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ON THE CONJUGACY CLASSES OF THE FINITY UNITARY GROUPS

BY

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On the conjugacy classes of the finite unitary groups

The conjugacy classes of all the classical groups have just been discussed by G. E. WALL [4] by a very ingenious method. In the case of the finite unitary groups we give here a simpler alternative method, which, we hope, throws more light on the question and shows more concretely the significance of the terms in the final formula.

I am greatly indebted to Dr. G. E. WALL, who kindly sent me a summary of his unpublished manuscript. I also wish to express my gratitude to Prof. R. BRAUER for many important discussions.

1. Let $\mathfrak{F} = GF(q^2)$ be the finite field consisting of q^2 elements, where q is a power of a prime. For $\alpha \in \mathfrak{F}$ we write $\bar{\alpha} = \alpha^q$. Then $\bar{\alpha}$ is called the *conjugate* of α. For any matrix M with elements in $\mathfrak{F}$ we write M^* = the conjugate transpose of M. Then a square matrix M with elements in $\mathfrak{F}$ is called *unitary* or *Hermitian*, if $MM^* = M^*M = 1$ or $M = M^*$, respectively. The group of all unitary matrices with elements in $GF(q^{2s})$ is denoted by $U(n, q^{2s})$ and, especially, $\mathfrak{U}_n = U(n, q^2)$. The group of all non-singular square matrices with elements in $GF(q^t)$ is denoted by $GL(n, q^t)$ and, especially, $\mathfrak{G}_n = GL(n, q^2)$. Let $u(n, q^{2s})$ and $g(n, q^t)$ be the number of elements in $U(n, q^{2s})$ and $GL(n, q^t)$, respectively, and, especially, $u_n = u(n, q^2)$, $g_n = g(n, q^2)$. As is well known ([1] p. 77 and 134)

$$u(n, q^{2s}) = q^{{}^1\!/_2 n(n-1)s} \prod_{i=1}^{n} (q^{si} - (-1)^i),$$

$$g(n, q^t) = q^{{}^1\!/_2 n(n-1)t} \prod_{i=1}^{n} (q^{ti} - 1).$$

By convention, $u(0, q^{2s}) = g(0, q^t) = 1$.

2. We begin with the following simple lemma.

Lemma 1. *The number of non-singular Hermitian matrices with elements in* $\mathfrak{F}$ *equals*

$$\frac{g_n}{u_n} = q^{{}^1\!/_2 n(n-1)} \prod_{i=1}^{n} (q^i + (-1)^i).$$

Proof. Every non-singular Hermitian matrix A with elements in $\mathfrak{F}$ can be written ([2] p. 16)

$$A = XX^* ,$$

where $X \in \mathfrak{G}_n$. Suppose that $A = X_1 X_1^*$ for some $X_1 \in \mathfrak{G}_n$. Write $Y = X^{-1} X_1$. Then $XYY^*X^* = XX^*$, so that $YY^* = 1$, i.e. Y is unitary. This clearly also holds conversely. Hence from the g_n products XX^* each u_n are equal. This proves the lemma.

It is perhaps of interest to note that the so called »polar decomposition» is not valid in the finite case. This is so because u_n is always even and hence $\mathfrak{U}_n$ contains involutions which are Hermitian. So the coset representatives of $\mathfrak{U}_n$ in $\mathfrak{G}_n$ cannot be chosen from Hermitian matrices only. It is also not true in the finite case that every Hermitian matrix has a square root.

