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ON FUNCTIONS OF CLASS *U*

BY

CHUJI TANAKA

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On functions of class U

1. Introduction. We begin with

Definition. *Let* $f(z)$ $(z=re^{i\theta})$ *be regular and bounded:* $|f(z)|<1$ *in* $|z|<1$. *If* $|f(e^{i\theta})|=1$ *almost everywhere on the arc* $A(e^{i\theta}: \Theta_1<\Theta<\Theta_2)$, *then we call* $f(z)$ *a function of class* $U(\Theta_1, \Theta_2)$. *Instead of* $U(0,2\pi)$, *we write simply* U.

R. Nevanlinna [14] was the first to point out the interest which lies in the class U. G. Hössjer [6], W. Seidel [24] and O. Frostman [2—4] have made important contributions to the theory of class U, which was extended in various directions by many authors. (K. Noshiro [16—18], M. Tsuji [26], A. J. Lohwater [10—12], M. Ohtsuka [20,21], O. Lehto [7—9], D. A. Storvick [25]).

In this note, we shall establish some new properties of functions of class U or $U(\Theta_1, \Theta_2)$. Our main theorems read as follows:

Theorem 1. *Suppose that* $f(z)\in U$, *and it has at least one singular point on* $|z|=1$. *Then the following propositions hold:*

(1) *the set* S *of singularities of* $f(z)$ *lying on* $|z|=1$ *coincides with the set of the limit points of the* $\alpha-$ *points* $(|\alpha|<1)$ *of* $f(z)$, *except for a set of values* α *of capacity zero.*

(2) $f(z)$ *is meromorphic in* $|z|\leqq\infty$, *except for the set* S *on* $|z|=1$, *which is of linear measure zero.*

Theorem 2. *Let* $f(z)$ $(z=re^{i\theta})$ *belong to class* $U(\Theta_1, \Theta_2)$. *If* $f(z)$ *is not regular on the arc* $A(e^{i\theta}: \Theta_1<\Theta<\Theta_2)$, *then the following propositions hold:*

(1) *the set* S *of singularities of* $f(z)$ *on* A *is the closure of the union* $M_1 \cup M_2$, *where* $M_1 = A \cap E(e^{i\theta} : \alpha\in R(f, e^{i\theta}))$[1], $M_2 = A \cap E(e^{i\theta} : \alpha= f(e^{i\theta}))$, *and* α *is any fixed point of modulus less than* 1.

(2) $f(z)$ *is meromorphic in the sector:* $\Theta_1<\Theta<\Theta_2$, $0\leqq r\leqq\infty$, *except for the set* S *of linear measure zero lying on* A.

(3) *if at least two values in* $|w|<1$ *are omitted by* $w=f(z)$ *in the neighborhood of* A, *then* S *is a perfect set, whose linear measure is zero but whose capacity is positive.*

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[1]) $R(f, e^{i\theta})$ is the range of values at $e^{i\theta}$, which is defined as the set of values a such that $\lim_{n\to\infty} z_n = e^{i\theta}$, $|z_n|<1$, $f(z_n) = a$.

Remark. Part (3) was proved by P. J. Myrberg [13] (K. Noshiro [19] p. 20) in the special case where $w=f(z)$ is a function which maps $|z|<1$ onto the universal covering surface of a domain obtained by excluding from $|w|<1$ a set of capacity zero with at least two points.

2. Lemmas. In order to establish our theorems, we need some lemmas.

Lemma 1. *For* $|a|<1$,

$$\left|\frac{z-a}{1-\bar{a}z}\right| < \exp\left\{\frac{2(1-|a|)}{\left|\frac{1}{\bar{z}}-a\right|^2}\right\}. \tag{2.1}$$

Proof: By the inequality $\log(1+x)\leqq x$ for $x\leqq 0$,we have for $|z|\leqq 1$, $|a|<1$,

$$\begin{aligned}\log|(z-a)/(1-\bar{a}z)| &= 1/2\cdot\log\{1+(|z|^2-1)\ (1-|a|^2)\ /\ |1-\bar{a}z|^2\}\\ &\leqq 1/2\cdot(|z|^2-1)\ (1-|a|^2)\ /\ |1-\bar{a}z|^2\\ &< 2(1-|a|)\ /\ |1/\bar{z}-a|^2.\end{aligned}$$

Hence, (2.1) holds for $|z|\geqq 1$, $|a|<1$. On the other hand, we evidently have

$$|(z-a)/(1-\bar{a}z)|<1 \quad \text{for } |z|<1\ ,\ |a|<1\ . \tag{2.2}$$

By (2.1) and (2.2), Lemma 1 is established.

