=1}^{r} \frac{1}{1 - z^{is}} = \sum_{v \mid s} v^r \mu(v)\, P(z, v) ,$$

where

$$P(z\,,v) = \prod_{i=0}^{r-1} \prod_{j=1}^{\frac{s}{v}-1} \left(1 - z^{\left(i\frac{s}{v}+j\right)v}\right).$$

We shall show that $M_1(z)$ is divisible by $\dfrac{1-z^s}{1-z}$. Clearly, this will imply the lemma.

Let $l\,(>1)$ be an arbitrary divisor of $s$ and let $\zeta_l$ be a primitive $l$-th root of unity. We have to show that $M_1(\zeta_l) = 0$ for each $l$.

If $P(\zeta_l, v) \neq 0$, then the least natural number $a$ which satisfies the condition $a\,v \equiv 0 \pmod l$ must be $\dfrac{s}{v}$. On the other hand, this number is, of course, $\dfrac{l}{(v, l)}$. Hence

$$v = (v\,,l)\,\frac{s}{l}.$$

If now $\left(l\,,\dfrac{s}{l}\right) > 1$, then there exists a prime $p$ such that $p \mid l$, $p \mid \dfrac{s}{l}$. But then $p \mid v$ and $p \mid (v, l)$, so that $p^2 \mid v$ and $\mu(v) = 0$. Hence we may assume that

$$\left(l\,,\frac{s}{l}\right) = 1\,.$$

Now

$$\zeta_l^v = \zeta_{\frac{s}{v}}^{\frac{s}{l}}$$

is a primitive $\dfrac{s}{v}$-th root of unity, so that

$$\prod_{j=1}^{\frac{s}{v}-1} (1 - \zeta_l^{jv}) = \frac{s}{v}$$

and

$$M_1(\zeta_l) = s^r \sum_{v \mid s} \mu(v) = 0\,.$$

This proves the lemma.

Put $v = \dfrac{w}{u}$. Then we can write the expression (14) in the form

$$D_\varrho(h^\varrho\,,t) = \frac{(-1)^{n-1}}{(1-x^n)\,(1-x^{\frac{n}{r}})^r} \sum_{u \mid \frac{n}{r}} R(u)\, \triangle^u_{t;\,h_1, h_2, \ldots, h_r}$$

$$\sum_{v \mid \frac{n}{ru}} v^r \mu(v)\,(1 - x^{uv})\,(1 - x^{2uv}) \ldots (1 - x^{n})$$

For each $u$ we use lemma 4 taking $s = \dfrac{n}{ru}$, $z = x^u$. It follows that $D_\varrho(h^\varrho, t)$ is a polynomial in $x$ with integral coefficients, as asserted.

Let now $e = (\ldots g^{\nu(g)} \ldots)$ be an arbitrary dual class

$$e = \pm \sum_{\varrho, m} \chi(m, e)\, B^\varrho(h^\varrho m)\,.$$

We denote

$$D(e, t) = \sum_{\varrho, m} \chi(m, e)\, D_\varrho(h^\varrho m, t)\,.$$

Again we write $x = -q$. Then, by our result above, $D(e, t)$ is a polynomial in $x$ with rational coefficients. We shall show that $D(e, t)$ multiplied by suitable factors of the form $1 - x^i$ is a polynomial in $x$ with integral coefficients. By the classical theorem of Gauss, it follows that $D(e, t)$ itself has integral coefficients, and this will prove our theorem.

We have

$$D(e, t) = \frac{1}{R(n)} \sum_{|\varrho| = n} \sum_m \left( \prod_{g \in \mathcal{G}} \frac{1}{z_{\varrho(m,g)}} \chi^{\nu(g)}_{\varrho(m,g)} \right) e_\varrho(x) \sum_{w \mid \varrho} \sum_{u \mid w}$$

$$\Phi_{\frac{n}{w}}(x^w)\, \mu\left(\frac{w}{u}\right) \left(\frac{w}{u}\right)^{p_\varrho} R(u)\, \triangle^u_{t;\, h_1 m, h_2 m, \ldots, h_{p_\varrho} m}$$

$$= \frac{1}{R(n)} \sum_{w \mid n} \sum_{u \mid w} \Phi_{\frac{n}{w}}(x^w)\, \mu\left(\frac{w}{u}\right) R(u) \sum_{\substack{|\varrho| = n \\ w \mid \varrho}} \sum_m$$

$$\triangle^u_{t;\, h_1 m, h_2 m, \ldots, h_{p_\varrho} m} \left( \prod_{g \in \mathcal{G}} \frac{1}{z_{\varrho(m,g)}} \left(\frac{w}{u}\right)^{p_{\varrho(m,g)}} \chi^{\nu(g)}_{\varrho(m,g)}\, e_{\varrho(m,g)}(x^{d(g)}) \right).$$

We shall now consider more closely the expression $\triangle^u_{t;\, h_1 m, h_2 m, \ldots, h_{p_\varrho} m}$ for fixed $u$, $\varrho$, and $m$. Let $\alpha$ be a fixed substitution of the dual $\varrho$-variables $Y^\varrho$ into $\mathcal{G}$, which belongs to the mode $m$. Let us consider first a certain fixed simplex $g$ and assume that the $h_i$'s are so numbered that $h_1, h_2, \ldots, h_{\varrho(m,g)}$ correspond to those dual $\varrho$-variables $Y^\varrho$, which are mapped onto $g$ by $\alpha$. Denote shortly $\sigma = \varrho(m, g)$, $p = p_\varrho$, $d = d(g)$, and $c =$ an arbitrary root of the simplex $g$. Let $a = (u, d)$ and $b = (w, d)$. Since $w \mid \varrho$, we must have $\dfrac{w}{b} \mid \sigma$, so that we can write $\sigma = \dfrac{w}{b}\tau$. Let $\tau = \{1^{t_1} 2^{t_2} \ldots\}$, so that $\sum t_i = p$ and $\sum i\, t_i = \dfrac{b}{w} |\nu(g)|$.