Let

$$f(t) = t^d + a_1 t^{d-1} + \ldots + a_d$$

be an arbitrary polynomial over $\mathfrak{F}$ with $a_d \neq 0$. Then we write

$$\tilde{f}(t) = \bar{a}_d^{-1} (\bar{a}_d t^d + \bar{a}_{d-1} t^{d-1} + \ldots + 1) .$$

Lemma 2. *Suppose that, identically,*

$$f(t) = \tilde{f}(t)$$

and that $f(t)$ is irreducible. Then d is odd and every root ξ of the equation $f(t) = 0$ satisfies the condition

$$\xi^{q^d+1} = 1 .$$

Proof. We have $\mathfrak{F}(\xi) = GF(q^{2d})$. The Galois group of $\mathfrak{F}(\xi)$ with respect to $\mathfrak{F}$ is a cyclic group of order d generated by

$$\tau : x \rightarrow x^{q^2} .$$

Our assumption implies that ξ^{-q} is also a root of the equation $f(t) = 0$. Hence $\xi^{-q} = \tau^i(\xi) = \xi^{q^{2i}}$ for some $i, 0 \leqq i \leqq d-1$. Thus we have $\xi^{q^{2i}+q} = 1$. Let ξ belong to the exponent e. Then $q^{2i-1} \equiv -1 \pmod{e}$ and $q^{2d} \equiv 1 \pmod{e}$. Let s be the g.c.d. of $2i-1$ and $2d$. Then we can choose integers x, y such that

$$x(2i-1) + y\, 2d = s .$$

Here x is odd, since s is odd. Raising the congruences to the powers x and y, respectively, and multiplying we get

$$q^s \equiv -1 \pmod{e} .$$

If d is even, we have

$$q^d \equiv 1 \pmod{e} .$$

But this means that $\tau^{1/2 d}(\xi) = \xi$, which is impossible. Hence d must be odd, and we have $q^d \equiv -1 \pmod{e}$. This proves the lemma.

Lemma 3. *If the characteristic polynomial of a square matrix A with elements in a perfect field $\mathfrak{K}$ is irreducible and g is an arbitrary polynomial over $\mathfrak{K}$, then the only solutions X, with elements in $\mathfrak{K}$, of the matrix equation*

$$AX - XA = g(A)$$

are of the form $X = f(A)$, where f is a polynomial over $\mathfrak{K}$ and moreover, $X = 0$ unless g vanishes identically.

Proof. From the hypothesis it follows that in a suitable algebraic extension field of $\mathfrak{K}$ we can transform A into diagonal form where the diagonal elements are all distinct. Then it is easy to see that the transform of X must also be diagonal and hence X can be expressed as a polynomial of A.

Lemma 4. *Let A and B be square matrices, not necessarily of the same order, with elements in a perfect field $\mathfrak{K}$ and suppose that their characteristic polynomials are irreducible. Let X be a solution, with elements in $\mathfrak{K}$, of the matrix equation*

$$AX = XB .$$

If A and B are similar, then X is either 0 or non-singular. If A and B are not similar, then X is 0.

Proof. This is a special case of the well known Schur's lemma.

3. Consider now an arbitrary matrix A in $\mathfrak{G}_n$. Suppose that there exists a matrix X in $\mathfrak{G}_n$ such that

$$XAX^{-1} \in \mathfrak{U}_n . \tag{1}$$

Write $\Gamma = X^*X$. Then we have

$$X^{*-1}A^*X^*XAX^{-1} = 1 ,$$

i.e.

$$A^*\Gamma A = \Gamma . \tag{2}$$

Conversely, if (2) is valid for a Hermitian matrix Γ, then as in the proof of lemma 1 we can write $\Gamma = X^*X$ for some $X \in \mathfrak{G}_n$, and we see that (1) is true. To each matrix A in $\mathfrak{G}_n$ which is similar to a matrix in $\mathfrak{U}_n$ we can thus associate a non-empty set $\Gamma(A)$ of Hermitian matrices satisfying

