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# A CONDITION FOR THE SELFADJOINTNESS OF A LINEAR OPERATOR

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## A condition for the selfadjointness of a linear operator

Let $T$ be a selfadjoint linear operator from a complex Hilbert space $\mathfrak{H}$ into $\mathfrak{H}$. It is well known that its spectrum lies on the real axis and that its resolvent

$$R_\lambda = (T - \lambda I)^{-1} \qquad (I = \text{the identity operator})$$

satisfies the inequality

$$|R_\lambda| \leqq \frac{1}{|\text{Im } \lambda|}$$

for every $\lambda$ with $\text{Im } \lambda \neq 0$. The purpose of this paper is to prove the following converse theorem:

*Let $T$ be a linear operator from a complex Hilbert space $\mathfrak{H}$ into $\mathfrak{H}$ such that*

(a) *the domain of definition $\mathfrak{D}_T$ of $T$ is dense in $\mathfrak{H}$,*
(b) *the imaginary axis, with the possible exception of the origin, belongs to the resolvent set of $T$,*
(c) *the resolvent of $T$ satisfies the inequality*

$$|R_{i\xi}| \leqq \frac{1}{|\xi|}$$

*for every real $\xi \neq 0$.*

*Then $T$ is selfadjoint.*

*Proof.* The inequality in (c) can be written

$$|Tx - i\xi x|^2 \geqq |\xi|^2 |x|^2 .$$

Since here

$$|Tx - i\xi x|^2 = |Tx|^2 + |\xi|^2|x|^2 + i\xi[(Tx , x) - (x , Tx)] ,$$

we thus have

$$|Tx|^2 + i\xi[(Tx , x) - (x , Tx)] \geqq 0$$

or

$$|Tx|^2 \geqq 2\xi \text{ Im } (Tx , x)$$

for every $x$ in $\mathfrak{D}_T$ and every real $\xi$. From this it immediately follows that $\text{Im}\,(Tx, x) = 0$ or

$$(Tx\,,x) = (x\,,Tx)$$

for every $x$ in $\mathfrak{D}_T$. By means of the polarization formula

$$4(Tx\,,y) = (T(x+y)\,,x+y) - (T(x-y)\,,x-y) + i(T(x+iy)\,,x+iy) - i(T(x-iy)\,,x-iy)$$

and the corresponding formula for $(x, Ty)$ we thus get

$$(Tx\,,y) = (x\,,Ty) \tag{1}$$

for every $x$ and $y$ in $\mathfrak{D}_T$. This shows that the adjoint operator $T^*$, which exists by virtue of (b), is an extension of $T$.

In order to prove conversely that $T$ is an extension of $T^*$ we first show that $R_i^* = R_{-i}$. From (1) it follows that

$$(Tx - ix\,,y) = (x\,,Ty + iy)$$

whenever $x$ and $y$ are in $\mathfrak{D}_T$. Denoting here

$$Tx - ix = u\,,\quad Ty + iy = v,\quad x = R_iu\,,\quad y = R_{-i}v$$

we thus get

$$(u\,,R_{-i}v) = (R_iu\,,v) \tag{2}$$

for every $u$ and $v$ in $\mathfrak{H}$. This shows that $R_i^* = R_{-i}$.

Now let $y$ be an arbitrary vector of $\mathfrak{D}_{T^*}$. If we set $T^*y = z$ we thus have

$$(Tx\,,y) = (x\,,z)$$

and, consequently,

$$(Tx - ix\,,y) = (x\,,z + iy)$$

for every $x$ in $\mathfrak{D}_T$. Denoting here

$$Tx - ix = u\,,\quad x = R_iu$$

we therefore see by virtue of the equation (2) that

$$(u\,,y) = (R_iu\,,z + iy) = (u\,,R_{-i}(z + iy))\,.$$

Since $u$ is arbitrary, we thus have

$$y = R_{-i}(z + iy) \in D_{T+iI} = D_T\,.$$

As a consequence, $\mathfrak{D}_{T^*} \subset \mathfrak{D}_T$, and our proof is complete.

It is well known that the spectrum of a unitary operator $U$ lies on the unit circle and that its resolvent satisfies the inequality

$$|R_\lambda| \leqq \frac{1}{|\,|\lambda| - 1|}\,.$$

It would be interesting to know whether from these two assumptions it conversely follows that $U$ is unitary.

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