By definition,

$$\triangle^u_{t:\, h_1 m, h_2 m, \ldots, h_{p_\varrho} m} = \sum_{j_1 = 0}^{u-1} \sum_{j_2 = 0}^{u-1} \cdots \sum_{j_p = 0}^{u-1} \delta^u_{t,\, \sum_{i=1}^{p} (h_i m) x^{j_i}}\,.$$

Consider now the following partial sum of this expression. Fix $j_{p+1}$, $j_{p+2}, \ldots, j_{p_Q}$ in some way, denote shortly

$$N = \sum_{i=p+1}^{p_Q} (h_i m)\, x^{j_i},$$

and then sum with respect to $j_1, j_2, \ldots, j_p$. So we get the expression

(17) $$\sum_{j_1=0}^{u-1} \sum_{j_2=0}^{u-1} \cdots \sum_{j_p=0}^{u-1} \delta^u_{t;\ \sum_{i=1}^{p} (h_i m)\, x^{j_i} + N}\,.$$

Now the set $\{(h_i m)\}$ $(i = 1, 2, \ldots, p)$ consists of the following polynomials in $x$:

$$f_j(x) = c\left(1 + x^d + x^{2d} + \ldots + x^{\left(j\frac{w}{b} - 1\right)d}\right) \qquad (j = 1, 2, \ldots),$$

$f_j(x)$ appearing $t_j$ times. We have

$$f_j(x) \equiv j\, f(x) \pmod{R(u)},$$

where

$$f(x) = c\,\frac{aw}{bu}\left(1 + x^d + x^{2d} + \ldots + x^{\left(\frac{u}{a} - 1\right)d}\right).$$

Furthermore,

$$f(x)(x^a - 1) \equiv 0 \pmod{R(u)}.$$

Hence the expression (17) is

(18) $$\left(\frac{u}{a}\right)^p \sum_{(j)} \delta^u_{t;\, f(x) P_j(x) + N}\,,$$

where $P_j(x)$ goes all the $a^p$ polynomials that can be formed from the matrix

$$\begin{pmatrix}
1 & \ldots & 1 & 2 & \ldots & 2 & 3 & \ldots \\
x & \ldots & x & 2x & \ldots & 2x & 3x & \ldots \\
x^2 & \ldots & x^2 & 2x^2 & \ldots & 2x^2 & 3x^2 & \ldots \\
\cdot & & \cdot & \cdot & & \cdot & \cdot & \\
\cdot & & \cdot & \cdot & & \cdot & \cdot & \\
\cdot & & \cdot & \cdot & & \cdot & \cdot & \\
\underbrace{x^{a-1} \ \ldots \ x^{a-1}}_{t_1} & & & \underbrace{2x^{a-1} \ \ldots \ 2x^{a-1}}_{t_2} & & & 3x^{a-1} & \ldots
\end{pmatrix}$$

so that one term is taken from each column and these $p$ terms are then summed. (Of course, these polynomials are not necessarily all different.) Let

$$P_j(x) = \sum_{(i)} i \sum_{l=1}^{t_i} x^{k_l} \tag{19}$$

be any such polynomial, the $k_l$'s being certain integers between 0 and $a-1$, determined by the choice of $P_j(x)$. Take auxiliary variables $y_{g0}, y_{g1}, \ldots, y_{g,a-1}$. Let the monomial

$$\prod_{(i)} \prod_{l=1}^{t_i} y^i_{g,k_l} \tag{20}$$

correspond to the polynomial (19). Then (19) $\leftrightarrow$ (20) determines a $1-1$ correspondence between the polynomials $P_j(x)$ and the terms in the product

$$S_\tau(y_{g0}, y_{g1}, \ldots, y_{g,a-1}) = \prod_{(i)} (y^i_{g0} + y^i_{g1} + \ldots + y^i_{g,a-1})^{t_i}, \tag{21}$$

and hence the number of times that a certain polynomial $P_j(x)$ appears is equal to the coefficient of the corresponding term in the product (21).

We assume now that the above procedure is carried out for each simplex $g$. We shall form a new expression $D_1(e, t)$ by replacing $\triangle^u_{t;h_1 m, h_2 m, \ldots, h_{p_\varrho} m}$ by the product

$$\prod_{g \in \mathcal{G}} \left(\frac{u}{(u, d(g))}\right)^{p_\varrho(m,g)} S_{\frac{(w, d(g))}{w} \varrho(m,g)} (y_{g0}, y_{g1}, \ldots, y_{g,(u,d(g))-1}) .$$

From the above considerations it follows that it is enough to prove the assertion for $D_1(e, t)$, i.e. we have to show that $D_1(e, t)$ multiplied by suitable factors of the form $1 - x^i$ is a polynomial in $x$ and $y_{gj}$ with integral coefficients

We have

$$D_1(e, t) = \frac{1}{R(n)} \sum_{w \mid n} \sum_{u \mid w} \Phi_{\frac{n}{w}}(x^w) \mu\left(\frac{w}{u}\right) R(u) \prod_{g \in \mathcal{G}} \Bigg[ \sum_{\substack{|\sigma| = |\nu(g)| \\ \frac{w}{(w,\, d(g))}}}^{\sigma}$$

$$\frac{1}{z_\sigma} \left(\frac{w}{(u, d(g))}\right)^{p_\sigma} S_{\frac{(w, d(g))}{w} \sigma} (y_{g0}, y_{g1}, \ldots, y_{g,(u,d(g))-1}) \chi^{\nu(g)}_\sigma e_\sigma(x^{d(g)}) \Bigg] .$$

It is clearly enough to prove the assertion for the expressions in square brackets. Let us again consider a fixed simplex $g$. Let it be the same as that considered above and let us also use the same notations that were used above. Then the expression we are interested in is $\Big($of course we may assume that $\frac{w}{b} \Big| \; |\nu(g)|$, otherwise the expression is $0\Big)$

$$\sum_{|\tau| = \frac{b}{w}|\nu(g)|} \frac{1}{z_\tau}\left(\frac{b}{a}\right)^{p_\tau} S_\tau(y_{g0}, y_{g1}, \ldots, y_{g,a-1})\, \chi^{\nu(g)}_{\frac{w}{b}\tau}\, e_\tau(x^{\frac{w}{b}d}) . \tag{22}$$

Now $\frac{b}{a}$ is an integer. Take $\frac{b}{a}$ auxiliary variables $x_1, x_2, \ldots, x_{\frac{b}{a}}$ and replace in (22) $\left(\frac{b}{a}\right)^{p_\tau}$ by $S_\tau(x_1, x_2, \ldots, x_{\frac{b}{a}})$. Then it is enough to prove the assertion for the new expression, because from the new expression we get (22) by taking $x_1 = x_2 = \ldots = x_{\frac{b}{a}} = 1$. We denote the products $y_{gi}x_j \left(i = 0, 1, \ldots, a-1; j = 1, 2, \ldots, \frac{b}{a}\right)$ in some order by $y_0, y_1, \ldots, y_{b-1}$. Then

$$S_\tau(x_1, x_2, \ldots, x_{\frac{b}{a}})\, S_\tau(y_{g0}, y_{g1}, \ldots, y_{g,a-1}) = S_\tau(y_0, y_1, \ldots, y_{b-1}) .$$