(2). We denote by $\gamma(A)$ the number of elements in this set. Let $\mathfrak{C}(A)$ be the centralizer of A in $\mathfrak{G}_n$ and let $c(A)$ be the number of elements in $\mathfrak{C}(A)$. Let $B \in \mathfrak{C}(A)$. Then

$$A^* B^* \Gamma B A = B^* A^* \Gamma A B = B^* \Gamma B$$

for all $\Gamma \in \Gamma(A)$. Hence it is easy to see that $\mathfrak{C}(A)$ can be considered as a permutation group of the set $\Gamma(A)$. Assume that this permutation representation is transitive. Let

$$A_1 = X_1 A X_1^{-1} \in \mathfrak{U}_n\,, \qquad \Gamma_1 = X_1^* X_1\,,$$

$$A_2 = X_2 A X_2^{-1} \in \mathfrak{U}_n\,, \qquad \Gamma_2 = X_2^* X_2\,.$$

From the transitivity assumption it follows that there exists $D \in \mathfrak{C}(A)$ such that $D^* \Gamma_1 D = \Gamma_2$. Denote $F = X_1 D X_2^{-1}$. Then $F^{-1} A_1 F = A_2$ and $F^* F = 1$ so that $F \in \mathfrak{U}_n$. By the proof of lemma 1, it is easy to see that the total number of matrices in $\mathfrak{G}_n$ transforming A into a matrix of $\mathfrak{U}_n$ is $\gamma(A) u_n$. On the other hand $X' A X'^{-1} = X'' A X''^{-1}$ if and only if $X''^{-1} X' \in \mathfrak{C}(A)$. As a result from our considerations we thus have

Lemma 5. *If $\mathfrak{C}(A)$, considered as a permutation group of the set $\Gamma(A)$, is transitive, then the elements of $\mathfrak{U}_n$ similar to A in $\mathfrak{G}_n$ (if there are any) form exactly one conjugacy class of $\mathfrak{U}_n$ and the number of elements in it equals*

$$\frac{\gamma(A)\, u_n}{c(A)}\,.$$

4. Let $f(t) = t^d + a_1 t^{d-1} + \ldots + a_d$ be an irreducible polynomial over $\mathfrak{F}$. Define the matrices

$$M(f) = M_1(f) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & 1 \\ -a_d & -a_{d-1} & \cdots & -a_1 \end{pmatrix},$$

the »companion matrix» of f, and

$$M_l(f) = \begin{pmatrix} M(f) & \cdot & & \cdot \\ 1_d & M(f) & & \cdot \\ \cdot & 1_d & & \cdot \\ \cdots & \cdots & \cdots & \cdots \\ , & \cdot & & M(f) \end{pmatrix},$$

with l diagonal blocks $M(f)$ and 1_d is the $d \cdot d$ identity matrix. Finally, if $\nu = \{l_1, l_2, \ldots, l_p\}$ is a partition of a positive integer k, whose p parts, written in descending order, are $l_1 \geqq l_2 \geqq \ldots \geqq l_p > 0$,

$$M_\nu(f) = \operatorname{diag}\,(M_{l_1}(f)\,, M_{l_2}(f)\;, \ldots, M_{l_p}(f))\,.$$

Let $A \in \mathfrak{G}_n$ have characteristic polynomial

$$f_1^{k_1} f_2^{k_2} \ldots f_N^{k_N}\,,$$

where $f_1, f_2, \ldots, f_N$ are distinct irreducible monic polynomials over $\mathfrak{F}$, $k_i \geqq 0$ $(i = 1, 2, \ldots, N)$ and, if $d_1, d_2, \ldots, d_N$ are the respective degrees of $f_1, f_2, \ldots, f_N$, $\sum_{i=1}^{N} k_i d_i = n$. Then A is similar in $\mathfrak{G}_n$ to a matrix

$$A_0 = \operatorname{diag}\,(M_{\nu_1}(f_1)\,, M_{\nu_2}(f_2)\,, \ldots, M_{\nu_N}(f_N))\,,$$

where $\nu_i = \{l_1^{(i)}, l_2^{(i)}, \ldots, l_{p_i}^{(i)}\}$ is a certain partition (whose parts are written in descending order) of k_i for $i = 1, 2, \ldots, N$ (cf. [3], p. 406).

