ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

306

ON THE GRAM DETERMINANT AND LINEAR TRANSFORMATIONS OF HILBERT SPACE

BY

KLAUS VALA

HELSINKI 1961
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1962.306

Communicated 8 September 1961 by R. and F. Nevanlinna

KESKUSKIRJAPAINO
HELSINKI 1961

On the Gram Determinant and Linear Transformations of Hilbert Space

1. *Introduction.* Recently Everitt proved two theorems concerning inequalities for Gram determinants of L^2-integrable functions [2]. The first of these teorems can be found — as the author remarks — in [1].

Moppert generalized these inequalities in [3]. He replaced the partial map by an arbitrary orthogonal projection of a Hilbert space. This was based on the following remark: Let E_1 and E_2 be two measurable sets and $E_1 \subset E_2$. Let $L^2(E_i)$ be the space of L^2-integrable functions in E_i $(i = 1, 2)$. The space $L^2(E_1)$ can then be considered as a subspace of $L^2(E_2)$. Let φ_{E_1} be the characteristic function of the set E_1. Then the map $f \to \varphi_{E_1} f$ is an orthogonal projection of the space $L^2(E_2)$ on the subspace $L^2(E_1)$.

In the present paper a generalization of the results of Everitt and Moppert will be given. The projections are here replaced by arbitrary bounded linear transformations. It is shown how the (generalized) Courant-Hilbert inequality can be deduced as a consequence of (generalized) Everitt inequality. A necessary and sufficient condition for equality in (generalized) Courant-Hilbert inequality will be given and finally a simple »geometric» interpretation is pointed out.

2. *The Gram determinant.* Let H be a Hilbert space[1]). The inner product of x and y is denoted by the symbol (x, y) and the norm of vector x by $|x|$. The Gram determinant of vectors $x_1, \ldots, x_n$ is

$$G(x_1, x_2, \ldots, x_n) = \begin{vmatrix} (x_1, x_1) & (x_1, x_2) & \ldots & (x_1, x_n) \\ (x_2, x_1) & (x_2, x_2) & \ldots & (x_2, x_n) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ (x_n, x_1) & (x_n, x_2) & \ldots & (x_n, x_n) \end{vmatrix} .$$

Obviously $G(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = G(x_1, x_2, \ldots, x_n)$, where σ is any permutation of numbers $1, 2, \ldots, n$. It is easy to see that $G(x_1, x_2, \ldots, x_n) \geqq 0$ and $G(x_1, x_2, \ldots, x_n) = 0$ if and only if the vectors x_i are linearly dependent.

---

1) The scalars may be real or complex. We do not assume that H is separable.

We denote the linear subspace generated by vectors $y_1, y_2, \ldots, y_k$ by $L(y_1, y_2, \ldots, y_k)$.

Let now $x_1, x_2, \ldots, x_n$ be arbitrary vectors in H. We set

$$x_n = x_n' + x_n'' ,$$

where x_n'' is the orthogonal projection of x_n on the space $L(x_1, x_2, \ldots x_{n-1})$. In other words: $(x_n', x_i) = 0$, when $i = 1, 2, \ldots, n-1$, and $x_n'' \in L(x_1, x_2, \ldots, x_{n-1})$. It is well-known that x_n' and x_n'' are unique. By bilinearity of the inner product and elementary properties of determinants we have

$$G(x_1, x_2, \ldots, x_n) = G(x_1, x_2, \ldots, x_{n-1}) \, |x_n'|^2 . \tag{1}$$

Repeating in the same manner we set $x_h = x_h' + x_h''$ $(h = n-1, n-2, \ldots, 2)$, where x_h'' is the orthogonal projection of x_h on the subspace $L(x_1, x_2, \ldots, x_{h-1})$, and we thus have

$$G(x_1, x_2, \ldots, x_n) = |x_1|^2 \, |x_2'|^2 \ldots |x_n'|^2 .$$

If the vectors x_i are linearly independent we have $x_1 \neq 0$ and $x_i' \neq 0$ $(i = 2, \ldots, n)$ and then $G(x_1, x_2, \ldots, x_n) > 0$.

3. *Inequalities.* From (1) we get immediately the inequality

$$G(x_1, x_2, \ldots, x_n) \leqq G(x_1, x_2, \ldots, x_{n-1}) \, |x_n|^2 , \tag{2}$$

where equality holds if and only if one (or both) of the following conditions is satisfied: 1) The vectors $x_1, x_2, \ldots, x_{n-1}$ are linearly dependent. 2) $x_n' = x_n$, this condition being equivalent to $|x_n'| = |x_n|$.