By [7], § 8.1, pp. 143—146, we have

$$\chi^{\nu(g)}_{\frac{w}{b}\tau} = \sum_{|\mu| = |\tau|} a_\mu \chi^\mu_\tau$$

for some integers $a_\mu$. Put $z = x^{\frac{w}{b}d}$ and $T = \frac{b}{w}|\nu(g)|$. Then we have to consider

$$\sum_{|\tau| = T} \frac{1}{z_\tau} \chi^\mu_\tau S_\tau(y_0, y_1, \ldots, y_{b-1})\, e_\tau(z) . \tag{23}$$

For $l = jb + i$ $(0 \leqq i \leqq b-1,\ j = 0, 1, 2, \ldots)$ we write

$$Z_l = z^j y_i .$$

Now (23) is a formal Schur function of degree $T$ in the infinity of variables $Z_l$ and hence it can be written in the form

$$\sum_\mu \sum_{\substack{i_1, i_2, \ldots, i_p \\ i_r \neq i_s}} c_\mu Z^{m_1}_{i_1} Z^{m_2}_{i_2} Z^{m_2}_{i_2} \ldots Z^{m_p}_{i_p} ,$$

where $p = p_\mu$, the $c_\mu$'s are integers, the outer sum is to be formed over all partitions $\mu = (m_1, m_2, \ldots, m_p)$ of $T$, and the inner sum is to be taken over all ordered $p$-tuples $(i_1, i_2, \ldots, i_p)$, where the $i_k$'s are different non-negative integers. It is enough to consider the sum

$$\sum_{\substack{i_1, i_2, \ldots, i_p \\ i_r \neq i_s}} Z^{m_1}_{i_1} Z^{m_2}_{i_2} \ldots Z^{m_p}_{i_p} . \tag{24}$$

Rearranging the terms in the sum (24) we can write it in the form

$$\sum_{l_1=0}^{b-1} \sum_{l_2=0}^{b-1} \cdots \sum_{l_p=0}^{b-1} \sum_{\substack{i_1, \ldots, i_p \\ i_r \neq i_s \\ i_r \equiv l_r \pmod b}} Z_{i_1}^{m_1} Z_{i_2}^{m_2} \ldots Z_{i_p}^{m_p} .$$

It is enough to consider the sum

$$\sum_{\substack{i_1, i_2, \ldots, i_p \\ i_r \neq i_s \\ i_r \equiv l_r \pmod b}} Z_{i_1}^{m_1} Z_{i_2}^{m_2} \ldots Z_{i_p}^{m_p} \tag{25}$$

for fixed $l_1, l_2, \ldots, l_p$. Suppose the notations to be so chosen that

$$\begin{aligned} l_1 &= l_2 = \ldots = l_{r_1} = L_1 , \\ l_{r_1+1} &= l_{r_1+2} = \ldots = l_{r_1+r_2} = L_2 , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_{r_1+r_2+\ldots+r_{t-1}+1} &= \ldots = l_{r_1+r_2+\ldots+r_t} = L_t , \end{aligned}$$

where the $L_i$'s are different and $r_1 + r_2 + \ldots + r_t = p$. Then (25) can be written in the form

$$y_{l_1}^{m_1} y_{l_2}^{m_2} \ldots y_{l_p}^{m_p} \sum_{\substack{i_1, i_2, \ldots, i_{r_1} \\ i_s \neq i_u}} \sum_{\substack{i_{r_1+1}, \ldots, i_{r_1+r_2} \\ i_s \neq i_u}} \cdots \sum_{\substack{i_{r_1+\ldots+r_{t-1}+1}, \ldots, i_{r_1+\ldots+r_t} \\ i_s \neq i_u}} z^{m_1 i_1 + m_2 i_2 + \ldots + m_p i_p} . \tag{26}$$

Hence, finally, it is enough to prove our assertion for a sum which is of the type

$$\sum_{\substack{i_1, i_2, \ldots, i_r \\ i_s \neq i_u}} z^{m_1 i_1 + m_2 i_2 + \ldots + m_r i_r} . \tag{27}$$

Let $\pi = \{1^{p_1} 2^{p_2} \ldots\}$ be a partition of $r$. Let $\mathcal{R}_\pi$ be an arrangement of the numbers $1, 2, \ldots, r$ into subsystems so that $p_d$ subsystems consist of $d$ elements $(d = 1, 2, \ldots)$. The number of such arrangements $\mathcal{R}_\pi$ is obviously

$$N_\pi = \frac{r!}{\prod_d (d!)^{p_d} p_d!} .$$

Corresponding to the arrangement $\mathcal{R}_\pi$ we define the sum

$$U(\mathcal{R}_\pi) = \sum{}' z^{m_1 i_1 + m_2 i_2 \ldots + m_r i_r}$$

as follows. We identify $i_a$ and $i_b$ if $a$ and $b$ belong to the same subsystem and after this we sum all the remaining variables $i_k$ from 0 to $\infty$. Then clearly each $U(\mathcal{R}_\pi)$ is a rational function of $z$ of the form

$$\prod_i (1 - z^{n_i})^{-1} ,$$

where $\sum n_i = \sum m_i$. Furthermore, we denote

$$U(\pi) = \sum_{\mathcal{R}_\pi} U(\mathcal{R}_\pi) ,$$

the sum being formed over all arrangements $\mathcal{R}_\pi$. Now the truth of our assertion rests on the following lemma, which allows us to express (27) in terms of the $U(\pi)$.

**Lemma 5.** *We have*

$$\sum_{\substack{i_1, i_2, \ldots, i_r \\ i_s \neq i_u}} z^{m_1 i_1 + m_2 i_2 + \ldots + m_r i_r} = \sum_{|\pi| = r} A_\pi U(\pi) , \tag{28}$$

*where*

$$A_\pi = \prod_d \left((-1)^{d-1} (d-1)!\right)^{p_d} \quad \textit{for} \quad \pi = \{1^{p_1} 2^{p_2} \ldots\} .$$