Let $m_j^{(i)}$ denote the number of parts in the partition ν_i that are equal to j $(j = 1, 2, \ldots, k_i)$. Then we have

$$c(A) = \prod_{i=1}^{N} \Big\{\exp\Big[2d_i \Big(\sum_{j=1}^{k_i} (j-1)\, m_j^{(i)2} + 2 \sum_{\substack{j,l=1\\ j<l}}^{k_i} j\, m_j^{(i)} m_l^{(i)}\Big]\prod_{j=1}^{k_i} g(m_j^{(i)}, q^{2d_i})\Big\}\,, \tag{3}$$

where we use the notation $\exp(x) = q^x$. (See [3], p. 410, Lemma 2.4, or [1], p. 235. The formula can easily be modified to the form (3).)

We shall now consider the non-singular Hermitian matrices Γ satisfying

$$A_0^* \,\Gamma A_0 = \Gamma\,. \tag{4}$$

Corresponding to the partitioning of A_0 we divide the matrix in the natural way into $(\sum_{i=1}^{N} k_i)^2$ blocks and denote by

$$\Gamma_{vw}^{rs}\ (r\,, s = 1\,, 2\,, \ldots, N\,;\quad v = 1\,, 2\,, \ldots, k_r\,;\quad w = 1\,, 2\,, \ldots, k_s)$$

the part of Γ in the same position as the intersection of the v-th horizontal strip of $M_{\nu_r}(f_r)$ and the w-th vertical strip of $M_{\nu_s}(f_s)$ in the matrix A_0.

Put

$$L_0^{(i)} = 0\,,\quad L_\varkappa^{(i)} = l_1^{(i)} + l_2^{(i)} + \ldots + l_\varkappa^{(i)} \qquad (i = 1\,, 2\,, \ldots, N\,;\ \varkappa = 1\,, 2\,, \ldots, p_i)\,.$$

We use the notation $\mathfrak{A}_i$ for the following set of indices

$$\{L_1^{(i)}, L_2^{(i)}, \ldots, L_{p_i}^{(i)}\}$$

and we write $\mathfrak{B}_i$ for the complement of the set $\mathfrak{A}_i$ in the set $\{1, 2, \ldots, k_i\}$. For shortness, write $M_i = M(f_i)$. Then (4) is equivalent to the following conditions:

$$(5)\qquad \begin{cases} M_r^* \Gamma_{vw}^{rs} M_s + \Gamma_{v+1,w}^{rs} M_s + M_r^* \Gamma_{v,w+1}^{rs} + \Gamma_{v+1,w+1}^{rs} \\ = \Gamma_{vw}^{rs}, \quad \text{if } v \in \mathfrak{B}_r, w \in \mathfrak{B}_s, \\ M_r^* \Gamma_{vw}^{rs} M_s + \Gamma_{v+1,w}^{rs} M_s = \Gamma_{vw}^{rs}, \quad \text{if } v \in \mathfrak{B}_r, w \in \mathfrak{A}_s, \\ M_r^* \Gamma_{vw}^{rs} M_s + M_r^* \Gamma_{v,w+1}^{rs} = \Gamma_{vw}^{rs}, \quad \text{if } v \in \mathfrak{A}_r, w \in \mathfrak{B}_s, \\ M_r^* \Gamma_{vw}^{rs} M_s = \Gamma_{vw}^{rs}, \quad \text{if } v \in \mathfrak{A}_r, w \in \mathfrak{A}_s. \end{cases}$$

Suppose that M_s and M_r^{*-1} are not similar in $\mathfrak{G}_n$. Then it is easy to deduce, by lemma 4, from these equations, that all Γ_{vw}^{rs} $(v = 1, 2, \ldots, k_r;$ $w = 1, 2, \ldots, k_s)$ vanish. Since we want to have a non-singular Γ, we must have for each M_s exactly one M_r^{*-1} similar to it. (There cannot be more than one, because no two $M_r' s$ are similar.) Of course, r may be equal to s.