Lemma 2. *Put*

$$\bar{a}(a-z)\ /\ |a|\ (1-\bar{a}z) = 1+f(z) \qquad (0<|a|<1)$$

and suppose that

$$|z-e^{i\varphi}|\leqq\varepsilon\ ,\quad |a-e^{i\varphi}|\geqq 2\delta \quad |1/\bar{a}-e^{i\varphi}|\geqq 2\delta\ , \tag{2.3}$$

where $0<1-|a|<\delta^2/4$, $0<2\varepsilon<\delta<1/2$.

Then

$$|f(z)|<\{(1-|a|)+\frac{4}{\delta^2}(1-|a|)^2\}\frac{2+\varepsilon}{\delta}.$$

Proof: By simple calculation,

$$|f(z)|<(1-|a|)\ \ (|(z-a)\ /\ (\bar{a}z-1)|)\ \ (1+|z|)\ /\ |z-a|\ .$$

Hence, by Lemma 1,

$$|f(z)|<(1-|a|)\ \exp\{2(1-|a|)\ /\ |1/\bar{z}-a|^2\}\cdot(1+|z|)\ /\ |z-a|\ . \tag{2.4}$$

Since $|z-e^{i\varphi}|\leqq\varepsilon$ means that $|1/\bar{z}-e^{i\varphi}|<2\varepsilon$ for $0<\varepsilon<1/2$, (2.3) yields

$$|z-a|>\delta\ ,\quad |1/\bar{z}-a|>\delta\ ,$$

so that, by (2.4)

$$|f(z)|<(1-|a|)\exp\{2(1-|a|)/\delta^2\}\cdot(2+\varepsilon)/\delta\ . \tag{2.5}$$

By the inequality $e^x<1+2x$ for $0<x<1/2$, we obtain on account of (2.3) and (2.5)

$$|f(z)|<(1-a|)\ \{1+4/\delta^2\cdot(1-|a|)\}\cdot(2+\varepsilon)/\delta\ ,$$

which was to be proved.

Lemma 3. *Put*

$$B(z)=\prod_{n=1}^{+\infty}\bar{a}_n/|a_n|\cdot(a_n-z)\ /\ (1-\bar{a}_n z)\ ,$$

where $\sum_{n=1}^{+\infty}(1-|a_n|)<+\infty$. *For* $B(z)$ *to be singular at* $z=e^{i\varphi}$, *it is necessary and sufficient that* $z=e^{i\varphi}$ *is a limit point of* $\{a_n\}$.

Proof: Since the zeros of $B(z)$ do not accumulate at a regular point, it is sufficient to prove that, if $z=e^{i\varphi}$ is not a limit point of $\{a_n\}$, $B(z)$ is regular at $z=e^{i\varphi}$.

Suppose that $z=e^{i\varphi}$ is not a limit point of $\{a_n\}$. Then there exists a positive constant δ such that

$$|a_n-e^{i\varphi}|\geqq 2\delta\ ,\quad |1/\bar{a}_n-e^{i\varphi}|\geqq 2\delta \tag{2.6}$$

for $0<\delta<1/2$, $n=1, 2, \ldots$

Put

$$\bar{a}_n/|a_n|\cdot(a_n-z)\ /\ (1-\bar{a}_n z)=1+f_n(z)\ .$$

By the convergence of $\sum(1-|a_n|)$, we can find a sufficiently large integer N such that

$$0<1-|a_n|<\delta^2/4 \qquad \text{for } n\geqq N\ . \tag{2.7}$$

If we choose ε such that $0<\varepsilon<\delta/2<1/4$, then by (2.6), (2.7) and Lemma 2,

$$|f_n(z)|<\{(1-|a_n|)+\frac{4}{\delta^2}(1-|a_n|)^2\}\frac{2+\varepsilon}{\delta}$$

for $|z-e^{i\varphi}|\leqq\varepsilon,\ n\geqq N$, so that, on account of the convergence of $\sum(1-|a_n|)$, it follows that $\sum f_n(z)$ is uniformly convergent for $|z-e^{i\varphi}|\leqq\varepsilon$. Since $f_n(z)$ is regular in $|z-e^{i\varphi}|\leqq\varepsilon$, $B(z)=\Pi(1+f_n(z))$ is also regular in $|z-e^{i\varphi}|\leqq\varepsilon$, which was to be proved.