*Proof of the lemma.* The left-hand side of (28) consists of powers of $z$ of the form

$$z^{m_1 i_1 + m_2 i_2 + \ldots m_r i_r} , \tag{29}$$

where the $i_k$'s range from 0 to $\infty$ and are all different. On the other hand, a term of this type appears in $U(\{1^r\})$ only and its coefficient is $A_{\{1^r\}} = 1$. Suppose now that we have a term of the type (29), where the $i_k$'s are not all different. Without loss of generality, we may assume that

$$\begin{cases} i_1 = i_2 = \ldots = i_{a_1} = I_1 , \\ i_{a_1+1} = i_{a_1+2} = \ldots = i_{a_1+a_2} = I_2 , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ i_{a_1+a_2+\ldots+a_{s-1}+1} = \ldots = i_{a_1+a_2+\ldots+a_s} = I_s , \end{cases} \tag{30}$$

where the $I_k$'s are all different and $a_1 + a_2 + \ldots + a_s = r$. Then (29) appears in $U(\mathcal{R}_\pi)$ if and only if $\mathcal{R}_\pi$ satisfies the following condition: If $i_a$ and $i_b$ belong to the same subsystem of the arrangement $\mathcal{R}_\pi$, then they also both belong to the same set

$$\mathcal{A}_k = \{i_{a_1+\ldots+a_{k-1}+1}, i_{a_1+\ldots+a_{k-1}+2}, \ldots, i_{a_1+\ldots+a_k}\}$$

for some $k$ $(1 \leqq k \leqq s)$. This condition can be expressed more simply by saying that $\mathcal{R}_\pi$ is a union

$$\mathcal{R}_\pi = \mathcal{R}_{\pi_1} \cup \mathcal{R}_{\pi_2} \cup \ldots \cup \mathcal{R}_{\pi_s},$$

where $\pi_k$ is a partition of $a_k$ and $\mathcal{R}_{\pi_k}$ is an arrangement of $\mathcal{S}_k$. Hence the coefficient of (29) (assuming that (30) is valid) on the right-hand side of (28) equals

$$\sum_{|\pi_1| = a_1} \sum_{|\pi_2| = a_2} \ldots \sum_{|\pi_s| = a_s} A_{\pi_1} N_{\pi_1} A_{\pi_2} N_{\pi_2} \ldots A_{\pi_s} N_{\pi_s}. \tag{31}$$

Now at least one of the $a_k$'s, say $a_1$, is $> 1$, and we have

$$\sum_{|\pi_1| = a_1} A_{\pi_1} N_{\pi_1} = \sum_{|\pi_1| = a_1} \frac{a_1!}{z_{\pi_1}} \chi_{\pi_1}^{\{1^{a_1}\}},$$

where $\chi$ denotes the character of the symmetric group $\mathfrak{S}_{a_1}$. By the character relations of $\mathfrak{S}_{a_1}$, this sum is 0, and hence also (31) is 0. This finishes the proof of the lemma and also the proof of the theorem.

5. Let us consider the special case that $n$ is an odd prime. In this case it is easy to see that the conditions (11) are satisfied for a prime $r$ if and only if $r = n$ and $q \equiv -1 \pmod{n}$. Hence the system $\mathcal{S}'_4$ consists of only one group $\mathfrak{N}\,\mathfrak{Z}$, where $\mathfrak{N}$ is the $n$-Sylowgroup of $\mathfrak{U}_n$. We also have

$$\frac{(-q)^n - 1}{q + 1} \equiv -n \pmod{n^2}.$$

Using the notation (7), we write

$$X_k = \operatorname{diag}(1, 1, \ldots, \omega_1, \ldots, 1) \qquad (1 \leqq k \leqq n),$$

where $X_k$ is a $n$ by $n$ diagonal matrix, the element in the $k$-th row and column being $\omega_1$, the other diagonal elements are 1. We also denote

$$Y = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Then $\mathfrak{N}\,\mathfrak{Z}$ is contained in the group $\mathfrak{B} = \{X_1, X_2, \ldots, X_n, Y\}$. This group is determined by the following defining relations

$$\begin{cases} X_k^{q+1} = 1, \quad X_k X_j = X_j X_k \quad (1 \leqq k, j \leqq n), \\ Y X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n} Y^{-1} = X_1^{a_2} X_2^{a_3} \ldots X_{n-1}^{a_n} X_n^{a_1}. \end{cases}$$

The elements of $\mathfrak{B}$ are divided into classes of conjugate elements as follows:

| Notation | Elements | No. of elements in the conj. class | Range of the parameters | No. of classes of the type in question |
|---|---|---|---|---|
| $C_1^{(k)}$ | $\{X_1^k X_2^k \ldots X_n^k\}$ | 1 | $k = 1, \ldots, q+1$ | $q+1$ |
| $C_2^{(a_1, a_2, \ldots, a_n)}$ | $\{X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n}$, $X_1^{a_2} X_2^{a_3} \ldots X_n^{a_1}$, $\ldots\ldots$ $X_1^{a_n} X_2^{a_1} \ldots X_n^{a_{n-1}}\}$ | $n$ | $a_i = 1, \ldots, q+1$ for $i = 1, \ldots, n$; not all $a_i$'s are equal | $\frac{1}{n}[(q+1)^n - q - 1]$ |
| $C_3^{(k, l)}$ | $\{Y^k X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n}\}$, where $l \equiv a_1 + \ldots + a_n$ (mod $q+1$) | $(q+1)^{n-1}$ | $k = 1, \ldots, n-1$, $l = 1, \ldots, q+1$ | $(n-1)(q+1)$ |

One sees immediately that the system of irreducible characters of $\mathfrak{B}$ is the following one:

| Class | $\psi_1^{(t, u)}$ $t = 1, 2, \ldots, n$ $u = 1, 2, \ldots, q+1$ | $\psi_n^{(b_1, b_2, \ldots b_n)}$ $b_i = 1, 2, \ldots, q+1$ for $i = 1, \ldots, n$; not all $b_i$'s are equal |
|---|---|---|
| $C_1^{(k)}$ | $\varepsilon^{nuk}$ | $n\, \varepsilon^{(b_1 + b_2 + \ldots + b_n) k}$ |
| $C_2^{(a_1, a_2, \ldots, a_n)}$ | $\varepsilon^{u(a_1 + a_2 + \ldots + a_n)}$ | $\sum\limits_{(i_1, \ldots, i_n)} \varepsilon^{b_{i_1} a_1 + b_{i_2} a_2 + \ldots + b_{i_n} a_n}$ |
| $C_3^{(k, l)}$ | $\varepsilon^{\frac{q+1}{n} tk + ul}$ | 0 |

Here $\varepsilon = \Theta(\omega_1)$ is a primitive $(q+1)$-st root of 1, and in $\sum\limits_{(i_1, \ldots, i_n)}$ the sum is over all systems $(i_1, i_2, \ldots, i_n)$ such that $\begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix}$ goes all the $n$ permutations $\begin{pmatrix} 1 & 2 & \ldots & n \\ 2 & 3 & \ldots & 1 \end{pmatrix}^j$ $(j = 1, 2, \ldots, n)$. The characteristic polynomial of an element belonging to class $C_3^{(k, l)}$ is

$$x^n - \omega_1^l ,$$

which splits into distinct linear factors in $\mathfrak{F}$, if $l \equiv 0 \pmod{n}$, but is irreducible in $\mathfrak{F}$, if $l \not\equiv 0 \pmod{n}$. Now it is easy to verify the truth of the following

**Theorem 3.** *The restriction of a basic C-function of type $\{n\}$ to $\mathfrak{B}$ is a character of $\mathfrak{B}$.*