Suppose next that M_s and M_r^{*-1} are similar in $\mathfrak{G}_n$. This means the same as

$$f_r = \tilde{f}_s .$$

We denote by

$$\Gamma^{rs}(\varkappa, \mu) \qquad (\varkappa = 1, 2, \ldots, p_r;\ \mu = 1, 2, \ldots, p_s)$$

the block of the matrix Γ consisting of all matrices Γ_{vw}^{rs} with

$$(6)\qquad \begin{cases} L_{\varkappa-1}^{(r)} + 1 \leqq v \leqq L_{\varkappa}^{(r)}, \\ L_{\mu-1}^{(s)} + 1 \leqq w \leqq L_{\mu}^{(s)}. \end{cases}$$

Denote the set of ordered pairs (v, w) satisfying (6) by $\mathfrak{B}_{\varkappa\mu}^{rs}$. Furthermore, we denote by Γ^{rs} the block consisting of all $\Gamma^{rs}(\varkappa, \mu)$ $(\varkappa = 1, 2, \ldots, p_r;$ $\mu = 1, 2, \ldots, p_s)$.

Now all the blocks $\Gamma^{rs}(\varkappa, \mu)$ have a »triangular» form, that is

$$(7)\qquad \Gamma_{vw}^{rs} = 0, \quad \text{if } (v, w) \in \mathfrak{B}_{\varkappa\mu}^{rs} \text{ and } v + w \geqq L_{\varkappa-1}^{(r)} + L_{\mu-1}^{(s)} + \min(l_{\varkappa}^{(r)}, l_{\mu}^{(s)}) + 2 .$$

This can be seen as follows. Without loss of generality we may assume that $l_{\varkappa}^{(r)} \leqq l_{\mu}^{(s)}$. For shortness, write

$$H = M_r^*, \qquad K = M_s$$

and

$$X = \Gamma^{rs}_{L^{(r)}_{\varkappa} L^{(s)}_{\mu}} , \qquad Y = \Gamma^{rs}_{L^{(r)}_{\varkappa} , L^{(s)}_{\mu} - 1} .$$

(We assume that $l^{(s)}_{\mu} \geqq 2$, otherwise our assertion is trivial.) Then, by (5), we have

$$\text{(8)} \qquad \begin{cases} H\,X\,K = X \,, \\ H\,Y\,K + H\,X = Y \,. \end{cases}$$

If X is not 0, then, by lemma 4, it must be non-singular, and we have

$$H^{-1}\,Y\,X^{-1} - Y\,X^{-1}\,H^{-1} = 1 \,.$$

But this contradicts lemma 3. Hence we must have $X = 0$. Continuing similarly, we can decide that every Γ^{rs}_{vw} with $v = L^{(r)}_{\varkappa}$, $w \geqq L^{(s)}_{\mu-1} + 2$ is zero. Then take $v = L^{(r)}_{\varkappa} - 1$, etc. Thus (7) follows.

Next we note that the partitions ν_r and ν_s are identical. Here, of course, we suppose that $r \neq s$. Firstly, the determinant of the matrix consisting of the blocks Γ^{rr}, Γ^{rs}, Γ^{sr}, Γ^{ss} is obviously a factor of the determinant of Γ and hence it must be $\neq 0$. From this it follows at once that $k_r = k_s$, for if $k_r < k_s$, say, we get a contradiction by taking the Laplace expansion of the determinant with respect to the first $d_r k_r$ rows. Secondly, suppose that

$$l^{(r)}_1 = l^{(s)}_1 \,,\, l^{(r)}_2 = l^{(s)}_2 \,, \ldots ,\, l^{(r)}_{\lambda-1} = l^{(s)}_{\lambda-1} \,,\, l^{(r)}_{\lambda} > l^{(s)}_{\lambda} \,.$$

Then from what we have proved it follows that

$$\Gamma^{rs}_{vw} = 0 \quad \text{for} \quad v = L^{(r)}_{\lambda} \,, \quad w \ \text{being arbitrary,}$$

which is impossible. This proves the assertion. For simplicity of notation we can now write $\Gamma_{vw}, \Gamma(\varkappa, \mu), L_{\lambda}, l_{\lambda}, d, k, p, m_j$ instead of Γ^{rs}_{vw}, $\Gamma^{rs}_{vw}(\varkappa, \mu), L^{(r)}_{\lambda}, L^{(s)}_{\lambda}$, etc.