Let T be a bounded linear transformation of H. We consider n vectors $x_1, x_2, \ldots, x_n$ and from now on we assume always that the vectors x_i are linearly independent. The vectors $Tx_1, Tx_2, \ldots, Tx_n$ may be linearly independent or not. We split the vector Tx_n in two orthogonal components $Tx_n = (Tx_n)' + (Tx_n)''$, where $(Tx_n)''$ is the orthogonal projection of Tx_n on the subspace $L(Tx_1, \ldots, Tx_{n-1})$. Let $x_n = x_n' + x_n''$ as before. $Tx_n = Tx_n' + Tx_n''$ and $Tx_n'' \in L(Tx_1, \ldots, Tx_{n-1})$. By the minimal property of the normal vector $(Tx_n)'$ we thus have

$$|(Tx_n)'| \leqq |Tx_n'| , \tag{3}$$

where the sign of equality holds if and only if $(Tx_n)' = Tx_n'$. Since $Tx_n = Tx_n' + Tx_n'' = (Tx_n)' + (Tx_n)''$ this condition is equivalent to $(Tx_n)'' = Tx_n''$.

Then we have by (1), (2) and (3)

$$G(Tx_1, \ldots, Tx_n) = G(Tx_1, \ldots, Tx_{n-1}) |(Tx_n)'|^2$$
$$\leqq G(Tx_1, \ldots, Tx_{n-1}) |Tx_n'|^2 \leqq |T|^2 G(Tx_1, \ldots, Tx_{n-1}) |x_n'|^2$$
$$= |T|^2 G(Tx_1, \ldots, Tx_{n-1}) \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_{n-1})}.$$

We thus obtain the

Theorem 1: *If the vectors $x_1, x_2, \ldots, x_n$ are linearly independent and the linear transformation T is bounded we have the following inequality*

$$G(Tx_1, \ldots, Tx_n) \leqq |T|^2 G(Tx_1, \ldots, Tx_{n-1}) \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_{n-1})}, \tag{4}$$

where the equality holds if and only if one (or both) of the following conditions is satisfied: 1) The vectors $Tx_1, \ldots, Tx_{n-1}$ are linearly dependent (in which case both sides in (4) are $= 0$). 2) $Tx_n' = (Tx_n)'$ and $|Tx_n'| = |T| |x_n'|$.

By repeated use of *(4)* we get first

$$G(Tx_1, \ldots, Tx_n) \leqq |T|^4 G(Tx_1, \ldots, Tx_{n-2}) \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_{n-2})},$$

and so on. Finally we have

$$G(Tx_1, \ldots, Tx_n) \leqq |T|^{2n-2} \frac{|Tx_1|^2}{|x_1|^2} G(x_1, \ldots, x_n).$$

Since G is symmetric we can replace the index 1 by any index i and we thus have

$$G(Tx_1, \ldots, Tx_n) \leqq |T|^{2n-2} \min_{1 \leq i \leq n} \frac{|Tx_i|^2}{|x_i|^2} G(x_1, \ldots, x_n). \tag{5}$$

From this we get immediately the less sharp inequality

$$G(Tx_1, \ldots, Tx_n) \leqq |T|^{2n} G(x_1, \ldots, x_n). \tag{6}$$

4. *Special cases.* Let $T = P =$ orthogonal projection, i.e. $P^2 = P = P^*$. $|Px| = |x|$ if and only if $Px = x$, thus the condition 2) in theorem 1 now reads: $(Px_n)' = Px_n' = x_n'$. We assert that this condition is equivalent to $(Px_n)' = x_n'$. Assume that the latter is the case. We have by (3): $|(Px_n)'| \leqq |Px_n'| \leqq |x_n'|$ and hence $|(Px_n)'| = |Px_n'| = |x_n'|$, but this implies $(Px_n)' = Px_n' = x_n'$.

We thus have the following result: When $T = P$ the inequality (4) has the form

$$G(Px_1, \ldots, Px_n) \leqq G(Px_1, \ldots, Px_{n-1}) \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_{n-1})}, \tag{7}$$

where equality holds if and only if one (or both) of the following conditions is satisfied: 1) The vectors $Px_1, \ldots, Px_{n-1}$ are linearly dependent. 2) $(Px_n)' = x_n'$.

The inequality (7) is due to Everitt and Moppert (cf. [2] and [3]).

When $T = P$ the inequality (6) becomes

$$G(Px_1, \ldots, Px_n) \leqq G(x_1, \ldots, x_n), \tag{8}$$

where equality holds (by inequality (5)) if and only if $Px_i = x_i$ $(i = 1, 2, \ldots, n)$. This is the Courant-Hilbert inequality (cf. [1] pp. 107—108).