*Proof.* For each $B^{\varrho}(h^{\varrho})$ we write

$$D_{\varrho}(h^{\varrho}\,;t\,,u) = \frac{1}{n(q+1)^n} \sum_{G\in\mathfrak{B}} B^{\varrho}(h^{\varrho})\,(G)\,\overline{\psi_1^{(t\,,\,u)}}\,(G)$$

and

$$D_{\varrho}(h^{\varrho}\,;b_1\,,b_2\,,\ldots,b_n) = \frac{1}{n(q+1)^n} \sum_{G\in\mathfrak{B}} B^{\varrho}(h^{\varrho})\,(G)\,\overline{\psi_n^{(b_1,\,b_2,\ldots,\,b_n)}}\,(G)\,.$$

We also put

$$\delta_a(b) = \begin{cases} 0\,, & \text{if } a \not\equiv 0 \pmod{b}\,,\\ 1\,, & \text{if } a \equiv 0 \pmod{b}\,.\end{cases}$$

By a straightforward computation we get (writing shortly $h$ instead of $h_{n1}$)

$$D_{\{n\}}(h^{\{n\}}\,;t\,,u) = \left[\frac{1}{n}\left(1+\frac{\Phi_{n-1}(-q)}{(q+1)^{n-1}}\right) - \delta_t(n)\right]\delta^1_{h\,,\,nu}$$
$$+\;(n\,\delta_t(n)-1)\,\delta_{h-nu}\big(n\,(q+1)\big)$$

and

$$D_{\{n\}}(h^{\{n\}}\,;b_1\,,b_2\,,\ldots,b_n) = \frac{\Phi_{n-1}(-q)}{(q+1)^{n-1}}\,\delta^1_{h,\,b_1+b_2+\ldots+b_n}\,.$$

Clearly, these expressions are both integers, and hence our theorem is proved.

6. **Characters of $\mathfrak{U}_2$.** The elements of $\mathfrak{U}_2$ are divided into classes of conjugate elements as follows:

| Notation | Canonical form in GL $(2, q^2)$ | No. of elements in the conj. class | Range of the parameters | No. of classes of the type in question |
|---|---|---|---|---|
| $C_1^{(k)}$ | $\begin{pmatrix}\varrho^{k(q-1)} & \\ & \varrho^{k(q-1)}\end{pmatrix}$ | 1 | $k=1\,,\ldots,q+1$ | $q+1$ |
| $C_2^{(k)}$ | $\begin{pmatrix}\varrho^{k(q-1)} & \\ 1 & \varrho^{k(q-1)}\end{pmatrix}$ | $q^2-1$ | $k=1\,,\ldots,q+1$ | $q+1$ |
| $C_3^{(k\,,\,l)}$ | $\begin{pmatrix}\varrho^{k(q-1)} & \\ & \varrho^{l(q-1)}\end{pmatrix}$ | $q(q-1)$ | $k\,,l=1\,,\ldots,q+1$, $k\neq l$; $C_3^{(k\,,\,l)}=C_3^{(l\,,\,k)}$ | $\frac{1}{2}\,q(q+1)$ |
| $C_4^{(k)}$ | $\begin{pmatrix}\varrho^{k} & \\ & \varrho^{-qk}\end{pmatrix}$ | $q(q+1)$ | $k=1\,,\ldots,q^2-2$, $k\not\equiv 0 \pmod{q-1}$ $C_4^{(k)}=C_4^{(-qk)}$ | $\frac{1}{2}\,(q^2-q-2)$ |

Here $\varrho$ denotes a primitive element of $\mathfrak{F}$. Let $\eta$ be a primitive $(q^2-1)$-st root of 1 and $\varepsilon = \eta^{q-1}$. Our assertion is that the table of irreducible characters of $\mathfrak{U}_2$ is the following one:

| Class | $\chi_1^{(t)}$ | $\chi_q^{(t)}$ | $\chi_{q-1}^{(t,u)}$ | $\chi_{q+1}^{(t)}$ |
|---|---|---|---|---|
| | $t = 1, \ldots, q+1$ | $t = 1, \ldots, q+1$ | $t, u = 1, \ldots, q+1$ $t \neq u$, $\chi_{q-1}^{(t,u)} = \chi_{q-1}^{(u,t)}$ | $t = 1, \ldots, q^2-2$, $t \not\equiv 0 \pmod{q-1}$; if $t_1 \equiv -tq$ $(\mathrm{mod}\ q^2-1)$, then $\chi_{q+1}^{(t_1)} = \chi_{q+1}^{(t)}$ |
| $C_1^{(k)}$ | $\varepsilon^{2tk}$ | $q\,\varepsilon^{2tk}$ | $(q-1)\,\varepsilon^{(t+u)k}$ | $(q+1)\,\varepsilon^{tk}$ |
| $C_2^{(k)}$ | $\varepsilon^{2tk}$ | $0$ | $-\varepsilon^{(t+u)k}$ | $\varepsilon^{tk}$ |
| $C_3^{(k,l)}$ | $\varepsilon^{t(k+l)}$ | $-\varepsilon^{t(k+l)}$ | $-\varepsilon^{tk-ul}-\varepsilon^{uk-tl}$ | $0$ |
| $C_4^{(k)}$ | $\varepsilon^{-tk}$ | $\varepsilon^{-tk}$ | $0$ | $\eta^{tk}+\eta^{-tkq}$ |

The most direct proof goes as follows. It is clear that the $\chi_1^{(t)}$'s are characters of $\mathfrak{U}_2$. The transformations which are represented with respect to the hyperbolic basis by matrices of the form

$$\begin{pmatrix} a & 0 \\ b & \bar{a}^{-1} \end{pmatrix}, \quad \text{with} \quad a\bar{b} + \bar{a}b = 0\,,$$

belong to $\mathfrak{U}_2$, and they form a subgroup of $\mathfrak{U}_2$ of order $q(q^2-1)$, which we denote by $\mathfrak{H}$. It is, in fact, the normalizer of a $p$-Sylowgroup of $\mathfrak{U}_2$ ($q$ = power of $p$). Let $\psi^{(t)}$ be the character of $\mathfrak{H}$ defined by

$$\psi^{(t)}\left(\begin{pmatrix} \varrho^k & 0 \\ b & \varrho^{-qk} \end{pmatrix}\right) = \eta^{tk} \qquad (t = 1, 2, \ldots, q^2-1)\,.$$

