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ON A METHOD OF EXTREMALIZATION IN THE CLASS S_k OF UNIVALENT FUNCTIONS

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On a method of extremalization in the class S_k of univalent functions

1. Let S be the class of normalized univalent functions

$$f(z) = z + a_2 z^2 + \dots \qquad (\,|z| < 1;\ f(0) = f'(0) - 1 = 0)\,.$$

All the known extremalization methods in this class encounter great difficulties in calculation when applied to higher coefficients. With a view to comprehension of the essential nature of these difficulties, and further to develop these methods, a subclass S_k of the general class S has been chosen. This class consists of univalent functions with bounded boundary-rotation. The problems in this class appear to be very similar to those of class S. However, the much simpler structure of class S_k makes the corresponding extremal problems markedly easier of solution. The methods developed for these simpler questions can provide ideas and inspiration for questions of more general type.

Our most recent study concerning extremalization in class S_k was [3]. In the present paper, the notations adopted there are employed. The extremal task was expressed in terms of Lagrange's coefficient λ. The conditions found were interpreted as equations of complex variables. This interpretation gave us the opportunity of using the operation of conjugation in simplifying the equations found. It was noticed that the necessary conditions implied the symmetry of the extremal domain in the non-trivial case. — Here, the trivial case means, that the extremal domain is a polygon with only two points. In this case, the corresponding extremal problem contains only two variables t_1 and t_2.

This extremalization method seems worthy of further consideration. In the following we will present still another way of eliminating variables in the necessary conditions (6) and (10) of [3].

2. The necessary conditions (6) — (10) form the following system of equations:

$$\begin{cases} s\,t_\mu + t_\mu^2 = \bar{s}\,t_\mu^{-1} + t_\mu^{-2}\,, \\ 2\,s\,t_\mu + t_\mu^2 + 2\,\bar{s}\,t_\mu^{-1} + t_\mu^{-2} - 2\,\lambda = 0\,. \end{cases}$$

Denote by t an arbitrary number belonging to the solution

$$t_1 , \ldots , t_N \qquad (|t_\nu| = 1) .$$

The necessary conditions for t read

$$f_1(t) = t^4 + s\,t^3 - \bar{s}\,t - 1 = 0 , \tag{1}$$

$$f_2(t) = t^4 + 2\,s\,t^3 - 2\,\lambda\,t^2 + 2\,\bar{s}\,t + 1 = 0 . \tag{2}$$

Here

$$s = \sum_1^N A_\nu\, t_\nu$$

and

$$\lambda = \lambda_1 , \lambda_2$$

are constants characterizing a fixed solution.

We derive new necessary conditions for the number t, writing

$$\begin{cases} F(t) = \alpha(t)\, f_1(t) + \beta(t)\, f_2(t) = 0 , \\ \overline{F(t)} = 0 . \end{cases} \tag{3}$$

Here $\alpha(t)$ and $\beta(t)$ are arbitrary finite functions, and $\overline{F(t)}$ signifies the conjugate number of $F(t)$. It is clear that all the roots of the system (3) need not satisfy the original system (1) — (2). However, it appears that the operations (3) are very easy to use, and they give for the unit root t new necessary conditions which strongly restrict its properties. These conditions applied to the corresponding domains appear to be strong enouch for the purposes of the extremalization

By repeated use of the operations (3), we derive the following necessary condition for the unit root t of the system (1) — (2)

$$K_1(s , \lambda)\, t + K_2(s , \lambda) = 0 , \tag{4}$$

where

$$K_1(s , \lambda) = (\lambda\,\bar{s} + 3\,s)\left(\frac{3}{2}\,s^2 - 2\,\lambda\right) - (\lambda s + 3\,\bar{s})\left(\frac{s\,\bar{s}}{2} - 2\right),$$

$$K_2(s , \lambda) = \left(\frac{s\,\bar{s}}{2} - 2\right)^2 - \left(\frac{3}{2}\,s^2 - 2\,\lambda\right)\left(\frac{3}{2}\,\bar{s}^2 - 2\,\lambda\right).$$

It can be seen immediately that for fixed λ, the number t is uniquely determined, except in the case when

$$K_1(s , \lambda) = K_2(s , \lambda) = 0 . \tag{5}$$

3. It appears, that this system implies the symmetry of the possible extremal domain relating to the solution of (1) — (2). The system (5) gives

$$\left(\frac{3}{2}\, s^2 - 2\,\lambda\right) e^{-i\omega} = 2 - \frac{|s|^2}{2} > 0 \qquad (k < 4)\,, \tag{6}$$

$$(\lambda\,\bar{s} + 3\,s)e^{i\omega} + (\lambda\,s + 3\,\bar{s}) = 0\,. \tag{7}$$

Here, ω is an undetermined constant. Two different possibilities of satisfying (7) exist:

I. $$\begin{cases} \lambda\,s + 3\,\bar{s} = 0\,, \\ 3\,s + \lambda\,\bar{s} = 0\,. \end{cases}$$

If $s = \bar{s} = 0$, we derive from the system (1) — (2)

$$\begin{cases} \lambda^2 = 1\,, \\ t^4 = 1\,. \end{cases}$$

In this case the corresponding domain is symmetrical and in fact the system (1) — (2) has solutions. However, the corresponding function $f(z)$ is, not the extremal function in this case. — An analogous phenomeneon was met in the general class S (cf. [1] and [2]).

If s and $\bar{s} \neq 0$, $\lambda = \pm\,3$ and the symmetry conditions

$$s = \pm\,\bar{s}$$

are valid.

II. $$\begin{cases} \lambda\,s + 3\,\bar{s} \neq 0\,, \\ 3\,s + \lambda\,\bar{s} \neq 0\,. \end{cases}$$

In this case, it can be seen in the same way that $\lambda = \pm\,3$, which implies the symmetry conditions $s = \pm\,\bar{s}$.

4. Thus we are repeatedly led to the result:

In the non-trivial case, the extremal domain for the coefficient a_3 in the class S_k is necessarily symmetrical.

The principal intention of the procedure presented is that of demonstrating that the operations (3) give an astonishingly simple means of deriving the necessary condition (4) for the solution of the system (1) — (2). Thus the difficulties encountered when using two variables t_1, t_2 in [3] can be avoided.

References.

[1] TAMMI, O.: On the maximalization of the coefficient a_3 of bounded schlicht functions. - Ann. Acad. Sci. Fenn. Ser. A. I n:o 149 (1953).

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