Lemma 4. $B(z)$ *is meromorphic in* $|z|\leqq\infty$, *except for a set of linear measure zero lying on* $|z|=1$.

Proof: For $|z| > 1$, we can put

$$1/B(z) = \prod_{n=1}^{+\infty} |a_n|/\bar{a}_n \cdot (1/z - \bar{a}_n) / (\bar{a}_n/z - 1) .$$

By the convergence of $\sum (1 - |\bar{a}_n|)$, $1/B(z)$ is regular in $|z|>1$ and it has zeros at $\{1/\bar{a}_n\}$. In other words, $B(z)$ is meromorphic in $|z|>1$, and it has poles at $\{1/\bar{a}_n\}$.

By Lemma 3, the set of singular points coincides with the set of the limit points of $\{a_n\}$. Hence it is sufficient to prove that the set E of the limit points of $\{a_n\}$ is of linear measure zero. By F. Riesz's theorem ([22]; R. Nevanlinna [14], p. 207) the radial limit $B(e^{i\varphi})$ is of modulus one for almost all φ. Hence, by Egoroff's theorem, we can find suitable positive constants ε, δ and a closed set E^* lying on $|z| = 1$ such that $m(CE^*) < \varepsilon$, $1/2 \leqq |B(re^{i\varphi})| < 1$ for $\varphi \in E^*$, $1 - \delta \leqq r < 1$. Therefore, the set E is contained in CE^*. Letting $\varepsilon \to 0$, we see that $m(E) = 0$, which proves Lemma 4.

Lemma 5. *If $f(z) \in U$, then, except for a set of values α, $|\alpha|<1$, of capacity zero, we can put*

$$(f(z) - \alpha) / (1 - \bar{\alpha} f(z)) = B(z) ,$$

where $B(z)$ is the Blaschke product extended over the α-points of $f(z)$.[1]

Lemma 6. *Let D be a simply connected domain of hyperbolic type, E a closed set of capacity zero contained in the boundary Γ, and z_0 a point of E. Suppose that $f(z)$ is regular and bounded in the common part of D and a certain neighborhood of z_0. We denote by $C_D(f, z_0)$ and $C_{\Gamma - E}(f, z_0)$ the interior cluster set and the boundary cluster set, respectively.*[2] *If $\Omega = C_D(f, z_0) - C_{\Gamma - E}(f, z_0)$ (open set) is not empty, then $w = f(z)$ takes every value, with one possible exception, belonging to Ω_n infinitely often in any neighborhood of z_0, where Ω_n is any component of Ω.*[3]

Lemma 7. *Put*

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{i\varphi} + z) / (e^{i\varphi} - z) \, d\mu(\varphi) ,$$

where

(i) $\displaystyle\int_{-\pi}^{+\pi} |d\mu(\varphi)| < + \infty$,

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1) Frostman [2], p. 111.

2) For these definitions, we refer to K. Noshiro's book [19], p. 1—2.

3) Noshiro [19], p. 25.

(ii) $\mu'(\varphi) = 0$ *almost everywhere on the arc* $A(e^{i\varphi} : \varphi_1 < \varphi < \varphi_2)$.

Then the following alternatives are possible:

(1) $F(z)$ *is regular on* A .

(2) $F(z)$ *has an enumerable number of poles of first order on* A .

(3) $\lim_{r \to 1} R[F(re^{i\varphi})] = \pm\infty$ *for a non-enumerable set of points on* A.[1)]

Proof: If $\mu(\varphi)$ is constant on A , we can put

$$F(z) = \frac{1}{2\pi} \int_{CA} (e^{i\varphi} + z) \,/\, (e^{i\varphi} - z) \; d\mu(\varphi) ,$$

where CA denotes the complementary set of A with respect to $(-\pi , +\pi)$. Hence, $F(z)$ is regular on A.

If $\mu(\varphi) \not\equiv$ constant on A , $\mu(\varphi)$ admits the following representation:

$$\mu(\varphi) = \mu_1(\varphi) + \mu_2(\varphi) + \mu_3(\varphi) ,$$

where all functions $\mu_i(\varphi)$ $(i = 1, 2, 3)$ are of bounded variation; $\mu_1(\varphi)$ is continuous and $\mu_1'(\varphi) = 0$ almost everywhere on A , $\mu_2(\varphi)$ is absolutely continuous and $\mu_3(\varphi)$ is a step-function. Since $\mu'(\varphi) = 0$ almost everywhere on A, $\mu_2(\varphi) = 0$ on A .