Then the character of $\mathfrak{U}_2$ induced by $\psi^{(t)}$ is

$$\psi^{(t)*} = \chi_{q+1}^{(t)}\,.$$

By Theorem 1 of Green ([6], p. 403), the following classfunction is a character of $\mathfrak{U}_2$

$$\varphi^{(s)}(x) = \begin{cases} 2\,\varepsilon^{sk} & , \ \text{if} \ x \in C_1^{(k)}\,, \\ 2\,\varepsilon^{sk} & , \ \text{if} \ x \in C_2^{(k)} \\ \varepsilon^{sk} + \varepsilon^{sl} & , \ \text{if} \ x \in C_3^{(k,l)} \\ \eta^{sk} + \eta^{-skq} & , \ \text{if} \ x \in C_4^{(k)}\,. \end{cases}$$

Then

$$\chi_{q-1}^{(t,u)} = \chi_1^{(u)}\left(\chi_{q+1}^{(t-u)} - \varphi^{(t-u)}\right) \qquad (t, u = 1, 2, \ldots, q+1)\,.$$

7. **Characters of $\mathfrak{U}_3$.** The elements of $\mathfrak{U}_3$ are divided into classes of conjugate elements as follows:

| Notation | Canonical form in GL $(2, q^2)$ | No. of elements in the conj. class | Range of the parameters | No. of classes of the type in question |
|---|---|---|---|---|
| $C_1^{(k)}$ | $\begin{pmatrix} \varrho^{k(q-1)} & & \\ & \varrho^{k(q-1)} & \\ & & \varrho^{k(q-1)} \end{pmatrix}$ | $1$ | $k = 1, \ldots, q+1$ | $q+1$ |
| $C_2^{(k)}$ | $\begin{pmatrix} \varrho^{k(q-1)} & & \\ 1 & \varrho^{k(q-1)} & \\ & & \varrho^{k(q-1)} \end{pmatrix}$ | $(q-1)(q^3+1)$ | $k = 1, \ldots, q+1$ | $q+1$ |
| $C_3^{(k)}$ | $\begin{pmatrix} \varrho^{k(q-1)} & & \\ 1 & \varrho^{k(q-1)} & \\ & 1 & \varrho^{k(q-1)} \end{pmatrix}$ | $q(q^2-1)(q^3+1)$ | $k = 1, \ldots, q+1$ | $q+1$ |
| $C_4^{(k,l)}$ | $\begin{pmatrix} \varrho^{k(q-1)} & & \\ & \varrho^{k(q-1)} & \\ & & \varrho^{l(q-1)} \end{pmatrix}$ | $q^2(q^2-q+1)$ | $k, l = 1, \ldots, q+1$<br>$k \neq l$ | $q(q+1)$ |
| $C_5^{(k,l)}$ | $\begin{pmatrix} \varrho^{k(q-1)} & & \\ 1 & \varrho^{k(q-1)} & \\ & & \varrho^{l(q-1)} \end{pmatrix}$ | $q^2(q-1)(q^3+1)$ | $k, l = 1, \ldots, q+1,$<br>$k \neq l$ | $q(q+1)$ |
| $C_6^{(k,l,m)}$ | $\begin{pmatrix} \varrho^{k(q-1)} & & \\ & \varrho^{l(q-1)} & \\ & & \varrho^{m(q-1)} \end{pmatrix}$ | $q^3(q-1)(q^3-q+1)$ | $k, l, m = 1, \ldots, q+1,$<br>$k < l < m$ | $\frac{1}{6}\, q(q^2-1)$ |
| $C_7^{(k,l)}$ | $\begin{pmatrix} \varrho^{k(q-1)} & & \\ & \varrho^{l} & \\ & & \varrho^{-ql} \end{pmatrix}$ | $q^3(q^3-1)$ | $k = 1, \ldots, q+1,$<br>$l = 1, \ldots, q^2-2,$<br>$l \not\equiv 0 \pmod{q-1}$;<br>if $l_1 \equiv -ql$<br>$\pmod{q^2-1}$, then<br>$C_7^{(k,l_1)} = C_7^{(k,l)}$ | $\frac{1}{2}(q-1)^2(q-2)$ |
| $C_8^{(k)}$ | $\begin{pmatrix} \tau^{k(q^3-1)} & & \\ & \tau^{q^2k(q^3-1)} & \\ & & \tau^{q^4k(q^3-1)} \end{pmatrix}$ | $q^3(q+1)^2(q-1)$ | $k = 1, \ldots, q^3,$<br>$k \not\equiv 0$<br>$\pmod{q^2-q+1}$;<br>if $k_1 \equiv q^2k$ or<br>$k_2 \equiv q^4k$<br>$\pmod{q^3+1}$, then<br>$C_8^{(k)} = C_8^{(k_1)} = C_8^{(k_2)}$ | $\frac{1}{3}\, q(q^2-1)$ |

Here $\tau$ denotes a primitive element of $GF(q^6)$ and $\varrho = \tau^{q^4+q^2+1}$ is a primitive element of $\mathfrak{F}$. Let $\zeta$ be a primitive $(q^3+1)$-st root of 1 and $\eta$ and $\varepsilon$ be primitive $(q^2-1)$-st and $(q+1)$-st roots of 1, respectively, such that

$$\varepsilon = \zeta^{q^2-q+1} = \eta^{q-1} .$$

Our assertion is that the table of irreducible characters of $\mathfrak{U}_3$ is as given in the following two tables.

| Class | $\chi_1^{(t)}$ | $\chi_{q^2-q}^{(t)}$ | $\chi_{q^3}^{(t)}$ | $\chi_{q^2-q+1}^{(t,u)}$ |
|---|---|---|---|---|
| | $t=1,\ldots,q+1$ | $t=1,\ldots,q+1$ | $t=1,\ldots,q+1$ | $t,u=1,\ldots,q+1,$ $t\neq u$ |
| $C_1^{(k)}$ | $\varepsilon^{3tk}$ | $(q^2-q)\,\varepsilon^{3tk}$ | $q^2\,\varepsilon^{3tk}$ | $(q^2-q+1)\,\varepsilon^{(t+2u)k}$ |
| $C_2^{(k)}$ | $\varepsilon^{3tk}$ | $-q\,\varepsilon^{3tk}$ | $0$ | $-(q-1)\,\varepsilon^{(t+2u)k}$ |
| $C_3^{(k)}$ | $\varepsilon^{3tk}$ | $0$ | $0$ | $\varepsilon^{(t+2u)k}$ |
| $C_4^{(k,l)}$ | $\varepsilon^{t(2k+l)}$ | $-(q-1)\,\varepsilon^{t(2k+l)}$ | $q\,\varepsilon^{t(2k+l)}$ | $-(q-1)\,\varepsilon^{(t+u)k+ul}$ $+\,\varepsilon^{2uk+tl}$ |
| $C_5^{(k,l)}$ | $\varepsilon^{t(2k+l)}$ | $\varepsilon^{t(2k+l)}$ | $0$ | $\varepsilon^{(t+u)k+ul}$ $+\,\varepsilon^{2uk+tl}$ |
| $C_6^{(k,l,m)}$ | $\varepsilon^{t(k+l+m)}$ | $2\,\varepsilon^{t(k+l+m)}$ | $-\,\varepsilon^{t(k+l+m)}$ | $\sum\limits_{(k,l,m)} \varepsilon^{tk+u(l+m)}$ |
| $C_7^{(k,l)}$ | $\varepsilon^{t(k-l)}$ | $0$ | $\varepsilon^{t(k-l)}$ | $\varepsilon^{tk-ul}$ |
| $C_8^{(k)}$ | $\varepsilon^{tk}$ | $-\,\varepsilon^{tk}$ | $-\,\varepsilon^{tk}$ | $0$ |

By $\sum\limits_{(x,y,z)}$ we mean a sum over the cyclic permutations of $x, y, z$ and $\sum\limits_{\{x,y,z\}}$ means a sum over all permutations of $x, y, z$.