We consider now an arbitrary block $\Gamma(\varkappa, \mu)$. By (5) and (7), we have

$$\text{(9)} \qquad H\,\Gamma_{vw}\,K = \Gamma_{vw}$$

for all Γ_{vw} with $v + w = L_{\varkappa-1} + L_{\mu-1} + \min(l_{\varkappa}, l_{\mu}) + 1$ (i.e. situated on the diagonal). By induction on $v + w$ we now show that (9) is valid for all pairs $(v, w) \in \mathfrak{B}^{rs}_{\varkappa\mu}$. We may therefore suppose (9) to be true for the pairs $(v+1, w)$, $(v, w+1)$, $(v+1, w+1)$. Put

$$X = H^{-1}\,\Gamma_{v+1,w}\,K + \Gamma_{v,w+1} + H^{-1}\,\Gamma_{v+1,w+1} \,,$$
$$Y = \Gamma_{vw} \,.$$

But now (8) is valid. Hence our argument above shows that $X = 0$ and the assertion follows.

Let F be a non-singular matrix satisfying

$$H\,F\,K = F \,.$$

In what follows we keep F fixed. In the case $r = s$ we have $H = K^*$ and so $K^* F K = F$, $K^* F^* K = F^*$. Take $\beta \in \mathfrak{F}$ such that $F_1 = \beta F + \bar{\beta} F^* \neq 0$. Then F_1 is Hermitian and $K^* F_1 K = F_1$. By lemma 4, F_1 is non-singular. Hence in the case $r = s$ we may assume that F is Hermitian.

By lemma 3, obviously, every solution X of the matrix equation $H X K = X$ can be written in the form $X = F f(K)$, where f is a polynomial over $\mathfrak{F}$. We can thus write

$$\Gamma_{vw} = F \gamma_{vw} ;$$

where each γ_{vw} is a polynomial of K. Then

$$\Gamma^{rs} = \operatorname{diag}(F, F, \ldots, F)(\gamma_{vw}) .$$

Furthermore, by (5), we have

$$(10) \qquad K \gamma_{v+1,w} + K^{-1} \gamma_{v,w+1} + \gamma_{v+1,w+1} = 0 , \quad \text{if} \quad v \in \mathfrak{B}_r , w \in \mathfrak{B}_s .$$

If, in particular, $v + w = L_{\varkappa-1} + L_{\mu-1} + \min(l_\varkappa, l_\mu) + 1$, that is, we have a γ_{vw} situated on the diagonal, then the term $\gamma_{v+1,w+1}$ in (10) is 0.

The matrix K generates in the total d-dimensional matrix algebra over $\mathfrak{F}$, a field isomorphic to $GF(q^{2d})$. For counting purposes we may therefore consider the matrix (γ_{vw}) as a $k \cdot k$ matrix with elements in $\mathfrak{F}_d = GF(q^{2d})$. Namely, in the case $r \neq s$ we dont have any difficulties, because the Hermitian nature of Γ only requires that $\Gamma^{sr} = \Gamma^{rs*}$. In the case $r = s$ lemma 4 implies that $K^{q^d+1} = 1$. Hence it is easy to see that the $dk \cdot dk$ matrix Γ^{rr} with elements in $\mathfrak{F}$ is Hermitian if and only if the $k \cdot k$ matrix (γ_{vw}) with elements in $\mathfrak{F}_d$ is Hermitian.