If $\mu_1(\varphi)$ is constant, then $\mu_3(\varphi)$ is certainly not constant. In this case, we can put

$$F(z) = \frac{1}{2\pi} \int_{CA} (e^{i\varphi} + z) \,/\, (e^{i\varphi} - z) \quad d\mu(\varphi) \tag{2.8}$$
$$+ \sum_{n=1}^{+\infty} (e^{i\varphi_n} + z) \,/\, (e^{i\varphi_n} - z) \cdot J_n ,$$

where $\{e^{i\varphi_n}\} \subset A$ and $\sum |J_n| < +\infty$. Therefore $F(z)$ has an enumerable number of poles of first order on A .

If $\mu_1(\varphi)$ is not constant, then $\mu_1'(\varphi) = \pm\infty$ at a non-denumerable set of points on A. (Schlesinger and Plessner [23], §43). Since $\mu_3(\varphi)$ is discontinuous at an enumerable set of points on A, there exists a non-denumerable set E of points on A such that $\mu_1'(\varphi) = \pm\infty$ and $\mu_3(\varphi)$ is continuous at $e^{i\varphi} \in E$. Therefore

$$\mu'(\varphi) = \pm\infty \text{ for } e^{i\varphi} \in E . \tag{2.9}$$

Since

$$R[F(z)] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} (1 - r^2) \,/\, (1 + r^2 - 2r \cos(\Theta - \varphi)) \; d\mu(\varphi) ,$$

1) $R(F(re^{i\varphi}))$ is the real part of $F(re^{i\varphi})$.

$z = re^{i\varphi}$, we have by (2.9) and by Fatou's theorem

$$\lim_{r \to 1} R[F(re^{i\varphi})] = \pm \infty \text{ for } e^{i\varphi} \in E .$$

Thus Lemma 7 is completely established.

3. Proof of Theorem 1. By Lemma 5, we can put

$$(f(z) - \alpha) / (1 - \bar{\alpha} f(z)) = B(z) , \tag{3.1}$$

except for an exceptional set of values α of capacity zero, where $B(z)$ is the Blaschke product extended over the α-points ($|\alpha| < 1$) of $f(z)$. If $z = e^{i\varphi}$ is not a limit point of α-points, then, by Lemma 3, $B(z)$ is regular and of modulus one at $z = e^{i\varphi}$. By (3.1)

$$f(z) = (B(z) + \alpha) / (1 + \bar{\alpha} B(z)) , \tag{3.2}$$

so that $f(z)$ is also regular at $z = e^{i\varphi}$. On the other hand, a limit point of α-points is evidently a singular point of $f(z)$. Thus, the set S of singularities of $f(z)$ lying on $|z| = 1$ coincides with the set of the limit points of α-points, except perhaps for exceptional values α of capacity zero, which proves statement (1) in Theorem 1.

By (1) and Lemma 3, the set S coincides with the set of singularities of $B(z)$ lying on $|z| = 1$. Hence by (3.2) and Lemma 4, $f(z)$ is meromorphic in $|z| \leqq \infty$, except for the set S of linear measure zero lying on $|z| = 1$, which proves statement (2) in Theorem 1.

4. Proof of Theorem 2 (1). Since the closure $\overline{M_1 \cup M_2}$ of $M_1 \cup M_2$ is evidently contained in the set S of singularities of $f(z)$ on A, to establish part (1) it is sufficient to prove that $f(z)$ is regular at the complementary set of $\overline{M_1 \cup M_2}$ with respect to A.

Taking account of

$$(f(z) - \alpha) / (1 - \bar{\alpha} f(z)) \in U(\Theta_1 , \Theta_2) ,$$

we can decompose it as follows (W. Seidel [24] p. 204)

$$(f(z) - \alpha) / (1 - \bar{\alpha} f(z)) = e^{i\beta} \cdot B(z) \exp(F(z)), \tag{4.1}$$

where β is a real constant, $B(z)$ the Blaschke product extended over the α-points of $f(z)$,

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} (e^{i\varphi} + z) / (e^{i\varphi} - z) \, d\mu(\varphi) ,$$

$\mu(\varphi)$ a monotonic non-increasing function of φ, and $\mu(\varphi) = 0$ almost everywhere on $A(e^{i\varphi}\colon \Theta_1 < \varphi < \Theta_2)$.