The case of the linear characters is clear. The transformations which are represented with respect to the hyperbolic basis by matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & \bar{a}^{-1} \end{pmatrix}, \quad \text{with} \quad \begin{cases} c\bar{c} = 1 , \\ a\bar{e} + b\bar{c} = 0 , \\ a\bar{d} + \bar{a}d + b\bar{b} = 0 , \end{cases}$$

| Class | $\chi^{(t,u)}_{q(q^2-q+1)}$ $t,u=1,\dots,q+1,$ $t\neq u$ | $\chi^{(t,u,v)}_{(q-1)(q^2-q+1)}$ $t,u,v=1,\dots,q+1,$ $t<u<v$ | $\chi^{(t,u)}_{q^3+1}$ $t=1,\dots,q+1,$ $u=1,\dots,q^2-2,$ $u\not\equiv 0 \pmod{q-1}$; if $u_1\equiv -uq \pmod{q^2-1}$, then $\chi^{(t,u)}=\chi^{(t,u_1)}$ | $\chi^{(t)}_{(q+1)(q^2-1)}$ $t=1,\dots,q^3,$ $t\not\equiv 0 \pmod{q^2-q+1}$; if $t_1\equiv tq^2$ or $t_2\equiv tq^4 \pmod{q^3+1}$, then $\chi^{(t)}=\chi^{(t_1)}=\chi^{(t_2)}$ |
|---|---|---|---|---|
| $C_1^{(k)}$ | $q(q^2-q+1)\,\varepsilon^{(t+2u)k}$ | $(q-1)(q^2-q+1)$ $\varepsilon^{(t+u+v)k}$ | $(q^3+1)\,\varepsilon^{(t+u)k}$ | $(q+1)(q^2-1)\,\varepsilon^{tk}$ |
| $C_2^{(k)}$ | $q\,\varepsilon^{(t+2u)k}$ | $(2q-1)\,\varepsilon^{(t+u+v)k}$ | $\varepsilon^{(t+u)k}$ | $-(q+1)\,\varepsilon^{tk}$ |
| $C_3^{(k)}$ | $0$ | $-\varepsilon^{(t+u+v)k}$ | $\varepsilon^{(t+u)k}$ | $-\varepsilon^{tk}$ |
| $C_4^{(k,l)}$ | $(q-1)\,\varepsilon^{(t+u)k+ul}$ $+\,q\,\varepsilon^{2uk+tl}$ | $(q-1)\sum_{(t,u,v)}\varepsilon^{(t+u)k+vl}$ | $(q+1)\,\varepsilon^{uk+tl}$ | $0$ |
| $C_5^{(k,l)}$ | $-\varepsilon^{(t+u)k+ul}$ | $-\sum_{(t,u,v)}\varepsilon^{(t+u)k+vl}$ | $\varepsilon^{uk+tl}$ | $0$ |
| $C_6^{(k,l,m)}$ | $-\sum_{(k,l,m)}\varepsilon^{tk+u(l+m)}$ | $-\sum_{\{t,u,v\}}\varepsilon^{tk+ul+vm}$ | $0$ | $0$ |
| $C_7^{(k,l)}$ | $\varepsilon^{tk-ul}$ | $0$ | $\varepsilon^{tk}(\eta^{ul}+\eta^{-qul})$ | $0$ |
| $C_8^{(k)}$ | $0$ | $0$ | $0$ | $-\zeta^{tk}-\zeta^{tkq^2}-\zeta^{tkq^4}$ |

belong to $\mathfrak{U}_3$, and they form a subgroup of $\mathfrak{U}_3$ of order $q^3(q+1)(q^2-1)$, which we denote again by $\mathfrak{H}$. It is the normalizer of a $p$-Sylowgroup of $\mathfrak{U}_3$. Let $\psi^{(t,u)}$ be the character of $\mathfrak{H}$ defined by

$$\psi^{(t,u)}\left(\begin{pmatrix}\varrho^l & 0 & 0\\ b & \varrho^{k(q-1)} & 0\\ d & e & \varrho^{-ql}\end{pmatrix}\right)=\varepsilon^{tk}\,\eta^{ul}\qquad \begin{matrix}(t=1,2,\dots,q+1,\\ u=1,2,\dots,q^2-1).\end{matrix}$$

Then the character of $\mathfrak{U}_3$ induced by $\psi^{(t,u)}$ is $\chi^{(t,u)}_{q^3+1}$. This character is irreducible, if $u\not\equiv 0 \pmod{q-1}$. We also have

$$\chi^{(t)}_{q^3}=\chi^{(t)}_1\,(\chi^{(q+1,q^2-1)}_{q^3+1}-\chi^{(q+1)}_1)\qquad (t=1,2,\dots,q+1).$$

Next, we have to prove that

$$\begin{aligned}&\chi^{(t,u)}_{q^2-q+1} && (t,u=1,2,\dots,q+1),\\ &\chi^{(t,u,v)}_{(q-1)(q^2-q+1)} && (t,u,v=1,2,\dots,q+1),\\ &\chi^{(t)}_{(q+1)(q^2-1)} && (t=1,2,\dots,q^3+1)\end{aligned}$$

are characters of $\mathfrak{U}_3$. We use the method indicated in § 3. Taking Theorem 2 into account, we see that it is enough to show that these functions are characters of the following subgroups of $\mathfrak{U}_3$: $\mathfrak{P} \times \mathfrak{Z}$, $\mathfrak{U}_1 \times \mathfrak{U}_2$, and $\mathfrak{B}$. (That the functions of the third type are characters of $\mathfrak{B}$ has already been proved in Theorem 3.) We omit the straightforward verification, that this is the case. Finally, we have

$$\chi^{(t)}_{q^2-q} = \chi^{(t(q^2-q+1))}_{(q+1)(q^2-1)} - \chi^{(t)}_{q^3} + \chi^{(t)}_1 \quad (t = 1, 2, \ldots, q+1),$$

$$\chi^{(t,u)}_{q(q^2-q+1)} = \chi^{(t,u,u)}_{(q-1)(q^2-q+1)} + \chi^{(t,u)}_{q^2-q+1} \quad (t, u = 1, 2, \ldots, q+1).$$

## Appendix

8. Let $N(s, q^2)$ denote the number of distinct $U$-irreducible polynomials $f \in \mathfrak{F}$ of degree $s$, which, by Lemma 3, is the same as the number of simplexes of degree $s$.