For each j $(1 \leqq j \leqq k)$ such that $m_j \geqq 1$ we define $\varkappa_j$ such that

$$l_{\varkappa_j+1} = \ldots = l_{\varkappa_j+m_j} = j .$$

Denote by P_j the block consisting of m_j^2 blocks $\Gamma(\varkappa, \mu)$ such that

$$\varkappa_j + 1 \leqq \varkappa , \mu \leqq \varkappa_j + m_j .$$

We also define a $m_j \cdot m_j$ matrix Q_j with elements in $\mathfrak{F}_d$, which we call the *principal matrix* associated with the block P_j, as follows. Let $(Q_j)_{\tau\sigma}$ be the element of Q_j in the τ-th row and σ-th column. If j is odd, put

$$(Q_j)_{\tau\sigma} = \gamma_{L_{\varkappa_j} + \frac{1}{2}(j+1) + j(\tau-1),\, L_{\varkappa_j} + \frac{1}{2}(j+1) + j(\sigma-1)} \cdot$$

Take an arbitrary fixed element β in $\mathfrak{F}_d$ such that $\beta^{q^d} + \beta K^2 \neq 0$. (If $K^2 \neq -1$ we may simply take $\beta = 1$.) Then in the case j is even we put

$$(Q_j)_{\tau\sigma} = \beta \gamma_{L_{\varkappa_j}+\frac{1}{2}j-1+j(\tau-1),\, L_{\varkappa_j}+\frac{1}{2}j+1+j(\sigma-1)} + \beta^{q^d} \gamma_{L_{\varkappa_j}+\frac{1}{2}j+1+j(\tau-1),\, L_{\varkappa_j}+\frac{1}{2}j-1+j(\sigma-1)} .$$

Using (10) and Laplace expansions of the determinant of the matrix (γ_{vw}) it is not hard to see that this determinant is a non-zero multiple of a power product of the determinants of the principal matrices (Cf. [1], p. 234 and, especially, the paper referred to in the footnote on that page). Hence in order to make (γ_{vw}) non-singular, we must make all the principal matrices non-singular.

We now distinguish between the cases $r \neq s$ and $r = s$. Suppose first that $r \neq s$. We count the number of possible (γ_{vw}) matrices (which is the same as the number of Γ^{rs} matrices). Consider first the block P_j. The principal matrix Q_j can be chosen in $g(m_j, q^{2d})$ ways. After the choice of Q_j the conditions (10) determine all elements γ_{vw} with

$$v + w = 2\, L_{\varkappa_j} + 1 + j\,(\tau + \sigma - 1) \qquad (\tau\,, \sigma = 1\,, 2\,, \ldots\,, m_j)\,.$$

Then we may choose freely $(j - 1)\, m_j^2$ elements γ_{vw}, say with

$$v = L_{\varkappa_j} + 1 + j\,(\tau - 1) \quad (\tau = 1\,, 2\,, \ldots\,, m_j)\,,$$
$$w = L_{\varkappa_j} + j\,\sigma \quad (\sigma = 1\,, 2\,, \ldots\,, m_j)\,.$$

This gives us

$$\exp\,(2d\,(j - 1)\, m_j^2)\, g(m_j\,, q^{2d})$$

possibilities. After this everything else in P_j is determined by the conditions (10). Outside the blocks P_j we may choose freely every γ_{vw} with $v = L_\varkappa + 1$ $(\varkappa = 1\,, 2\,, \ldots\,, p)$, except the known zeros, of course. This gives us

$$\exp\,(4d \sum_{\substack{j,l=1\\ j<l}}^{k} j m_j m_l)$$

possibilities. The total number is thus

$$\exp\,\{2d\,(\sum_{j=1}^{k} (j - 1)\, m_j^2 + 2 \sum_{\substack{j,l=1\\ j<l}}^{k} j\; m_j m_l)\} \prod_{j=1}^{k} g(m_j\,, q^{2d})\,.$$