5. *Investigation of inequality (6).* Assume that $|T| > 0$, that is, T is not identically $= 0$. Then, by (5), a necessary condition for equality in (6) is $|Tx_i| = |T||x_i|$ $(i = 1, 2, \ldots, n)$. We shall see that this condition is also sufficient. For this purpose we first prove the following

Lemma: *Let $x_1, x_2, \ldots, x_n$ be linearly independent vectors in the Hilbert space H and let T be a bounded linear transformation of H and $|T| > 0$. If $|Tx_i| = |T||x_i|$ $(i = 1, 2, \ldots, n)$, so $T = |T|V$, where V maps the subspace $L = L(x_1, x_2, \ldots, x_n)$ isometrically on the subspace $T(L)$.*

Proof. Let E be the orthogonal projection of H on the subspace L and let x and y be in L. We have

$$(ET^*Tx, y) = (T^*Tx, Ey) = (T^*Tx, y) = (x, T^*Ty) = (Ex, T^*Ty)$$
$$= (x, ET^*Ty).$$

Thus the restriction $(ET^*T)_L$ of transformation ET^*T to the subspace L is self-adjoint (and completely continuous, the dimension of L being finite). We have for x_i $(i = 1, 2, \ldots, n)$

$$|T|^2|x_i|^2 = |Tx_i|^2 = (Tx_i, Tx_i) = (T^*Tx_i, x_i) = (T^*Tx_i, Ex_i) = (ET^*Tx_i, x_i)$$
$$\leqq |ET^*Tx_i||x_i| \leqq |ET^*T||x_i|^2 \leqq |T^*||T||x_i|^2 = |T|^2|x_i|^2.$$

It follows that there is equality everywhere in the above inequalities and thus we have $|ET^*T| = |T|^2$ and $|ET^*Tx_i| = |ET^*T||x_i|$. As the transformation $(ET^*T)_L$ is self-adjoint and completely continuous this implies (cf. [4] p. 229) that every vector x_i is an eigenvector of $(ET^*T)_L$ and $ET^*Tx_i = |T|^2x_i$. Since the vectors x_i form a base of L it follows that $(ET^*T)_L = |T|^2I_L$, where I_L denotes the identity mapping of L.

Let now x and y be in L. We have

$$(Tx, Ty) = (T^*Tx, y) = (T^*Tx, Ey) = (ET^*Tx, y) = |T|^2(x, y).$$

The restriction of the transformation $\dfrac{T}{|T|}$ to L thus maps L isometrically on the subspace $T(L)$. Q.E.D.

By this lemma we are now able to prove the following

Theorem 2: *Let the vectors* $x_1, x_2, \ldots, x_n$ *be linearly independent and* $|T| > 0$. *Then* $G(Tx_1, \ldots, Tx_n) = |T|^{2n} G(x_1, \ldots, x_n)$ *if and only if* $|Tx_i| = |T||x_i|$ $(i = 1, 2, \ldots, n)$.

As already stated, the *necessity* follows from (5).

Sufficiency. By lemma we have $T = |T|V$, where the restriction of V to $L = L(x_1, x_2, \ldots, x_n)$ maps L isometrically on $T(L)$. We set again $x_i = x_i' + x_i''$, where x_i'' is the orthogonal projection of x_i on the subspace $L(x_1, x_2, \ldots, x_{i-1})$ $(i = 2, \ldots, n)$. We have

$$G(Tx_1, \ldots, Tx_n) = G(Tx_1, Tx_2', \ldots, Tx_n') = |Tx_1|^2 |Tx_2'|^2 \ldots |Tx_n'|^2$$
$$= |T|^{2n} |x_1|^2 |x_2'|^2 \ldots |x_n'|^2 = |T|^{2n} G(x_1, x_2', \ldots, x_n')$$
$$= |T|^{2n} G(x_1, x_2, \ldots, x_n) .$$

The proof is thus completed.

6. *A remark.* The above lemma may have some interest in itself. Let e.g. A be a contraction of n-dimensional (n is finite) linear space R^n, that is, $|Ax| \leqq |x|$ for every $x \in R^n$. If then $|Ax_i| = |x_i|$ $(i = 1, 2, \ldots, n)$ and the vectors x_i are linearly independent, so, by lemma, A is a unitary transformation of R^n.

7. *A »geometric» interpretation.* The determinant

$$G(x_1, \ldots, x_n | y_1, \ldots, y_n) = \det (x_i, y_k) \qquad (1 \leqq i, k \leqq n)$$

defines an inner product in the n-th exterior power of H (cf. [5] p. 20). The above considerations can be properly interpreted in terms of exterior powers of linear transformations. Thus, for example, the exterior power of an orthogonal projection is an orthogonal projection (in the metric defined by the determinant G, cf. [5] p. 36). Interpreted in this way, the Courant-Hilbert inequality (8) states only the simple fact that the norm of any orthogonal projection of a vector is at most equal to the norm of the vector.

Bibliography

[1] COURANT, R. and HILBERT, D.: Methods of mathematical physics, Vol. I. - New York—London (1953).

[2] EVERITT, W. N.: Inequalities for Gram determinants. - Quart. J. Math., Oxford Ser., Vol. 8, No 31 (1957).

[3] MOPPERT, C. F.: On the Gram determinant. - Quart. J. Math., Oxford Ser., Vol. 10, No 39 (1959).

[4] RIESZ, F. et SZ. NAGY, B.: Leçons d'analyse fonctionelle. - Budapest (1953).

[5] VALA, K.: Sur la puissance extérieure d'un espace linéaire. - Ann. Acad. Sci. Fenn. A. I. 233 (1956).

Printed December 1961.