**Theorem 4.** $N(s, q^2) = \frac{1}{s}\left[\sum_{k \mid s} \mu(k) q^{\frac{s}{k}} - c_s\right]$, *where*

$$c_s = \sum_{k \mid s} \mu(k)(-1)^{\frac{s}{k}} = \begin{cases} -1, & \text{if } s = 1, \\ 2, & \text{if } s = 2, \\ 0 & \text{otherwise}. \end{cases}$$

**Corollary.** *The number of distinct $U$-irreducible polynomials of degree $s$ over $\mathrm{GF}(q^2)$ equals the number of distinct irreducible polynomials of degree $s$ over $\mathrm{GF}(q)$, except in the cases $s = 1, s = 2$.*

The proof of this fact follows immediately if we take $a = 0$ in the formula (12′).

If we denote the generating functions for the number of classes of $\mathrm{GL}(n, q)$ and $\mathrm{U}(n, q^2)$ by $\vartheta_G(q, x)$ and $\vartheta_U(q^2, x)$, respectively, then it follows from Theorem 4, by the formula of Green ([6], p. 408), that

$$\frac{\vartheta_G(q, x)}{\vartheta_U(q^2, x)} = \prod_{i=1}^{\infty} \frac{1 - x^i}{1 + x^i},$$

in accordance with the result of Wall ([11]).

9. If one wants to apply the procedure of § 3, it seems that one of the most difficult subgroups to consider is the $p$-Sylowgroup $\mathfrak{P}$. It is therefore desirable to study more thoroughly the structure and representations of this group, especially, compared with those of the $p$-Sylowgroup of $\mathrm{GL}(n, q)$.

Let $\mathfrak{P}_3^G$ and $\mathfrak{P}_3^U$ be the $p$-Sylowgroups of $\mathrm{GL}(3, q)$ and $\mathfrak{U}_3$, respectively. Then the following is valid.

**Theorem 5.** *$\mathfrak{P}_3^G$ and $\mathfrak{P}_3^U$ are isomorphic if and only if $p > 2$.*

*Proof.* We can take

$$\mathfrak{P}_3^G = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \middle| \; a, b, c \in \mathrm{GF}(q) \right\}$$

and

$$\mathfrak{P}_3^U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & -\bar{a} & 1 \end{pmatrix} \middle| \; \begin{array}{l} a, b \in \mathrm{GF}(q^2), \\ b + \bar{b} + a\bar{a} = 0 \end{array} \right\}.$$

If $p = 2$, then $\mathfrak{P}_3^G$ contains more involutions than $\mathfrak{P}_3^U$. Suppose $p > 2$. Take $0 \neq \alpha \in \mathrm{GF}(q^2)$ such that $\alpha + \bar{\alpha} = 0$. Then the mapping $\sigma : \mathfrak{P}_3^G \to \mathfrak{P}_3^U$, defined by

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \xrightarrow{\sigma} \begin{pmatrix} 1 & 0 & 0 \\ a + c\alpha & 1 & 0 \\ -\frac{1}{2}(a^2 - \alpha^2 c^2) + (2b - ac)\alpha & -a + c\alpha & 1 \end{pmatrix},$$

gives the required isomorphism.

Let $\mathfrak{P}_n = \mathfrak{P}_n^G$ be the $p$-Sylowgroup of $\mathrm{GL}(n, q)$. We may take $\mathfrak{P}_n$ to be the group of matrices of the form

$$A = \begin{pmatrix} 1 & 0 & 0 \ldots 0 \\ a_{21} & 1 & 0 \ldots 0 \\ a_{31} & a_{32} & 1 \ldots 0 \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots} \\ a_{n1} & a_{n2} & \ldots\ldots 1 \end{pmatrix}, \tag{32}$$

We define

$$\varkappa(A) = \dim \ker (A - \mathrm{I}).$$

From the results of Green ([6], p. 431) it follows that the function $\chi$ defined by

$$\chi(A) = \Phi_{\varkappa(A)-1}(q)$$

is a character of $\mathfrak{P}_n$. In what follows we give a direct simple proof of this fact.

For $n = 1$ our assertion is trivial. Suppose it is true for $\mathfrak{P}_{n-1}$. We denote by $\mathfrak{K}$ the group of matrices (32) with $a_{ij} = 0$ for $j \geqq 2$, and by $\mathfrak{P}$ the group of matrices (32) with $a_{i1} = 0$ for $i = 2, \ldots, n$. Then $\mathfrak{K}$ and $\mathfrak{P}$ are subgroups of $\mathfrak{P}_n$, $\mathfrak{K}$ is normal in $\mathfrak{P}_n$, and $\mathfrak{P}_n$ is a semidirect product of $\mathfrak{K}$ and $\mathfrak{P}$. Furthermore, $\mathfrak{P}$ is isomorphic to $\mathfrak{P}_{n-1}$. For $A \in \mathfrak{P}_n$ we have a unique decomposition

$$A = K P , \qquad K \in \mathfrak{K} , P \in \mathfrak{P} .$$

Now the induction assumption gives us a character of $\mathfrak{P}$, which we denote by $\chi'$. Since $\mathfrak{P} \cong \mathfrak{P}_n/\mathfrak{K}$ we can consider $\chi'$ as a character of $\mathfrak{P}_n$ and we have

$$\chi'(\mathrm{A}) = \Phi_{\varkappa(P)-2}(q) .$$

On the other hand, let $\chi'^*$ denote the character of $\mathfrak{P}_n$ induced by the character $\chi'$ of the subgroup $\mathfrak{P}$. By the formula of Frobenius, we have

$$\chi'^*(A) = \sum_{G \in \mathfrak{K}} \chi'_0(GAG^{-1}) .$$

By means of elementary vector space theory, one can verify that there exist matrices $G \in \mathfrak{K}$ such that $GAG^{-1} \in \mathfrak{P}$ if and only if

$$\mathrm{im}\,(A - \mathrm{I}) = \mathrm{im}\,(P - \mathrm{I})$$

and that in this case the number of such matrices is

$$q^{\varkappa(A)-1} .$$

Hence

$$\chi = \chi' - \chi'^* .$$

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