Suppose next that $r = s$. Then we have to count the number of all possible non-singular Hermitian matrices (γ_{vw}). Consider first the block P_j. For the principal matrix Q_j we have, by lemma 1,

$$\frac{g(m_j\,, q^{2d})}{u(m_j\,, q^{2d})}$$

possibilities. After the choice of Q_j, again, the conditions (10) determine all elements γ_{vw} with

$$v + w = 2\,L_{\varkappa_j} + 1 + j\,(\tau + \sigma - 1) \qquad (\tau, \sigma = 1\,, 2\,, \ldots, m_j)\,.$$

Now it is easy to see that every other »diagonal» with $v + w =$ constant inside a block $\Gamma(\varkappa, \mu)$ with $\varkappa = \mu$ can be chosen in q^d ways. Namely, if the number of elements in it is odd and the middle term is γ, say, we must have $\gamma = \gamma^{q^d}$ and hence q^d possible γ's; if, on the other hand, this number is even and γ', γ'' are the middle terms, we consider $\beta\gamma' + \beta^{q^d}\gamma''$. In the other blocks the argument is the same as above, but now we have only one half the number of free choices compared with the above case, because of the Hermitian nature of Γ^{rs}. Hence the total number is in this case

$$\exp\left\{2d\left(\tfrac{1}{2}\sum_{j=1}^{k}(j-1)\,m_j^2 + \sum_{\substack{j,l=1\\ j<l}}^{k} j\ m_j m_l\right)\right\}\prod_{j=1}^{k}\frac{g(m_j\,,q^{2d})}{u(m_j\,,q^{2d})}\,.$$

5. Without going into details we shall now sketch the proof that $\mathfrak{C}(A_0)$ operates transitively on $\Gamma(A_0)$. The most straightforward proof goes as follows. Take a fixed $\Gamma_0 \in \Gamma(A_0)$ as simple as possible, e.g. such that all the blocks $\Gamma(\varkappa, \mu)$ with $\varkappa \neq \mu$ are zero. Then write down the most general form of a matrix $B \in \mathfrak{C}(A_0)$. Then by a counting argument somewhat analogous to that carried through above, one can see that the number of matrices $B^*\Gamma_0 B$, $B \in \mathfrak{C}(A_0)$ is the same as the total number $\gamma(A_0)$ of matrices in the set $\Gamma(A_0)$.

6. By lemma 5, we can now state the final result as follows.

Theorem. *The matrix* $A \in \mathfrak{G}_n$ *is similar to a matrix of* $\mathfrak{U}_n$ *if and only if for every index* r $(1 \leqq r \leqq N)$ *there is exactly one index* s $(1 \leqq s \leqq N)$ *such that*

$$f_r = \tilde{f}_s \quad \text{and} \quad \nu_r = \nu_s\,.$$

In this case the elements of $\mathfrak{U}_n$ *similar to* A *in* $\mathfrak{G}_n$ *form exactly one conjugacy class of* $\mathfrak{U}_n$ *and the number of elements in it equals*

$$u(n\,,q^2)\prod_{i=1}^{N}\left[\exp\left\{2d_i\left(\tfrac{1}{2}\sum_{j=1}^{k}(j-1)\,m_j^{(i)2} + \sum_{\substack{j,l=1\\ j<l}}^{k} j\ m_j^{(i)}m_l^{(i)}\right)\right\}\right.$$
$$\left.\prod_{j=1}^{k_i}\Theta\,(m_j^{(i)}\,,q^{2d_i})\right]^{-1}\,,$$

where

$$\Theta\,(m_j^{(i)}\,,q^{2d_i}) = \begin{cases} u(m_j^{(i)}\,,q^{2d_i})\,, & \text{if } f_i = \tilde{f}_i\,,\\ [g(m_j^{(i)}\,,q^{2d_i})]^{1/2}\,, & \text{if } f_i \neq \tilde{f}_i\,,\end{cases}$$

and $\exp(x)$ *stands for* q^x.

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