Put

$$e^{i\theta} \in A \cap C(\overline{M_1 \cup M_2}) . \tag{4.2}$$

Since $z = e^{i\theta}$ is not a limit point of the α-points, by Lemma 3, $B(z)$ is regular and of modulus one at $z = e^{i\theta}$. In a sufficiently small neighborhood of $z = e^{i\theta}$, the following three cases are possible by Lemma 7.

(1) $F(z)$ is regular at $z = e^{i\theta}$, and $\lim R[F(re^{i\theta})] = 0$ as $r \to 0$.

(2) $F(z)$ has an enumerable number of poles $\{e^{i\varphi_n}\}$ of first order, and $\lim R[F(re^{i\varphi_n})] = -\infty$ as $r \to 0$. (Since $\mu(\varphi)$ is non-increasing, in (2.8) we have $J_n < 0$, so that by arguments entirely similar to those applied by W. Seidel ([24], pp. 205—206), we conclude that $\lim R[F(re^{i\varphi})] = -\infty$).

(3) $\lim R[F(re^{i\varphi})] = -\infty$ for a non-enumerable set of points. (For, $\mu(\varphi)$ is non-increasing).

In both cases (2) and (3), the right-hand side of (4.1) tends to zero, so that there exists at least one point $z = e^{i\varphi}$ in a sufficiently small neighborhood of $z = e^{i\theta}$ such that $\lim f(re^{i\varphi}) = \alpha$, which is contrary to (4.2). Hence, only case (1) is possible.

Thus, the right-hand side of (3.1) is regular and of modulus one at $z = e^{i\theta}$. Since

$$f(z) = (G(z) + \alpha) \,/\, (1 + \bar{\alpha}G(z)) ,$$

where $G(z) = e^{i\beta} \cdot B(z) \cdot \exp(F(z))$, $f(z)$ is also regular at $z = e^{i\theta}$, which was to be proved.

5. Corollaries of Theorem 2 (1). As an immediate consequence of Theorem 2 (1), we obtain

Corollary 1. *Let $f(z)$ belong to class $U(\Theta_1, \Theta_2)$. If $f(z)$ is not regular on the arc A ($e^{i\theta} : \Theta_1 < \Theta < \Theta_2$), and α ($|\alpha| < 1$) is omitted by $f(z)$ in a neighborhood of A, then there exists at least one point $z = e^{i\varphi}$ on A such that*

$$\lim_{r \to 1} f(re^{i\varphi}) = \alpha .$$

It is an extension of W. Seidel's theorem ([24], p. 205).

If we apply Theorem 2 (1) to a sequence of arcs $\{A_n\}$ containing a singular point $z_0 = e^{i\theta_0}$ whose lengths tend to zero as $n \to \infty$, then we obtain the following result of Seidel ([24], p. 211).

Corollary 2. *Let $w = f(z)$ belong to class $U(\Theta_1, \Theta_2)$. If $f(z)$ is singular at $z_0 = e^{i\theta_0}$ on the arc $A(e^{i\theta} : \Theta_1 < \Theta < \Theta_2)$, then the cluster set of $f(z)$ at z_0 is $|w| \leqq 1$.*

Let $z_0 = e^{i\theta_0}$ be an isolated singular point on A. Then there exists a sufficiently small arc A_0 containing z_0 such that $f(z)$ is regular on A_0 except for $z = z_0$. Applying Corollary 1 to the arc A_0, we conclude that if a value α ($|\alpha| < 1$) is omitted it is the radial limit at z_0, so that there exists at most one omitted value. Hence we obtain

Corollary 3. *Let* $w = f(z)$ *belong to class* $U(\Theta_1, \Theta_2)$. *If there exists an isolated singular point* $z_0 = e^{i\theta_0}$ *on the arc* $A(e^{i\theta}: \Theta_1 < \Theta < \Theta_2)$, *then at most one value in* $|w| < 1$ *is omitted by* $f(z)$ *in the neighborhood of* z_0.

As an immediate consequence of Corollary 3, we have

Corollary 4. *Let* $w = f(z)$ *belong to class* $U(\Theta_1, \Theta_2)$. *If* $f(z)$ *is not regular on the arc* $A(e^{i\theta}: \Theta_1 < \Theta < \Theta_2)$, *and at least two values in* $|w| < 1$ *are omitted by* $f(z)$ *in the neighborhood of* A, *then the set of singularities of* $f(z)$ *on* A *is perfect.*

This is an extension of W. Seidel's theorem. ([24], p. 213).

6. Proof of Theorem 2 (2)—(3). Since $\lim |f(re^{i\theta})| = 1$ almost everywhere on A, by Egoroff's theorem we can find suitable positive constants ε, δ, δ' and a closed set E contained in A such that

$$m(A-E) < \varepsilon,$$

$$0 \leqq |\alpha| < 1-\delta \leqq |f(re^{i\theta})| < 1 \quad \text{for } 1-\delta' \leqq r < 1,\ \Theta \in E.$$

Hence, by Theorem 2 (1), the set S of singularities on A is contained in $A-E$. Letting $\varepsilon \to 0$, we have $m(S) = 0$. Then, with the function relation $f(z) = 1/\overline{f(1/\bar{z})}$ for $|z| \geqq 1$, $f(z)$ can be continued analytically beyond the open arcs A_n, where $A-S = \bigcup A_n$, which proves (2).

If at least two values in $|w| < 1$ are omitted by $w = f(z)$ in the neighborhood of A, then by Theorem 2 (1) and Corollary 4, S is a perfect set of linear measure zero. Suppose that S is of capacity zero. Then, by Lemma 6 and Corollary 2, $f(z)$ assumes every value in $|w| < 1$ infinitely often with one possible exception in the neighborhood of S, which is contrary to the hypothesis. Hence S is of positive capacity, which proves (3).

7. An application of Lemma 7. By Lemma 7, we can establish

Corollary 5. *If* $w = f(z)$ *belongs to class* $U(\Theta_1, \Theta_2)$, *and it is not regular on the arc* $A(e^{i\theta}: \Theta_1 < \Theta < \Theta_2)$, *then every value on* $|w| = 1$ *is assumed by* $f(z)$ *infinitely often on* A.

Corollary 5 is an extension of W. Seidel's theorem ([24], p. 208), but it is a special case of Caldéron-Domingues-Zygmund's theorem [1] (M. Ohtsuka [21], p. 299).

Proof: For any real λ, $F(z) = (f(z) + e^{i\lambda}) \,/\, (f(z) - e^{i\lambda})$ is a regular function in $|z| < 1$ such that

$$R(F(z)) < 0 \text{ in } |z| < 1,$$

$$R(F(z)) = 0 \text{ almost everywhere on } A.$$

Hence, by Herglotz's theorem [5] (R. Nevanlinna [15], p. 196) we can put

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} (e^{i\varphi} + z) \,/\, (e^{i\varphi} - z)\, d\mu(\varphi) + i\beta,$$

where β is a real constant, $\mu(\varphi)$ a monotonic non-increasing function of φ and $\mu'(\varphi) = 0$ almost everywhere on $A(e^{i\varphi} : \Theta_1 < \Theta < \Theta_2)$.

By Lemma 7, the following cases are possible:

(1) $F(z)$ is regular on A.

(2) $F(z)$ has a finite number of poles $\{e^{i\varphi_n}\}$ $(n = 1, 2, \ldots, k)$ of first order on A.

(3) $F(z)$ has an infinite number of poles $\{e^{i\varphi_n}\}$ of first order on A.

(4) $\lim\limits_{r \to 1} R(F(re^{i\varphi})) = -\infty$ for a non-enumerable set of points on A.

In case (1), $f(z) = e^{i\lambda} \cdot (F(z) + 1) / (F(z) - 1)$ is also regular on A. For, if $F(z) \to 1$ as $z \to z_0 \in A$, $f(z)$ is unbounded in the neighborhood of z_0, which is contrary to the hypothesis. Hence, case (1) is impossible.

In case (2), $f(z)$ tends to $e^{i\lambda}$ uniformly as $z \to e^{i\varphi_n}$ $(n = 1, 2, \ldots, k)$, so that, by Corollary 2, $z = e^{i\varphi_n}$ $(n = 1, 2, \ldots, k)$ is a regular point. Hence, $f(z)$ is regular on A, which is evidently impossible. Therefore, only cases (3) and (4) are possible. Hence, $e^{i\lambda}$ is assumed infinitely often on A. Since $e^{i\lambda}$ is arbitrary, Corollary 5 is proved.

(Waseda University, Tokyo)

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